Quantifying Ocean Mixing from Hydrographic Data

Jan David Zika

September 2009

Supervisors: Trevor McDougall, Bernadette Sloyan and Matthew England

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy
Declaration

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgment is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project’s design and conception or in style, presentation and linguistic expression is acknowledged.
Abstract

The relationship between the general circulation of the ocean and, along-isopycnal and vertical mixing is explored. Firstly, advection down isopycnal tracer gradients is related to mixing in specific regions of the ocean. Secondly, a general inverse method is developed for estimating both mixing and the general circulation.

Two examples of down gradient advection are explored. Firstly the region of Mediterranean outflow in the North Atlantic. Given a known transport of warm salty water out of the Mediterranean Sea and the mean hydrography of the eastern North Atlantic, the vertical structure of the along-isopycnal mixing coefficient, $K$, and the vertical mixing coefficient, $D$, is revealed. Secondly, the Southern Ocean Meridional Overturning Circulation, SMOC, is investigated. There, relatively warm salty water is advected southward, along-isopycnals, toward fresher cooler surface waters. The strength and structure of the SMOC is related to $K$ and $D$ by considering advection down along-isopycnal gradients of temperature and potential vorticity. The ratio of $K$ to $D$ and their magnitudes are identified.

A general tool is developed for estimating the ocean circulation and mixing; the tracer-contour inverse method. Integrating along contours of constant tracer on isopycnals, differences in a geostrophic streamfunction are related to advection and hence to mixing. This streamfunction is related in the vertical, via an analogous form of the depth integrated thermal wind equation. The tracer-contour inverse method combines aspects of the box, beta spiral and Bernoulli methods. The tracer-contour inverse method is validated against the output of a layered model and against in-situ observations from the eastern North Atlantic. The method accurately reproduces the observed mixing rates and reveals their vertical structure.
Acknowledgments

In 2005, I was investigating the possibility of a career in Physical Oceanography. I came to the office of a man known to me as Trevor ‘the Theorist’. His office seemed to be the continuation of a corridor of overhanging shelves overloaded with ship-reports and complicated instrumentation. The room was dimly lit. Trevor had proudly clad his walls with rejection letters from scientific journals. He described to me a tantalising project. The project became the beginnings of my research as an oceanographer, and the fantastic journey that this thesis has been. Along the way we entrained Bernadette. She brought us back to reality and made us see the big picture when those corridor walls had all but fallen in. I thank both Trevor and Bernadette for their wonderful and tireless supervision.

Matt England has been a great confidant to the North, as has Steve Rintoul on my excursions, both physically and academically, to the South. While in Woods Hole in 2007 I learned a great deal from Steve Thorpe and my experiences at GFD have contributed greatly to my formation as a scientist. Time spent with Steve Griffies and Robert Hallberg has helped me keep a third foot in the numerical modelling space and I thank them for their data, time and friendship. I am also grateful to Susan Wijffels, Nathan Bindoff, Andreas Thurnherr, Carl Wunsch, Jean-Baptiste Salèe and Peter McIntosh for valuable comments and discussions on aspects of this thesis.

Over the years I shared an office, a supervisor, a few climbing trips, a journey to Israel and of course a few beers with my good friend Andreas Klocker. I have gained a lot from my friendship and collaboration with Paul Durack particularly through our work, along with Ben Galton-Fenzi and Andrew Meijers, on re-establishing the Society of Sub-Professional Oceanographers (Southern Hemisphere Branch). Thanks also to Willem Sijp and Michael
Bates for fruitful discussions at UNSW.

Although I have not subjected them to reading this thesis, except this section, I would like to thank my girlfriend Ella and my parents Paul and Katrina, for their loving support over the years.
Contents

Statement of Originality i

Abstract ii

Acknowledgments iii

Supporting Publications xxiv

1 General Introduction 1

1.1 Vertical mixing . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2
1.2 Along-Isopycnal Mixing . . . . . . . . . . . . . . . . . . . . . 3
1.3 Inverse Methods . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
1.4 This Thesis . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
  1.4.1 Chapter 2 . . . . . . . . . . . . . . . . . . . . . . . . . . 7
  1.4.2 Chapter 3 . . . . . . . . . . . . . . . . . . . . . . . . . . 8
  1.4.3 Chapter 4 . . . . . . . . . . . . . . . . . . . . . . . . . . 9
  1.4.4 Chapter 5 . . . . . . . . . . . . . . . . . . . . . . . . . 10
  1.4.5 Chapter 6 . . . . . . . . . . . . . . . . . . . . . . . . . 11

2 Vertical and Lateral Mixing Processes Deduced from the
  Mediterranean Water Signature in the North Atlantic 12

2.1 Abstract . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 13
3 Diagnosing the Southern Ocean Overturning from Tracer Fields

3.1 Abstract .......................................................... 52
3.2 Introduction ...................................................... 53
3.3 Water Mass Equation and Cross-Contour Flow ............... 57
3.4 The Southern Ocean Overturning .............................. 59
3.5 The Residual Mean Overturning and Bolus Velocity ....... 69
3.6 Diapycnal flow and the 1-D balance .......................... 74
3.7 The relative role of vertical and along-isopycnal mixing .... 76
3.8 Discussion and conclusions ..................................... 78

4 A Tracer-Contour Inverse Method for Estimating Ocean Circulation and Mixing

4.1 Abstract .......................................................... 84
4.2 Introduction ...................................................... 85
4.3 Tracer-contours .................................................. 87
4.4 Application of the tracer-contour equations ................. 93
4.4.1 The Hallberg Isopycnal Model and Study Regions ....... 94
4.4.2 Contours Only ............................................... 95
4.4.3 The thickness-weighted mean box inverse method ....... 100
4.4.4 The Bernoulli streamfunction inverse method .......... 107
4.5 Effect of random error ........................................ 108
<table>
<thead>
<tr>
<th>Appendix: Chapter 2</th>
<th>147</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.1 The three extra terms in equation (2.20)</td>
<td>147</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Appendix: Chapter 3</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.1 Derivation of the Water Mass Equation</td>
<td>150</td>
</tr>
<tr>
<td>B.2 Derivation of the Density Equation</td>
<td>153</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Appendix: Chapter 4</th>
<th>155</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.1 Derivation of the water mass equation</td>
<td>155</td>
</tr>
<tr>
<td>C.2 Derivation of the density equation</td>
<td>158</td>
</tr>
<tr>
<td>C.3 Derivation of an isopycnal advective-diffusive balance equation for a conservative tracer $C$</td>
<td>160</td>
</tr>
<tr>
<td>C.4 Reference level streamfunction at a fixed pressure</td>
<td>162</td>
</tr>
<tr>
<td>C.5 Defining tracer-contours</td>
<td>163</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Appendix: Chapter 5</th>
<th>165</th>
</tr>
</thead>
<tbody>
<tr>
<td>D.1 Uncertainties and Sensitivity Analysis</td>
<td>165</td>
</tr>
<tr>
<td>D.2 Mixing lengths, heights and thermal wind</td>
<td>169</td>
</tr>
</tbody>
</table>

References | 171 |

References | 171 |
List of Figures

2.1 Salinity - conservative temperature diagram of three vertical “casts” of the atlas hydrography near the Gulf of Cadiz at locations illustrated in Fig.2.2. The four dashed lines in are straight-line approximations to where the four neutral density surfaces lie in this figure. . . . . . . . . . . . . . . . . . . . . . . . . . . . 25

2.2 Contours of salinity on the neutral density surface $\gamma^n = 27.70$ kg m$^{-3}$. Also shown are the locations of “casts” (Fig.2.1) of the atlas hydrography near the Gulf of Cadiz. . . . . . . . . . . . . . . . . 26

2.3 Contours of $\Theta_0$ on the $\gamma = 27.70$ kg m$^{-3}$ density surface. These five contours define the western-most edge of five areas, each of which extends eastwards as far as $351^\circ$E. The colour bar and contours are of conservative temperature (°C). . . . . 27

2.4 The volume flux of Mediterranean Water flowing into the North Atlantic as a function of neutral density layer (following Baringer and Price, 1997). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 30

2.5 Ten examples of the linear equation (2.20) relating $K$ and $D$ (for the known values of $FQ$). The five lines that nearly coincide at the larger values of $D$ and $K$ are for the five contours of the 27.65 - 27.70 density layer while the other five lines are for the 27.70 - 27.725 density layer. . . . . . . . . . . . . . . . 33
2.6 The dianeutral diffusivity $D$ for each density layer, determined assuming that both $D$ and $K$ are constant in each layer. . . . 34

2.7 The along-isopycnal diffusivity $K$ for each density layer, determined assuming that both $D$ and $K$ are constant in each layer. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 35

2.8 Using the values of $D$ and $K$ for each layer (from Table 2.3 and Figure 2.6 and Figure 2.7), this figure shows the fractional contribution to the right hand side of (2.20) from the term in $D$ and the term in $K$ for each of the five equations for each layer. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 39

2.9 Contours of $\theta_0$ on the $\sigma_2=36.2$ kg m$^{-3}$ density layer. These five contours define the western-most edge of five areas. The eastern most edge is also shown where the flux into a given area is the sum of flux values through that boundary. The colour bar and contours are of potential temperature (°C). . . 42

2.10 Ten examples of the linear equation (2.20) relating $K$ and $D$ (for the known values of $FQ$). Unlike the hydrographic atlas data the lines representing (2.20) for the $\sigma_2 = 36.3$ kg m$^{-3}$ layer (grey lines) are close to parallel suggesting $K$ and $D$ should not be determined for each layer independently. . . . . 44

2.11 The dianeutral diffusivity $D$ for each set of areas in the vertical, determined assuming that both $D$ and $K$ are constant within all the areas in the vertical, assuming $F = 1$ (grey and dashed), $F \neq 1$ (black) and the explicit value in HIM (small dashed). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 46
2.12 The epineutral diffusivity $K$ for each set of areas in the vertical, determined assuming that both $D$ and $K$ are constant within all the areas in the vertical, assuming $F = 1$ (grey and dashed), $F \neq 1$ (black) and the explicit value in HIM (small dashed).

3.1 Temperature gradient and positive curvature on an isopycnal. Darker grey represents warmer. Along-Isopycnal Mixing (bent arrows) acts to smooth out the temperature gradient. In the case of no vertical mixing there must be an up gradient advection $\mathbf{v} \cdot \mathbf{n}_\Theta$ (thick grey arrow) if the curvature is to be maintained in steady state.

3.2 A $\Theta - S$ curvature exists down the water column (solid line). Vertical mixing (curved arrows) acts to smooth this curvature. Temperature and salinity must be advected by $\mathbf{v} \cdot \mathbf{n}_\Theta$ (solid arrows) along-isopycnals to maintain the curvature in steady state.

3.3 (a) Colormap of conservative temperature (°C) along the ACC on $\gamma_n = 27.7 \text{ kg m}^{-3}$ with positions of the northern, central and southern contours shown (dashed lines). (b) Temperature and salinity of northern (red), central (green) and southern contours (blue) whose extent is fully circumpolar between neutral densities $\gamma_n = 27.2 \text{ kg m}^{-3}$ and $\gamma_n = 28 \text{ kg m}^{-3}$.
3.4 Contributions to layer cross-contour transport from the along-isopycnal mixing term (taking $K$ to be $200 \text{ m}^2 \text{s}^{-1}$; black bars) and the vertical mixing terms (taking $D$ to be $2 \times 10^{-4} \text{ m}^2 \text{s}^{-1}$; white bars). Transports are across the southern (a), central (b) and the northern contours (c) of the ACC. The temperature and salinity of each contour is marked with a circle in of Fig.3.3b. Positive values are with increasing temperature (northward for the layers shown).

3.5 Terms contributing to the cumulative cross-contour transport between $\gamma_n = 27.4 \text{ kg m}^{-3}$ and $\gamma_n = 28 \text{ kg m}^{-3}$ due to the along-isopycnal mixing term ($K = 200 \text{ m}^2 \text{s}^{-1}$; solid line) and the vertical mixing term ($D = 2 \times 10^{-4} \text{ m}^2 \text{s}^{-1}$; dot-dashed line). Transport across the (a) southern, (b) central and (c) northern contours of the ACC. Positive values are with increasing temperature (northward for the layers shown).
3.6 Black contours show the northward transport (Sv) of UCDW between $\gamma_n = 27.4$ kg m$^{-3}$ and $\gamma_n = 28$ kg m$^{-3}$ into the northern side of the ACC from the sum of terms in (3.2) for various values of $K$ and $D$. Grey shading across the centre of the figure represents abyssal estimates of $D$ from both Munk (1966) and Ledwell et al. (1993). Below the x-axis, colored bars show relevant estimates of lateral or along-isopycnal diffusivities from McKeague et al. (2005) (black bar-$K^y$; grey bar-$K^x$), Phillips and Rintoul (2000) (red bar), Gille (2003) (green bar) and Naveira-Garabato et al. (2007) (blue bar). UCDW transport estimates across hydrographic sections at 30-40$^\circ$S are shown from Lumpkin and Speer (2007) (20 Sv; green contour) and Sloyan and Rintoul (2001) (52 Sv; red contour). Blue lines represent estimates of the ratio $K/D$ below $27.7\gamma_n$, made in section 3.7, plus or minus one standard deviation. Taking the spread between the green and red contours to be a reasonable error range for the UCDW transport, the predicted range for the circumpolarly averaged mixing coefficients is cross hatched in light blue.
3.7 (a) Potential vorticity \( (PV = f/h \text{ for } h = 0.1 \text{kg m}^{-3}/\rho_z) \) on \( \gamma_n = 27.7 \text{ kg m}^{-3} \) overlayed with the northern \( PV \) contour along the ACC. (b) The depth of layers between \( \gamma_n = 27.5 \text{ kg m}^{-3} \) and \( \gamma_n = 27.8 \text{ kg m}^{-3} \) (grey shaded area) along the path of the contour in (a). In the shaded area the mean geostrophic flow does not contribute to transport across the ACC \( (\oint \vec{v} dx_{PV} = 0) \) in isopycnal coordinates. Above and below the shaded area, the net geostrophic transport due to the mean flow may be present in each isopycnal layer. In depth coordinates there is no contribution to the overturning from the mean geostrophic flow above the shallowest topographic feature and at latitudes of Drake Passage \( (\oint dx = 0 \text{ where } x \text{ is the zonal coordinate}) \). Below topography a zonal pressure gradient may exist allowing for meridional transport due to the mean geostrophic flow. Note: (b) is a cross section of isopycnals at latitudes and longitudes of the contour shown in (a) not the depths of the individual PV contours on each isopycnal.

3.8 Contributions to bolus transport for \( K = 200 \text{ m}^2 \text{s}^{-1} \) from the thickness gradient term (black bars) and the beta gradient term (white bars) in (3.8). (a) Southern contour, (b) central contour and (c) northern contours of the ACC. Positive values are with increasing \( PV \) (northward for the layers shown).
3.9 (a) Diapycnal velocity ($\log_{10}$, m s$^{-1}$) due to the diapycnal diffusivity (second term on right hand side of (3.10)) for $D = 2 \times 10^{-4}$ m$^2$ s$^{-1}$ on $\gamma_n = 27.7$ kg m$^{-3}$. Overlayed are the northern and southern temperature contours of the ACC. (b) The accumulated transport through the $\gamma_n = 27.7$ kg m$^{-3}$ surface, between the two temperature contours due to both the first and second terms in (3.10) for $K = 200$ m$^2$ s$^{-1}$ and $D = 2 \times 10^{-4}$ m$^2$ s$^{-1}$ (dot dashed and solid, respectively).

3.10 (a) Along contour average position of the volumes defined between circumpolar contours. (b) Ratio of $K$ to $D$ determined using (3.17) for layers bounded by contours below 500 m along the ACC. Estimates represented by open circles are from equation (3.17) for the entire volumes shown in (a). Open squares and filled circles in (b) define values using volumes between open squares and filled circles respectively in (a). Estimates represented in (b) by open triangles are from equation (3.15). The solid line in (b) is the mean of all the estimates on each layer (equally weighted) and the shaded region represents ± one standard deviation, $\sigma$. The outlier on $\gamma_n = 27.9$ kg m$^{-3}$ is removed as it is more than 3$\sigma$ from the mean.
4.1 The difference in streamfunction $\Delta \Psi^\gamma$ on an isopycnal $\gamma$ may be related to the along-isopycnal and vertical mixing coefficients $K$ and $D$ using (4.9). Just as thermal wind relates velocities in the vertical, the difference $\Delta \Psi^\gamma$ may be related to $\Delta \Psi^{\gamma_0}$ on the reference level $\gamma_0$ using (4.11). Incorporating many contours on many isopycnals into the inversion makes the system overdetermined and allows $K$, $D$ and $\Psi^{\gamma_0}(x,y)$ to be estimated.

4.2 a: Start points (green crosses) and end points (red circles) of tracer-contours on the deepest and shallowest isopycnals defined in North Pacific region, joined by blue lines. The reference level is shown with an open gray mesh and the sea floor with a filled gray mesh. b, c and d: Terms in (4.12), scaled by the length, $\Delta x$, and the average Coriolis frequency of the contour, $\tilde{f}$, to cast each term in units of velocity. b: $K_0 \int_{x_1}^{x_2} f \left( 1/\lambda^h + 1/\lambda^\perp \right) dx / (\Delta x \tilde{f})$ using $K_0 = 500 m^2 s^{-1}$, c: $D_0 \int_{x_1}^{x_2} \left( f / \lambda^\gamma \right) dx / (\Delta x \tilde{f})$ using $D_0 = 2 \times 10^{-5} m^2 s^{-1}$ and d: $\mathcal{C} / (\Delta x \tilde{f})$, which is zero on the reference level at $\sigma_2 = 36.49$.

4.3 Same as Fig.4.2 but for South Atlantic with the reference level at $\sigma_2 = 36.95$ and using $D_0 = 1 \times 10^{-4} m^2 s^{-1}$.

4.4 Scatter plot of mixing terms in (4.12) for North Pacific region. That is, $K_0 \int_{x_1}^{x_2} f \left( 1/\lambda^h + 1/\lambda^\perp \right) dx / (\Delta x \tilde{f})$ vs. $D_0 \int_{x_1}^{x_2} \left( f / \lambda^\gamma \right) dx / (\Delta x \tilde{f})$ using $K_0 = 500 m^2 s^{-1}$ and $D_0 = 2 \times 10^{-5} m^2 s^{-1}$. Gray shading represents the potential density, $\sigma_2$, of the tracer-contours.

4.5 Same as Fig.4.4 but for South Atlantic region and using $D_0 = 1 \times 10^{-4} m^2 s^{-1}$. 

xvi
Mixing coefficient solutions for the North Pacific region using: (i) the tracer-contour equations (dotted), (ii) only the box equations and a reference level minimization (dashed) and (iii) combining tracer-contours and box equations with no reference level minimization (solid). The implicit coefficient in HIM is also shown (grey). a: Vertical mixing coefficient $D$. b: Along-isopycnal mixing coefficient $K$. For $D$ all the inverse solutions are virtually identical, while for $K$ only the box inversion solution is visibly different from the HIM value of $600 \text{ m}^3 \text{s}^{-1}$. 

Same as Fig.4.6, but for South Atlantic region. For both $D$ and $K$ all the inverse solutions overlay the model value except for the box inversion (dashed line).

Solution for reference level velocity component directed into the region ($\mathbf{v}_{ref} \cdot \mathbf{m}$) on the reference level ($\sigma_2 = 36.9 \text{ kg m}^{-3}$) for the North Pacific region, using: (i) only the tracer-contour equations (dotted), (ii) only the box equations (dashed), (iii) combining tracer-contours and boxes (solid), and (iv) assuming the flow is non-diffusive (Bernoulli inversion; dot-dashed). The velocity in HIM is also shown (grey). Positive values are directed into the region.

Same as Fig.4.8, but for South Atlantic region.
4.10 Mixing coefficient solutions for the tracer-contour inverse method for the North Pacific region with ±10% error (solid) and the implicit coefficient in HIM (dot dashed), shaded areas represent the standard deviation of the 100 ensemble results. a: Vertical mixing coefficient $D$. b: Along-isopycnal mixing coefficient $K$.  

4.11 As in Fig. 4.10 but for the South Atlantic region. 

4.12 Solution for reference level velocity component directed into the region ($v_{ref} \cdot m$) on the reference level ($\sigma_2 = 36.9\,kgm^{-3}$) for the North Pacific region ($\sigma_2 = 36.9\,kgm^{-3}$) and the HIM velocity (dashed) and diagnosed from the streamfunction solution of the tracer-contour inverse method (solid). Shaded areas represent the standard deviation of the 100 ensemble results. Positive values are directed into the region. 

4.13 As in Fig. 4.12, but for South Atlantic region. 

4.14 Condition number and standard deviation of error in diffusivities $K$ and $D$ and velocity $v_{ref} \cdot m$ on the reference level versus the relative box to tracer-contour weighting, $W_{Box}/W_{Contour}$ (Logarithmic scale). At each relative weighting, 100 inversions are conducted, each with 10% random error added to the each equation. There is a weak reference level minimization included in the box inverse method equations. Left (a): North Pacific region. Right (b): South Atlantic region.
5.1 Pressure on the $\gamma^\text{rf} = 26.525 \text{ kg m}^{-3}$ ($\sigma_0 \approx 26.75 \text{ kg m}^{-3}$) surface from the climatology of DW09. Also marked is the region considered in this study (solid black rectangle), the site of mooring C from the Subduction Experiment (black cross; Joyce et al., 1998) and the NATRE study regions showing the approximate SF$_6$ tracer extent during the fall of 1992 (dark grey rectangle), the spring of 1993 (medium grey rectangle) and November of 1994 (light grey rectangle; Ledwell et al., 1998).

5.2 Schematic showing how the tracer-contour inverse method is implemented. Mixing at points along a tracer contour between $(x_1,y_1)$ and $(x_2,y_2)$ on the isopycnal surface $\gamma$, is related to a geostrophic streamfunction on the reference pressure $p_0$ using (5.3), shown in green. The unit vector $\mathbf{n}$ is normal to the contour on the isopycnal. The tracer $C$, typically volume, conservative temperature anomaly or salinity anomaly, is conserved on an isopycnal layer bounded by sections using (5.4), shown in blue. Properties are advected or mixed through the upper and lower bounding surfaces of the layer and across each section. The velocity across a section is related to the streamfunction on the reference surface using a form of the thermal wind equation (Appendix B). The unit vector $\mathbf{m}$ is normal to the section. In this study $\Psi^{p_0}$ is solved for each cast $(i)$ and the mixing coefficients, $K$ and $D$, are solved for on each layer.
5.3 Vertical mixing coefficient, $D$, as determined using the tracer-contour inverse method (solid line) and the standard error of the inversion (light grey shading). Also shown are the estimates of $D$ from Ledwell et al. (1993), Ledwell et al. (1998), and Ferrari and Polzin (2005) and their reported uncertainties.

5.4 Along-isopycnal mixing coefficient, $K$, as determined using the tracer-contour inverse method (solid line) and the standard error of the inversion (light grey shading). Also shown are the estimates of $K$ from Joyce et al. (1998), Ledwell et al. (1998), Jenkins (1998), Armi and Stommel (1983), Spall et al. (1993) and Zika and McDougall (2008) and their reported uncertainties.

5.5 Top row: Cross-section velocity, $\mathbf{v} \cdot \mathbf{m}$, on $\gamma_{rf} = 26.525$ kg m$^{-3}$ ($\sigma_0 \approx 26.75$ kg m$^{-3}$). Length of arrows represents magnitude of $\mathbf{v} \cdot \mathbf{m}$ while the direction indicates whether the flow is into or out of the study region. Each section has a corresponding velocity figure starting from western section (far left) moving clockwise and to the southern section (far right). The grey box indicates the study region. Ledwell et al. (1998) release an SF$_6$ tracer at $\sigma_0 = 26.75$ kg m$^{-3}$ which is advected to the South-East, consistent with the flow field observed here. Bottom Row: Cross-section of velocity on neutral density surfaces $\mathbf{v} \cdot \mathbf{m}$ (positive values are directed out of the study region). As in the top row each section has a corresponding velocity figure.
D.1 Vertical mixing coefficient, $D$, (left) and along-isopycnal mixing coefficient, $K$, (right) as determined using the tracer-contour inverse method. Black lines represent the estimated value while the area between the grey lines represent the standard error. a and b: for the standard settings (solid line), no bolus velocity (dotted line) and including an unsteady term (dashed line). c and d: for the standard settings (solid line), increasing the weight on the contour equations (dashed line) and increasing the weight on the box equations (dotted line). e and f: as in c and d but using the climatology of GK.
List of Tables

2.1 Table of contour values $S_0$ and $\Theta_0$, the values of salinity (psu) and temperature ($^\circ$C) coming into each layer from the Mediterranean Sea, $S_M$ and $\Theta_M$, and the length, $L$, along these contours of the largest area of Fig.2.3. .......................... 29

2.2 Based on the hydrography and the volume flux of Mediterranean Water (from Baringer and Price (1997)) this table shows the coefficients of $K$ and $D$ in (2.20) (the first two parts of the table) and the left-hand side of (2.20) (the third part of the table) for the forty different control volumes we consider in this paper. The first two parts of the table have units: m psu K; the third has units: m$^3$ s$^{-1}$ psu K. .......................... 32

2.3 The diffusivities and their standard errors, with both $D$ and $K$ being treated as constant along each density layer. The eight layers are treated independently, and in each layer $D$ and $K$ are determined as the only two unknowns in an overdetermined set of five equations corresponding to the five different $S_0$ contours. .......................... 36
2.4 The diffusivities determined from HIM output and their standard errors, with both $D$ and $K$ being treated as constant for a set of areas in the vertical. The five contour locations are treated independently, and in each contour $D$ and $K$ are determined as the only two unknowns in an overdetermined set of four equations corresponding to the four different layers.
Supporting Publications


Chapter 1

General Introduction

The ocean is highly variable in time and space, due to a plethora of turbulent processes. Such processes range from 1mm-10m scale turbulent mixing, to 1km-100km scale eddy activity to larger spatial and temporal scale climate modes. Despite this variability, the ocean displays persistent features. These features, such as the shape of temperature-salinity profiles, are evident from hydrographic observations. Historically, such observations have led descriptive oceanographers to a qualitative understanding of the ocean circulation (Wieist, 1935; Sverdrup et al., 1942). The time mean hydrography has also been used to give global estimates of turbulent vertical mixing (Munk, 1966). The fundamental question asked in this thesis is: *Can the ocean circulation and mixing be accurately inferred from mean hydrographic observations?*

In this introduction, a brief discussion will be given of vertical and along-isopycnal mixing, their importance and existing observational estimates. Methods presently used to infer the general circulation and mixing from hydrographic data will then be described. The approach taken in this thesis is introduced and the key findings of each chapter are summarized.
1.1 Vertical mixing

Vertical mixing, $D$, controls the diapycnal component of the meridional overturning circulation, MOC, (Munk and Wunsch, 1998). That is, without diapycnal mixing, dense water, formed at high latitudes cannot cross isopycnal surfaces [except through nonlinear processes which are measurably small and themselves require lateral mixing McDougall (1987b)]. Lack of knowledge of vertical mixing, its magnitude and spatial variation, leads to large uncertainties in our knowledge of the volume, heat, freshwater, nutrient and tracer transports as well as climate sensitivity and climate feedbacks. One of the many such uncertainties relates to the sensitivity of the Atlantic meridional overturning circulation, AMOC, to shutdown. Manabe and Stouffer (1988) find that for certain magnitudes and spatial structures of $D$, the AMOC is much more likely to shutdown under future climate scenarios.

Munk (1966) discusses the balance between diapycnal upwelling and vertical mixing. Given 25 Sv ($\text{Sv}=10^6 \text{ m}^3 \text{ s}^{-1}$) of diapycnal upwelling below 1000m, Munk showed that the area average vertical diffusivity, $D$, must be approximately $10^{-4} \text{ m}^2 \text{ s}^{-1}$. In his original paper, Munk assumed a 1-dimensional steady state advective-diffusive balance for potential temperature and salinity separately, with respect to a vertical velocity. This approach now seems crude, but it can be shown that this 1-D approach holds for density, $\rho$, with some diapycnal velocity, $w^\gamma$, such that

$$w^\gamma = \frac{D_{\rho_z z}}{\rho_z} + D_z$$  \hspace{1cm} (1.1)

with the sole assumption being that effects due to the nonlinear equation of state are small (see Appendix B.2), (1.1) holding even when the ocean is unsteady and inhomogeneous in 3 dimensions.

Munk’s analysis assumes a total diapycnal transport of 25 Sv. The total
fraction of the MOC that is upwelled through isopycnals from the abyssal ocean is still unknown.

Individual observations of $D$ have largely been much smaller than those predicted by Munk. Using a SF$_6$ tracer, released below the thermocline in the eastern North Atlantic, Ledwell et al. (1993) estimate $D$ to be $O(10^{-5}$ m$^2$ s$^{-1}$). Microstructure measurements have revealed weak mixing in most regions [$O(10^{-5}$ m$^2$ s$^{-1}$), Gregg (1987)]. Using microstructure measurements, Polzin et al. (1997) find a strengthening of $D$ close to rough topography in the Brazil Basin. Vertical mixing can also be inferred indirectly from estimates of the kinetic energy dissipation rate by assuming small-scale mixing can be related to internal wave finestructure. Using such methods (Kunze et al., 2006, shear strain methods) and (Sloyan, 2006, strain methods) demonstrate that $D$ is generally intensified over rough topography and in energetic regions such as the Southern Ocean [$O(10^{-4}$-$10^{-3}$ m$^2$ s$^{-1}$)]. As direct estimates of diapycnal mixing are sparse, infrequent and commonly made above 1000m, it is as yet unclear whether vertical mixing in these energetic regions is sufficient to induce global diapycnal upwelling, below 1000m, of order 25 Sv. Also, Klocker and McDougall (2009b) have recently demonstrated the importance of various non-linear equation-of-state processes in the global upwelling of water across isopycnals.

1.2 Along-Isopycnal Mixing

The along-isopycnal mixing coefficient, $K$, is that which mixes tracers such as temperature and salinity or potential vorticity along isopycnal surfaces. The magnitude of $K$, like $D$, is thought to strongly control the overturning circulation and climate sensitivity (Gnanadesikan, 1999). Again, as an
example, Sijp et al. (2006) found that the stability of their climate model’s AMOC is a strong function of $K$.

Few direct estimates exist for $K$ in the oceans. The tracer released in the eastern North Atlantic (Ledwell et al., 1998), mixed horizontally with a rate of order 1000 m$^2$ s$^{-1}$ over the 18 months. Horizontal mixing at the sea surface has been estimated to be equal to or greater than that determined by Ledwell (i.e. 1000-10,000 m$^2$ s$^{-1}$; Zhurbas and Oh, 2004; Marshall et al., 2006; Sallée et al., 2008). A depth dependence of the mixing coefficient, particularly that which mixes potential vorticity or interface height (commonly referred to as $\kappa$ in the Eulerian vertical coordinate parameterization of Gent et al., 1995), is reinforced by adjoint inversions (Ferraira et al., 2005) and eddy resolving models (Eden and Greatbatch, 2008). It should be noted that calculations of $K$, in eddy resolving simulations, are not trivial, even with complete knowledge of a model’s full velocity and density fields (Eden et al., 2007). It is unsurprising that different methods for estimating the along-isopycnal diffusivity give different results, considering they are measuring different effects of mixing: dispersion of particles, stirring of tracers, reduction of isopycnal slopes etc. In this thesis, mixing represents all processes that transport properties (temperature, salinity, volume) which are not described by the mean flow.

The component of the MOC that is not upwelled through isopycnals in the ocean interior, must be upwelled, along isopycnals to the surface of the Southern Ocean. This flow must cross tracer contours on isopycnals, with velocity $v^\perp$. When the tracer considered is conservative temperature, $\Theta$ [a variable much like potential temperature, $\theta$, McDougall (2003)] or salinity, the overturning is subject to the following advective-diffusive balance equa-
\begin{equation}
\nu^\perp = K \frac{\nabla_\gamma \cdot (h \nabla_\gamma \Theta)}{h \nabla_\gamma \Theta} + D \frac{\Theta_z^2}{\nabla_\gamma S} \frac{R_\rho}{R_\rho - 1} \frac{d^2 S}{d \Theta^2} \tag{1.2}
\end{equation}

Here the steady state and a linear equation of state assumption have been made (see McDougall, 1984, and Appendix B.1). The operator \( \nabla_\gamma \) is the along-isopycnal gradient and \( R_\rho \) is the stability ratio \( (R_\rho = \nabla_\gamma S \Theta_z / \nabla_\gamma \Theta S_z) \). Equation (1.2) relates the along-isopycnal component of the MOC to mixing. So, mixing controls the strength of both the diapycnal component (1.1) and the along-isopycnal component of the MOC.

Similar to the analysis of Munk (1966), who relates the diapycnal component of the MOC to mixing, the along-isopycnal component of the MOC is explored in Chapter 3. In Chapter 3 it is found that in order to upwell less than 50 Sv of dense water, the along-isopycnal mixing coefficient, \( K \), must be less than 500 m\(^2\) s\(^{-1}\) below the thermocline at latitudes of the Antarctic Circumpolar Current.

### 1.3 Inverse Methods

Inverse methods are tools used to diagnose the ocean circulation from hydrographic data. Such methods seldom give insight into mixing processes. The most well known method is the box inverse method of Wunsch (1978). In the box method oceanic sections bound regions of the oceans, or entire ocean basins. In each region, mass, heat, salt and other properties are conserved on isopycnal layers. The unknowns in such methods are reference level velocities and diapycnal fluxes of properties. Box inversions are always underdetermined and the final solution for such parameters as the mixing coefficients, depends largely on the choice of constraints and weighting of those constraints. This sensitivity is particularly strong when mixing coefficients
are considered as unknowns in inverse studies. As pointed out by Wunsch (2006) there is ‘sometimes’ insufficient information to resolve some parts of the inverse solution, particularly the vertical mixing coefficient.

It is easy to see why box inverse methods struggle to resolve $D$. Consider the use of conservation of volume, for instance. Assuming lateral velocity, $\mathbf{v}$, were known very well (i.e. to within a small error $\epsilon$), one would have to differentiate this velocity to get the lateral divergence $\nabla \gamma \cdot \mathbf{v}$, amplifying $\epsilon$. If one is interested in the diapycnal velocity $w^\gamma$ it can be related through volume conservation, only through its vertical derivative, such that $\nabla \gamma \cdot (\mathbf{v} h) = -[w^\gamma]_l^u$. Only then can $w^\gamma$ be related, through further differentiation, to $D$ via (1.1).

Inverse models have tended to perform better when individual diapycnal property fluxes are considered (Sloyan and Rintoul, 2000) or when observed mixing coefficients are included as ‘knowns’ (St.Laurent et al., 2001).

Other inverse methods exist, such as the beta spiral method of Stommel and Schott (1977) and the Bernoulli method of Killworth (1986). As discussed in Chapter 4, these two methods rely on spiraling in the vertical of the along-isopycnal potential vorticity gradient (and more recently the temperature gradient). In their general application, these two methods assume mixing is small or zero. The two methods have had difficulty adequately describing mixing. Recent adjoint and gridded methods (Wunsch and Heimbach, 2007; Herbei et al., 2008) have shown promise but their ability to infer both the along-isopycnal and vertical mixing coefficients has remained elusive.
1.4 This Thesis

This thesis approaches the challenge of estimating vertical and along-isopycnal mixing in the ocean. This is done in two ways: (i) by relating known transports, down tracer gradients, on isopycnals to along-isopycnal mixing, $K$, and diapycnal mixing, $D$ and (ii) by developing, testing and applying, a new inverse method, which accurately estimates $K$, $D$ and the circulation from the mean hydrography. Chapters 3 and 4 are associated with approach (i) while Chapters 4 and 5 relate to approach (ii). Here each chapter is summarised.

1.4.1 Chapter 2

In Chapter 2 the transport of warm salty water from the Mediterranean Sea, into the North Atlantic Ocean, is related to along-isopycnal and diapycnal mixing. In this study, a finite volume approach is taken. A linear combination of conservation equations is considered for control volumes bounded laterally by contours of conservative temperature, $\Theta$, and vertically by isopycnals. A ‘known’ transport of Mediterranean water is used to infer $K$ across the bounding $\Theta$ contours and $D$ across the bounding isopycnals. Both $K$ and $D$ are assumed to be constant on each layer. It is found that $D$ is $O(10^{-4} \text{ m}^3 \text{ s}^{-1})$ in the outflow region. These values are high relative to open ocean estimates. As a large fraction of the area considered is near the coastlines of western Europe and northwestern Africa, our results are consistent with the hypothesis that mixing is strong in coastal regions. $K$ is found to be $O(300 \text{ m}^3 \text{ s}^{-1})$ consistent with observations in these depth ranges in the North Atlantic. Given observations of lateral mixing near the surface and in the pycnocline are much larger, as discussed in Section 1.2, our results support the hypothesis that there is a strong depth dependence to $K$. 
Uniquely, the linear combination of conservation equations eliminates the diapycnal velocity, \( w^\gamma \), and the vertical derivative of the vertical mixing coefficient, \( D_z \). Although the methodology of using equations for finite volumes is not pursued in later chapters, the work in this chapter gave an insight into the benefit of taking linear combinations of conservation statements and following contours of constant tracer, on isopycnals. The work in this chapter has been published in the Journal of Physical Oceanography.

### 1.4.2 Chapter 3

The Southern Ocean Meridional Overturning is discussed in Chapter 3. An advective-diffusive balance equation is derived in advective form, rather than divergence form. This equation is integrated along \( \Theta \) contours on isopycnals in the Southern Ocean, relating the transport on isopycnals to vertical and along-isopycnal mixing. Specifically, the magnitude and spatial structure of southward flowing Upper Circumpolar Deep Water, UCDW, is related to \( K \) and \( D \). The isopycnal flow across contours of constant \( PV \) and the flow across isopycnal surfaces is also considered and related to mixing. By integrating \( f u^\perp \) along contours of constant \( \Theta \) and \( S \), which span the entirety of the Antarctic Circumpolar Current, a balance is found between \( K \) and \( D \). We thus infer an average mixing ‘aspect ratio’, \( K/D \), of \( 2 \pm 1 \times 10^6 \). Given observed transports of UCDW of between 20 Sv and 50 Sv we estimate that \( K = 300 \pm 150 \text{ m}^2 \text{ s}^{-1} \) and \( D = 1 \pm 0.5 \times 10^{-4} \text{ m}^2 \text{ s}^{-1} \) below 500 m in the Southern Ocean. These estimates are again consistent with the hypothesis of a strong depth dependence of \( K \). The strong vertical mixing inferred is consistent with the few in situ observations and with the view that diapycnal processes play an important role in the Southern Ocean Meridional Overturning Circulation.
In this chapter we relate $K$ and $D$ to advection down isopycnal tracer gradients. A precise formalism is presented where the mean geostrophic and eddy induced velocities are considered. This formalism is continued in the following chapters where tracer contours are used to develop a general inverse method for estimating $K$, $D$ and the mean circulation. The work in this Chapter is in press in Journal of Physical Oceanography.

1.4.3 Chapter 4

Chapter 4 sees the presentation of a new inverse method, the *tracer-contour inverse method*. As in Chapter 3, tracer contours on isopycnals are considered. The flow across contours, on isopycnals, is related to differences in a geostrophic streamfunction. This geostrophic streamfunction is related in the vertical, through an analogous form of thermal wind. Taking many contours on many layers, an overdetermined inverse method may be developed.

The method is applied to the output of a layered ocean model in the North Pacific and South Atlantic basins. The method accurately resolves the known mixing coefficients, $K$ and $D$, and the mean circulation. The method is compared both practically and theoretically to the box, beta spiral and Bernoulli inverse methods.

The method is formulated so that a streamfunction is defined along ‘sections’ such that the tracer contour and box methods can be combined. The method performs best when the large scale conservation constraints of the box inverse method are retained. The method is less sensitive to error than conventional techniques and mixing terms are leading order in the analysis. When the tracer is $\Theta$ and $S$, the along-isopycnal component of the thermohaline overturning circulation is explicitly considered.

In this chapter the necessary requirements for an inverse method to infer
the mean circulation and mixing are identified. We find that in order to infer
the mean circulation there must be either spiraling of the isopycnal tracer
gradient in the vertical or there must be a component of the thermal wind
vector \( \mathbf{v}_z = (N^2/f \rho g) \nabla \gamma p \times \mathbf{k} \) down the along-isopycnal tracer gradient. If
the thermal wind vector has a component down the isopycnal tracer gradient
there must be changes in pressure along isopycnal tracer contours. Hence,
where the tracer considered is \( \Theta \) (or equivalently \( S \)) there must be nonzero
neutral helicity, \( H^n \) \( (H^n \propto \nabla \gamma p \times \nabla \gamma \Theta \cdot \mathbf{k}) \). The work in this chapter is in

1.4.4 Chapter 5

In Chapter 5 we apply the tracer-contour inverse method to mean hydro-
graphic data. The method is applied in the eastern North Atlantic, where
in situ observations of \( K \) and \( D \) have been made. The method accurately
reproduces the observed mixing rates from the hydrographic data alone. The
circulation inferred is consistent with observations also.

The strong depth dependence of the along isopycnal mixing coefficient,
\( K \), eluded to in the previous chapter, is revealed. In the North Atlantic,
\( K \) reduces from \( O(1000 \text{ m}^2 \text{ s}^{-1}) \), in the upper 500m, to \( O(100 \text{ m}^2 \text{ s}^{-1}) \),
below 1000m depth. The estimates presented are consistent with recent parameterisations for coarse resolution models.

In the eastern North Atlantic, distant from continental boundaries, weak
vertical mixing is observed, \( O(10^{-5} \text{ m}^2 \text{ s}^{-1}) \), consistent with in situ obser-
vations. A depth dependence of the vertical mixing coefficient is observed,
increasing gradually between 500m and 1800m depth. In Chapter 2 and
Chapter 3, strong vertical mixing, \( O(10^{-4} \text{ m}^3 \text{ s}^{-1}) \), is inferred near continen-
tal boundaries and along the ACC respectively. It is clear that the methods
presented in this thesis are able to resolve both strong mixing in coastal and energetic regions (Chapters 2 and 3) and weak mixing in more placid open ocean regions (Chapter 4).

The results of the inversion are largely insensitive to changes to the method. The estimates of $K$ and $D$ are robust, regardless of the subjective choices of parameterisation, equation weighting and data sources.

Chapter 5 sees the culmination of ideas developed and tested in Chapters 2, 3 and 4; the method being validated against the most robust of tests; in situ observations. Chapter 5 constitutes a manuscript, submitted for publication in the Journal of Physical Oceanography.

1.4.5 Chapter 6

Finally some conclusions are highlighted and future research directions are suggested.
Chapter 2

Vertical and Lateral Mixing Processes Deduced from the Mediterranean Water Signature in the North Atlantic

2.1 Abstract

The conservation equations of heat, salt and mass are combined in such a way that a rather simple relation is found between the known volume flux of Mediterranean Water entering the North Atlantic and the effects of lateral and vertical mixing processes. The method is a form of inverse method in which the only unknowns are the vertical and lateral diffusivities. For each isohaline contour on each neutral density surface we develop one equation in two unknowns, arguing that other terms that cannot be evaluated are small. By considering several such isohaline contours the method becomes overdetermined for each density layer and we find results for both the vertical and lateral diffusivity that vary smoothly in the vertical, giving some confidence in the method.

It is found that $D$ is $O(10^{-4} \text{ m}^3 \text{ s}^{-1})$ in the outflow region. These values are high relative to open ocean estimates. As a large fraction of the area considered is near the coastlines of western Europe and western North Africa, our results are consistent with the hypothesis that mixing is strong in coastal regions. $K$ is found to be $O(300 \text{ m}^3 \text{ s}^{-1})$ consistent with observations in these depth ranges in the North Atlantic. Given observations of lateral mixing near the surface and in the pycnocline are much larger, our results support the hypothesis that there is a strong depth dependence to $K$. 
2.2 Introduction

The motivation for the present work dates back to the paper of McDougall (1984) where the conservation equations of heat and salt were combined in a form that did not include the dianeutral advection, leaving a balance between lateral advection along a density surface and a specific combination of the effects of mixing of both heat and salt. This motivation will be described in this introduction for the advective form of the conservation equations. In order to make this approach workable as an inverse method in the ocean we need to implement this combination of conservation equations in the divergence form, and this is tackled in section 2.3.

The conservation equations for salinity $S$ and conservative temperature $\Theta$ [which is proportional to potential enthalpy and represents the “heat content” per unit mass of seawater, see McDougall (2003)] are

\[
S_t|_\gamma + \mathbf{V} \cdot \nabla_\gamma S + w^n S_z = h^{-1} \nabla_\gamma \cdot (h K \nabla_\gamma S) + (DS_z)_z \quad (2.1)
\]

and

\[
\Theta_t|_\gamma + \mathbf{V} \cdot \nabla_\gamma \Theta + w^n \Theta_z = h^{-1} \nabla_\gamma \cdot (h K \nabla_\gamma \Theta) + (D\Theta_z)_z. \quad (2.2)
\]

These equations have been written in the advective form and with respect to neutral density ($\gamma^n$) surfaces (Jackett and McDougall, 1997) so that $w^n$ is the vertical velocity through neutral density surfaces (i.e. the dianeutral velocity) and $\mathbf{V}$ is the thickness-weighted horizontal velocity obtained by temporally averaging the horizontal velocity between closely-spaced neutral density surfaces. Similarly, the salinity and conservative temperature are the thickness-weighted values obtained by averaging between closely spaced pairs of neutral density surfaces (McDougall and McIntosh, 2001). In these
equations $h(x, y)$ is the thickness between two closely-spaced neutral density surfaces. The mixing processes that appear on the right-hand sides are simply lateral mixing of passive tracers (with diffusivity K) along the density surfaces and vertical small-scale turbulent mixing (with diffusivity, D). We have not included double-diffusive convection nor double-diffusive interleaving. Note that the distinction between conservative temperature and potential temperature is not central to this paper and we will frequently refer to $\Theta$ as simply temperature.

Following McDougall (1984) we note that the temporal and spatial gradient of $S$ and $\Theta$ along neutral tangent planes are related through the relevant thermal expansion coefficient ($\alpha = -\rho^{-1} \rho_\Theta |_{S,p}$) and haline contraction coefficient ($\beta = -\rho^{-1} \rho_S |_{\Theta,p}$), that is,

$$\beta S_t |_\gamma = \alpha \Theta_t |_\gamma \quad \text{and} \quad \beta \nabla_\gamma S = \alpha \nabla_\gamma \Theta$$ (2.3)

Strictly these relations hold locally in neutral tangent planes but here we will assume that they hold in neutral density surfaces since McDougall and Jackett (2005,b) have shown that neutral density surfaces are everywhere almost tangent to neutral tangent planes.

The diabatic advection term can be eliminated between (2.1) and (2.2) by multiplying the equations by $\Theta_z$ and $S_z$, respectively and subtracting. On doing this and using (2.3) we find (similarly to McDougall, 1984)

$$S_t |_\gamma + \mathbf{V} \cdot \nabla_\gamma S = D a g N^{-2} (\Theta_z S_{zz} - S_z \Theta_{zz})$$

$$+ \frac{R_\rho}{(R_\rho - 1)} h^{-1} \nabla_\gamma \cdot (h K \nabla_\gamma S)$$

$$- \frac{\alpha/\beta}{(R_\rho - 1)} h^{-1} \nabla_\gamma \cdot (h K \nabla_\gamma \Theta)$$ (2.4)

where the buoyancy frequency $N$ is defined by $N^2 = g(\alpha \Theta_z - \beta S_z)$, and $R_\rho = \alpha \Theta_z / \beta S_z$ is the stability ratio of the vertical stratification. For completeness
we also write down another important linear combination of (2.1) and (2.2),
namely the one formed by cross-multiplying these equations by $\alpha$ and $\beta$
giving what might be loosely called the “density” conservation equation, or
the equation for the dianeutral velocity, namely,

$$w^* = D_z + D g N^{-2}(\alpha \Theta_{zz} - \beta S_{zz})$$

$$-K g N^{-2}(C_b \nabla \Theta \cdot \nabla \Theta + T_b \nabla \Theta \cdot \nabla p), \quad (2.5)$$

where the cabling coefficient and the thermobaric coefficient (McDougall,
1987b, 1991b) are

$$C_b = \frac{\partial \alpha}{\partial \Theta} + 2\frac{\alpha \partial \alpha}{\beta \partial \beta} - \frac{\alpha^2 \partial \beta}{\beta^2 \partial S} \quad \text{and} \quad T_b = \frac{\partial \alpha}{\partial p} - \frac{\alpha \partial \beta}{\beta \partial p} \quad (2.6)$$

and represent the non-linear nature of the equation of state of seawater.

The motivation for the present study comes from the rather simple bal-
ance in (2.4) between the lateral advection of salinity along isopycnals (the
left-hand side) and the terms on the right that depend respectively on the
vertical and lateral diffusivity. We describe this balance as “simple” for three
reasons. Firstly (2.4) does not contain any dependence on the dianeutral ve-
locity $w^*$, nor secondly, on the vertical derivative of the vertical diffusivity,
$D_z$. Thirdly, the term involving $D$ in (2.4) is proportional to the curvature
of the $S - \Theta$ diagram, which is much less subject to numerical noise caused
by vertical heave than the second derivatives $\Theta_z$ and $S_z$ in (2.1), (2.2) and
2.5; recall that

$$\frac{d^2 S}{d \Theta^2} = \Theta_z^{-3}(\Theta_z S_{zz} - S_z \Theta_{zz}) \quad (2.7)$$

For these three reasons we expect the diapycnal mixing term in (2.4) can
be evaluated from oceanographic data with less uncertainty than the diapyc-
nal mixing and advection terms in (2.1) and (2.2). Quantitative knowledge of
the flow out of the Mediterranean Sea into the North Atlantic, as described by Baringer and Price (1997), would seem a suitable way of estimating the left-hand side of (2.4), so that it would seem that the eastern North Atlantic may be a region of the ocean where we could exploit the rather simple balance between advection and mixing processes, as given by (2.4), perhaps being able to perform an inverse model without the need to consider unknown reference level velocities which are the traditional unknowns in oceanographic inversions. This idea is the motivation for the next section where we form rather specific combinations of the divergence forms of the basic conservation equations in order to arrive at an equation that satisfies the overall flux constraints over a finite volume, but has the above three essential features of the “simple” curvature form conservation equation, (2.4).

2.3 A careful linear combination of conservation equations

The three basic conservation (Boussinesq) equations, namely continuity, conservation of salinity and conservation of conservative temperature, can be written in divergence form with respect to neutral density coordinates as (Griffies, 2007)

\[
\begin{align*}
\frac{h_t}{\gamma} + \nabla \cdot (h \mathbf{V}) + [w^\gamma]_t^u &= 0 \quad (2.8) \\
(hS)_t\gamma + \nabla \cdot (h \mathbf{Vs}) + [w^\gamma S]_t^u &= \nabla \cdot (hK \nabla \gamma S) + [DS_z]_t^u, \quad (2.9)
\end{align*}
\]

and

\[
\begin{align*}
(h\Theta)_t\gamma + \nabla \cdot (h \mathbf{V}\Theta) + [w^\gamma \Theta]_t^u &= \nabla \cdot (hK \nabla \gamma \Theta) + [D\Theta_z]_t^u. \quad (2.10)
\end{align*}
\]
Here the superscripts $u$ and $l$ refer to the upper and lower interfaces bounding each layer and $h$ is the vertical distance (layer thickness) between these bounding density interfaces. In the above we have ignored double-diffusive convection in that the only diapycnal mixing is done with the same diffusivity for salinity as for conservative temperature. The last term in (2.9) [and in 2.10] is the difference in the diapycnal flux of salt (and temperature) across the upper and lower density interfaces.

For each layer we will consider a series of control volumes bounded by the source of Mediterranean Water in the east and a series of contours of salinity and conservative temperature, $S_0$ and $\Theta_0$. The $S_0$ and $\Theta_0$ contours coincide because they are on a density surface. We multiply (2.8) by the constant values $S_1$ and $\Theta_1$ (which we may well take to be different to $S_0$ and $\Theta_0$, see later) and subtract these equations from (2.9) and (2.10) respectively obtaining

\[
(h[S - S_1])_t |_{\gamma} + \nabla_{\gamma} \cdot (hV[S - S_1]) = \nabla_{\gamma} \cdot (hK\nabla_{\gamma} S) + [DS_z]_l^{u} - [w^\gamma[S - S_1]]_l^{u}
\]

(2.11)

and

\[
(h[\Theta - \Theta_1])_t |_{\gamma} + \nabla_{\gamma} \cdot (hV[\Theta - \Theta_1]) = \nabla_{\gamma} \cdot (hK\nabla_{\gamma} \Theta) + [D\Theta_z]_l^{u} - [w^\gamma[\Theta - \Theta_1]]_l^{u}.
\]

(2.12)

Here we have moved the dianeutral advection terms to the right-hand side, since they only occur as a result of mixing processes and we wish to place all the mixing influences on the right-hand side. This step of subtracting
off the constant reference values $S_1$ and $\Theta_1$ from the salinity and conservative temperature makes the divergence form of the conservation equations behave more like the advective forms (2.1) and (2.2). The resulting conservation equations for the anomaly variables are less sensitive to errors in the continuity equation 2.8 (McDougall, 1991a; McIntosh and Rintoul, 1997; Ganachaud et al., 2000; Sloyan and Rintoul, 2000). Since the lateral and vertical diffusion terms in (2.11) and (2.12) appear as gradients of $S$ and $\Theta$ we can regard (2.11) and (2.12) as exactly the same form of conservation equation as 2.9 and 2.10 except for the anomaly variables $[S - S_1]$ and $[\Theta - \Theta_1]$. 

Now we ignore the temporal derivative term and we spatially integrate (2.11) and (2.12) over the area from the Gulf of Cadiz to the $S_0$ contour (which almost exactly coincides with a $\Theta$ contour $\Theta_0$, on a neutral density surface). The volume flux into this layer from the Gulf of Cadiz is $Q$ which we will take as known from the work of Baringer and Price (1997). We will take the flow out across the $S_0$ contour to be $cQ$ where we expect $c$ to be not too different to unity because we will be considering contours that are not too distant from the Mediterranean outflow. In what follows we will assume that $0.7 < c < 1.0$. Taking the salinity and temperature of the Mediterranean outflow into this layer to be $S_M$ and $\Theta_M$, the area integrals of the left-hand sides of (2.11) and (2.12) i.e. the area integrals of $\nabla \gamma \cdot (hV[S - S_1])$ and $\nabla \gamma \cdot (hV[S - S_1])$ are

$$LHS_S = -Q[S_M - S_1] + cQ[S_0 - S_1] \quad \text{and} \quad (2.13)$$

$$LHS_{\Theta} = -Q[\Theta_M - \Theta_1] + cQ[\Theta_0 - \Theta_1]. \quad (2.14)$$

The values of $Q$, $S_M$ and $\Theta_M$ represent the values of the Mediterranean outflow after it has reached its equilibrium depth and has begun to spread out neutrally into the interior of the North Atlantic. That is, the values will be
chosen as representative of the flow after the intense dieneutral mixing and
tenainment in the down-slope flow has occurred. This is done so that our
values of $D$ and $K$ represent interior mixing processes in the North Atlantic
rather than representing intense mixing and dilution in the outflow plume.

It is convenient to rearrange these equations as

$$
LHS_S = -FQ[S_M - S_1] \quad \text{and} \quad LHS_{\Theta} = -FQ[\Theta_M - \Theta_1],
$$

(2.15)

where

$$
F = 1 - \frac{[S_1 - S_0]}{[S_M - S_0]}(1 - c) \approx 1 - \frac{[\Theta_1 - \Theta_0]}{[\Theta_M - \Theta_0]}(1 - c),
$$

(2.16)

The second approximate equality follows because the salinity and temper-
ture contrasts here are all measured along a neutral density surface and so
are related though the thermal expansion coefficient and the saline contrac-
tion coefficient which do not change substantially in this part of the North
Atlantic. The values of $S_1$ and $\Theta_1$ that we will choose will mean that $F$ is
little different to unity in our region of interest.

Now we come to the area integration of the right-hand sides of (2.11) and
(2.12). The lateral diffusion terms become the lateral diffusion across the
boundary of the control volume along the $S_0$ contour, namely,

$$
\int_{S_0} hK\nabla_S \cdot n dl \quad \text{and} \quad \int_{S_0} hK\nabla_\Theta \cdot n
$$

(2.17)

where $n$ is the outward unit vector in two dimensions, normal to the contour
and is the element of length along the contour.

Before area-integrating the diapycnal mixing and advection terms in (2.11)
and (2.12), we rewrite these terms in (2.11) as
\[
\frac{1}{2} (D^u + D^l)[S_z]^u - [w^\gamma]^u \left( \frac{1}{2} [S^u + S^l] - S_1 \right) \\
- \frac{1}{2} \left( (w^\gamma)^u + (w^\gamma)^l \right) [S]^u + [D]^u \frac{1}{2} \left( S^u_z + S^l_z \right)
\] (2.18)

The first term, \(0.5[D^u + D^l][S_z]^u\) corresponds to the term \(DS_zz\) in (2.1) while the last term in (2.18) corresponds to \(D_z S_z\) in (2.1). The first of the dianeutral advection terms in (2.18) does not have an analogous term in (2.1) while the other term, \(-[(w^\gamma)^u + (w^\gamma)^l][S]^u\) corresponds to \(w^\gamma S_z\) in (2.1). The next step in section 2.2 (where we were concerned with differential equations at a point) was to cross multiply the \(S\) and \(\Theta\) equations by the vertical gradients of temperature and salinity. This step in section 2.2 eliminated the terms in \(D_z S_z\) and \(w^\gamma S_z\), so simplifying the effects of diapycnal mixing into a single term that was proportional to the curvature of the vertical \(S - \Theta\) cast. We follow a similar procedure here, with the aim of minimizing the influence of dianeutral advection and of the vertical variation of \(D\) on the final equation. Only the contribution from the first term in (2.18) will be significant in the final equation.

We form the area-averaged vertical difference variables

\[
\Delta S = \langle S^u - S^l \rangle \quad \text{and} \quad \Delta \Theta = \langle \Theta^u - \Theta^l \rangle
\] (2.19)

and form \(\Delta S\) times the area integral of (2.12) minus \(\Delta \Theta\) times the area integral of (2.11) giving [using (2.15), (2.17), (2.18) and an equation corresponding to (2.18) for \(\Theta\)]

\[
FQ(\Delta \Theta[S_M - S_0] - \Delta S[\Theta_M - \Theta_0]) \approx \int_{S_0} hK(\Delta S\nabla\gamma \Theta - \Delta S\nabla\gamma \Theta) \cdot n dl
\]
\[ + A \left\langle D\left(\Theta^u_z - \Theta^l_z\right) \Delta S - (S^u_z - S^l_z) \Delta \Theta \right\rangle. \]  \hfill (2.20)

In addition there are three other terms that should appear on the right-hand side of (2.20), namely equations (A1) - (A3) of Appendix A.1. We have omitted these terms here because in Appendix A.1 we show them to be negligible. Here A is the area from the Gulf of Cadiz to the salinity contour and the angle brackets indicate an area average over area A. This equation is the one we use in this paper to deduce magnitudes for both the diffusivities \( D \) and \( K \) given knowledge of \( F Q \) and of the hydrography in the eastern North Atlantic. We have used the symbol \( D \) in (2.20) for the average of the diapycnal diffusivity at the lower and upper interfaces, \( 0.5(D^u + D^l) \). In what follows we will take the diapycnal diffusivity \( D \) to be constant along a layer. This essentially means assuming the diffusive flux can be represents by a coefficient \( D \) times an area averaged vertical gradient (e.g. \( S_z \)). It can also be noted from (2.20) that the numerical magnitudes that we obtain for both \( D \) and \( K \) in each layer are both directly proportional to the effective lateral advection, \( F Q \).

Equation (2.20) corresponds to (2.4) in the advective approach, and notice that like (2.4), (2.20) does not contain dianeutral advection and that diapycnal mixing enters proportionally to the mean curvature of the \( S - \Theta \) diagram of the vertical water columns, averaged over the area \( A \) (since \( (S^u_z - S^l_z) \) and \( \Delta \Theta \) correspond to \( S_{zz} \) and \( \Theta_{zz} \) respectively). Vertical heave contributes considerable noise when evaluating \( S_{zz} \) and \( \Theta_{zz} \) but, through cancellation, does not contribute significantly to the combination of terms \( (\Theta_z S_{zz} - S_z \Theta_{zz}) \), especially when the atlas data has been isopycnally averaged [as is the case with the atlas data of Gouretski and Koltermann (2004) which we use]. Individual casts from one time hydrographic surveys display fine jagged features associated with mixing processes themselves. In the formalism considered
here, we are interested in the mean effect of these mixing processes, which set the mean $S - \Theta$ curvature.

Equation (2.20) is in fact the conservation equation for the variable

$$\Delta S(\Theta - \Theta_1) - \Delta \Theta(S - S_1).$$

(2.21)

The choice of $\Theta_1$ and $S_1$ as the volume averaged conservative temperature and salinity of the control volume ensures that the term (A1) is minimized while the choice of the ratio $\Delta S/\Delta \Theta$ as the ratio of the volume averaged vertical gradients of salinity and temperature ensures that the vertical gradient of (2.21) is on average zero. This property minimizes the contribution of both dieneutral advection and the vertical gradient of $D$ [terms (A2) and (A3)] to (2.20).

Note that at the level of maximum Med Water influence where (2.20) reduces to

$$FQ[S_M - S_0] = -K \int_{S_0} h \nabla \gamma S \cdot n dl - DA \left< S_u^m - S_l^l \right>.$$  

(2.22)

In comparing this equation with (the negative of) the area integral of the original salinity conservation equation (2.9), it can be seen that in this special case where $\Delta S = 0$ the price that has been paid for eliminating the influence of dieneutral advection across both the upper and lower interfaces is simply to introduce the factor $F$.

### 2.4 Diffusivities from hydrographic atlas data

Hydrographic data from the region of the North Atlantic near the Straits of Gibraltar have been taken from the atlas of Gouretski and Koltermann (2004). We have used the atlas at a horizontal resolution of half a degree of longitude and latitude. Eight layers were chosen bounded by nine neutral
density surfaces and in each of these layers we consider five areas from the Straits of Gibraltar to different salinity contours. In Fig.2.1 and Fig.2.2 we show $S - \Theta$ curves from the three locations ($38^\circ$N, $350^\circ$E), ($38^\circ$N, $345^\circ$E) and ($38^\circ$N, $340^\circ$E). The eastern-most cast has the salinity maximum close to a neutral density of $\gamma = 27.7$ kg m$^{-3}$, but the salinity maximum rises to less dense surfaces as one moves along the tongue to the west. As explained by McDougall and Giles (1987), such a diapycnal migration of a salinity maximum should not automatically be interpreted as implying asymmetric vertical mixing processes since salinity itself is an asymmetric variable. Another way of looking at this issue is to form a salinity anomaly variable which represents the presence of Mediterranean Water with respect to a background North Atlantic water mass field that exists in the rest of the North Atlantic. This variable has isolines that rise about $15^\circ$C for every unit increase in salinity, and the maximum of this anomaly variable does not so obviously move across isopycnals.

Based on the curvature of the casts at ($38^\circ$N, $350^\circ$E) and ($38^\circ$N, $345^\circ$E) we see that the peak salinity occurs at a density of approximately $\gamma = 27.65$ kg m$^{-3}$ and we take this to be the central neutral density of the Mediterranean outflow. We will henceforth consider the hydrography between neutral densities of 27.50 kg m$^{-3}$ and 27.80 kg m$^{-3}$ in order to capture the majority of the outflow, consistent with the work of Baringer and Price (1997).

Looking closely at the $S - \Theta$ cast at ($38^\circ$N, $350^\circ$E) we notice a ‘dented’ section between the neutral densities 27.6 and 27.7. This dent represents a discontinuity in the curvature and is consistent with the observation that the Mediterranean Water does tend to have a strong signature on two distinct density surfaces (sometimes called two “cores”) close to the outflow region (as reviewed by Baringer and Price (1997)). In order to minimize the influence
Figure 2.1: Salinity - conservative temperature diagram of three vertical “casts” of the atlas hydrography near the Gulf of Cadiz at locations illustrated in Fig.2.2. The four dashed lines in are straight-line approximations to where the four neutral density surfaces lie in this figure.
Figure 2.2: Contours of salinity on the neutral density surface $\gamma^n = 27.70$ kg m$^{-3}$. Also shown are the locations of “casts” (Fig.2.1) of the atlas hydrography near the Gulf of Cadiz.
Figure 2.3: Contours of $\Theta_0$ on the $\gamma = 27.70$ kg m$^{-3}$ density surface. These five contours define the western-most edge of five areas, each of which extends eastwards as far as 351°E. The colour bar and contours are of conservative temperature ($^\circ$C).

Of this vertical variation of curvature we choose contours well away from the Gulf of Cadiz, that is, we choose contours that extend west of 340°E. At their eastern most point, the isopycnals where approximately flat, giving confidence that the volumes considered are not in a gravity current regime.

On the density surface $\gamma = 27.7$ kg m$^{-3}$ we show (Fig.2.2) the five $S_0$ contours that we use as bounds for five overlapping areas on this density surface. For the other density interfaces used in this study we chose contours at the locations of the arrow tips in Fig.2.3, thus forming regions of roughly equivalent areas on each density surface. In this way we have chosen $S_0$
contours passing through longitudes 332°E, 336.5°E, 338°E, 340°E, and 341.5°E each at 35.5°N, on neutral density surfaces of 27.5, 27.55, 27.6, 27.65, 27.7, 27.725, 27.75, 27.775 and 27.8. Thus we have five sets of areas on each of eight density layers, giving 40 equations of the form (2.20). Note that the density intervals were chosen to give approximately equal vertical spacing (in dbar) between layers. We have limited the area of our calculations to being west of 351°E in order to exclude the region of complicated geometry and complex mixing processes close to the Straits of Gibraltar and north of the Iberian Peninsula. Factoring in these areas would marginally increase the surface area and contour length but we would expect it produce a negligible change in our results. Table 2.1 gives the areas, and lengths along the S₀ and Θ₀ contours corresponding to the largest area on Fig.2.3. The salinity and conservative temperature of the Mediterranean Water entering each layer, S_M and Θ_M, are taken to be the maximum values close to (36°N, 352°E) which is close to the eastern most point of our data set and the position of maximum salinity and conservative temperature.

Baringer and Price (1997) describe the Mediterranean outflow as it descends from the Straits of Gibraltar. After the bulk of the mixing and entrainment has occurred, the volume flux of "Mediterranean Water" into the North Atlantic is estimated as 1.52 Sv (Sv ≡ 10⁶ m³ s⁻¹). Based on this paper we take this as the transport of the outflow once it has stopped actively entraining and that this outflow was distributed vertically over a range of neutral densities of 0.75 kg m⁻³ centred at approximately γ = 27.65 kg m⁻³. Plots of salinity and temperature on our surfaces affirm this range of densities. Hence we approximate the distribution of the flow as a sine wave with zeros at neutral densities of 27.65±0.375 kg m⁻³ and an integral between this region equalling 1.52 Sv. Figure 2.4 shows the volume flux that is assumed
<table>
<thead>
<tr>
<th>Neutral density (kg m(^{-3}))</th>
<th>(S_0)</th>
<th>(\Theta_0)</th>
<th>(S_M)</th>
<th>(\Theta_M)</th>
<th>(A)</th>
<th>(L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>27.50-27.55</td>
<td>35.531</td>
<td>9.520</td>
<td>36.072</td>
<td>11.701</td>
<td>2.199</td>
<td>3.955</td>
</tr>
<tr>
<td>27.55-27.60</td>
<td>35.536</td>
<td>9.239</td>
<td>36.124</td>
<td>11.614</td>
<td>2.245</td>
<td>3.969</td>
</tr>
<tr>
<td>27.60-27.65</td>
<td>35.542</td>
<td>8.950</td>
<td>36.178</td>
<td>11.524</td>
<td>2.259</td>
<td>4.041</td>
</tr>
<tr>
<td>27.65-27.70</td>
<td>35.547</td>
<td>8.652</td>
<td>36.236</td>
<td>11.441</td>
<td>2.233</td>
<td>4.090</td>
</tr>
<tr>
<td>27.70-27.75</td>
<td>35.543</td>
<td>8.388</td>
<td>36.299</td>
<td>11.361</td>
<td>2.246</td>
<td>4.074</td>
</tr>
<tr>
<td>27.75-27.775</td>
<td>35.530</td>
<td>8.161</td>
<td>36.336</td>
<td>11.256</td>
<td>2.272</td>
<td>4.011</td>
</tr>
<tr>
<td>27.775-27.80</td>
<td>35.510</td>
<td>7.902</td>
<td>36.320</td>
<td>11.034</td>
<td>2.286</td>
<td>3.972</td>
</tr>
<tr>
<td>27.80-27.825</td>
<td>35.480</td>
<td>7.588</td>
<td>36.247</td>
<td>10.582</td>
<td>2.320</td>
<td>3.882</td>
</tr>
</tbody>
</table>

Table 2.1: Table of contour values \(S_0\) and \(\Theta_0\), the values of salinity (psu) and temperature (°C) coming into each layer from the Mediterranean Sea, \(S_M\) and \(\Theta_M\), and the length, \(L\), along these contours of the largest area of Fig.2.3.
Figure 2.4: The volume flux of Mediterranean Water flowing into the North Atlantic as a function of neutral density layer (following Baringer and Price, 1997).

to flow into each density bin of width 0.05 kg m$^{-3}$. We use only the central half of the density range shown in Figure 2.4, and the actual volume flux into our densest three layers is half that shown simply because those layers have a density “width” of only 0.025 kg m$^{-3}$ Baringer and Price’s estimate has been gathered from mooring data and although it is indeed not exact we take it as a known for the purposes of this study. The sensitivity of changes in $Q$ of ± 10% is explored later in this section and it is emphasised that the solutions for K and D are proportional to the magnitude of $Q$.

In Table 2.2 we present the three relevant coefficients that appear in (2.20) for each of the 40 areas we consider (5 contours times 8 layers). The upper
set of 40 numbers are of \( \int_{S_0} h(\Delta S \nabla_{\gamma} \Theta - \Delta \Theta \nabla_{\gamma} S) \cdot \mathbf{n} dl \), which multiplies \( K \) in (2.20), the middle set are of \( A \left( (\Theta^u_z - \Theta^l_z) \Delta S - (S^u_z - S^l_z) \Delta \Theta \right) \) which multiplies \( D \) in (2.20) while the third set of coefficients are of \( Q(\Delta \Theta[S_m - S_0] - \Delta S[\Theta_M - \Theta_0]) \) which multiplies \( F \) in (2.20).

We argue in appendix A.1 that \( F \) lies in the range \( 0.85 < F < 1.0 \), or \( F = 0.95 \pm 0.075 \). In practice this means assuming that the transport into the control volumes in the Gulf of Cadiz is around 10%-20% of the transport across the contours in the North Atlantic Interior. In what follows we will actually take \( F \) to be unity because we consider the uncertainty in estimating \( Q \) to be larger than that associated with estimating \( F \), sensitivity to this choice is tested. Hence we will consider the left-hand side of (2.20) to be known for each of our 40 equations. As a first step, one might assume that the diffusivities \( K \) and \( D \) do not vary in space, either laterally, or vertically.

Then we have forty equations in just two unknowns and we find that the over-determined least-squares solution is \( K = 481 \pm 114 \text{ m}^2 \text{s}^{-1} \) and \( D = 2.2 \times 10^{-5} \pm 7.5 \times 10^{-5} \text{ m}^2 \text{s}^{-1} \). The estimate of the standard error of \( K \) was found by substituting \( D = 2.2 \times 10^{-5} \text{ m}^2 \text{s}^{-1} \) into each of the 40 equations and similarly, the estimate for the standard error of \( D \) was found by substituting \( K = 481 \text{ m}^2 \text{s}^{-1} \) into each of the 40 equations. The spatially constant diapycnal diffusivity, at \( D = 2.2 \times 10^{-5} \pm 7.5 \times 10^{-5} \text{ m}^2 \text{s}^{-1} \) while being relatively small, is also rather poorly known in that even its sign is in doubt.

The rather uncertain value we found for the spatially invariant diapycnal diffusivity raises the question of whether this is simply indicative of the resolving power of the method and the hydrography, or whether this result was indicative of a real variation of the diffusivity in space. Noting that each of our forty equations are simply linear relations between \( K \) and \( D \), we plotted
<table>
<thead>
<tr>
<th>Neutral density</th>
<th>Longitude of contour</th>
</tr>
</thead>
<tbody>
<tr>
<td>layer (kg m(^{-3}))</td>
<td>332(^\circ)E</td>
</tr>
<tr>
<td>(\int_{S_0} h(\Delta S \nabla \gamma \Delta \Theta - \Delta \Theta \nabla \gamma S) \cdot \mathbf{n} dl)</td>
<td></td>
</tr>
<tr>
<td>27.50-27.55</td>
<td>32.79</td>
</tr>
<tr>
<td>27.55-27.60</td>
<td>39.32</td>
</tr>
<tr>
<td>27.60-27.65</td>
<td>44.88</td>
</tr>
<tr>
<td>27.65-27.70</td>
<td>56.04</td>
</tr>
<tr>
<td>27.70-27.725</td>
<td>17.29</td>
</tr>
<tr>
<td>27.725-27.75</td>
<td>18.71</td>
</tr>
<tr>
<td>27.75-27.775</td>
<td>20.39</td>
</tr>
<tr>
<td>27.775-27.80</td>
<td>20.52</td>
</tr>
<tr>
<td>(A \left( (\Theta_u^u - \Theta_l^l) \Delta S - (S_u^u - S_l^l) \Delta \Theta \right) \times 10^{-7})</td>
<td></td>
</tr>
<tr>
<td>27.55-27.60</td>
<td>12.15</td>
</tr>
<tr>
<td>27.60-27.65</td>
<td>10.03</td>
</tr>
<tr>
<td>27.65-27.70</td>
<td>10.72</td>
</tr>
<tr>
<td>27.70-27.725</td>
<td>5.971</td>
</tr>
<tr>
<td>27.725-27.75</td>
<td>8.130</td>
</tr>
<tr>
<td>27.75-27.775</td>
<td>12.55</td>
</tr>
<tr>
<td>(Q(\Delta \Theta [S_M - S_0] - \Delta S[\Theta_M - \Theta_0]))</td>
<td></td>
</tr>
<tr>
<td>27.50-27.55</td>
<td>21514</td>
</tr>
<tr>
<td>27.55-27.60</td>
<td>27531</td>
</tr>
<tr>
<td>27.60-27.65</td>
<td>30899</td>
</tr>
<tr>
<td>27.65-27.70</td>
<td>34900</td>
</tr>
<tr>
<td>27.70-27.725</td>
<td>9457</td>
</tr>
<tr>
<td>27.725-27.75</td>
<td>9446</td>
</tr>
<tr>
<td>27.75-27.775</td>
<td>10338</td>
</tr>
<tr>
<td>27.775-27.80</td>
<td>10124</td>
</tr>
</tbody>
</table>

Table 2.2: Based on the hydrography and the volume flux of Mediterranean Water (from Baringer and Price (1997)) this table shows the coefficients of \(K\) and \(D\) in (2.20) (the first two parts of the table) and the left-hand side of (2.20) (the third part of the table) for the forty different control volumes we consider in this paper. The first two parts of the table have units: m psu K; the third has units: m\(^3\) s\(^{-1}\) psu K.
Evidence of variation in $D$ and $K$ with Depth

Figure 2.5: Ten examples of the linear equation (2.20) relating $K$ and $D$ (for the known values of $FQ$). The five lines that nearly coincide at the larger values of $D$ and $K$ are for the five contours of the 27.65 - 27.70 density layer while the other five lines are for the 27.70 - 27.725 density layer.

These forty linear relations on the D-K diagram and noticed that the straight lines for the five contours from each density layer tended to intersect while these points of (near) intersection are different from one layer to the next. This is illustrated in Figure 2.5 which shows ten such straight lines, five from each of two different layers. There is a strong tendency for the lines from an individual layer to nearly cross at a point and this strongly suggested that the method has sufficient resolving power to be able to estimate both $D$ and $K$, and that these diffusivities were not constant in the vertical.

Hence we performed least-squares fits assuming that the diffusivities $K$
and $D$ were constant along each layer. In this way we performed eight such fits with five equations in each fit, and the resulting values of the diffusivities for the eight layers are shown in Figure 2.6 and Figure 2.7 and in Table 3. The errors bars and for these diffusivities are obtained by taking the constant value of the other diffusivity (either $K$ or $D$) in each layer and the five equations (2.20) then give five values of the diffusivity in question (either $D$ or $K$) from which the standard deviation is calculated.

The first thing to notice in Figure 2.6 and Figure 2.7 is that the diffusivities are all positive and they vary quite smoothly from one layer to the next rather than varying randomly in the vertical. The standard error of these diffusivities is quite small, especially for the dieneutral diffusivity $D$. The
Figure 2.7: The along-isopycnal diffusivity $K$ for each density layer, determined assuming that both $D$ and $K$ are constant in each layer.
Table 2.3: The diffusivities and their standard errors, with both $D$ and $K$ being treated as constant along each density layer. The eight layers are treated independently, and in each layer $D$ and $K$ are determined as the only two unknowns in an overdetermined set of five equations corresponding to the five different $S_0$ contours.

<table>
<thead>
<tr>
<th>Neutral density layer (kg m$^{-3}$)</th>
<th>$D$ (m$^2$ s$^{-1}$)</th>
<th>$\sigma_D$ (m$^2$ s$^{-1}$)</th>
<th>$K$ (m$^2$ s$^{-1}$)</th>
<th>$\sigma_K$ (m$^2$ s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>27.50-27.55</td>
<td>17.5×10$^{-5}$</td>
<td>0.7×10$^{-5}$</td>
<td>110.9</td>
<td>20.4</td>
</tr>
<tr>
<td>27.55-27.60</td>
<td>18.3×10$^{-5}$</td>
<td>0.4×10$^{-5}$</td>
<td>137.1</td>
<td>8.25</td>
</tr>
<tr>
<td>27.60-27.65</td>
<td>19.8×10$^{-5}$</td>
<td>1.1×10$^{-5}$</td>
<td>213.7</td>
<td>23.3</td>
</tr>
<tr>
<td>27.65-27.70</td>
<td>15.0×10$^{-5}$</td>
<td>0.3×10$^{-5}$</td>
<td>328.6</td>
<td>5.92</td>
</tr>
<tr>
<td>27.70-27.725</td>
<td>8.88×10$^{-5}$</td>
<td>1.0×10$^{-5}$</td>
<td>262.9</td>
<td>20.3</td>
</tr>
<tr>
<td>27.725-27.75</td>
<td>6.41×10$^{-5}$</td>
<td>0.9×10$^{-5}$</td>
<td>250.0</td>
<td>23.0</td>
</tr>
<tr>
<td>27.75-27.775</td>
<td>4.32×10$^{-5}$</td>
<td>0.5×10$^{-5}$</td>
<td>244.6</td>
<td>17.5</td>
</tr>
<tr>
<td>27.775-27.80</td>
<td>4.17×10$^{-5}$</td>
<td>0.2×10$^{-5}$</td>
<td>198.3</td>
<td>8.82</td>
</tr>
</tbody>
</table>
error bars here do not reflect any uncertainty in the estimate of the effective transport \( FQ \) but rather they reflect the resolving power of the signature of the tracer patterns in the averaged hydrography to be able to distinguish between dianeutral and epineutral diffusion for a given \( FQ \). We find the rather small standard errors of these diffusivities remarkable and surprising.

The vertical structure of the diapycnal diffusivity in Figure 2.6 and Figure 2.7 exhibits a rather strong vertical variation from the core of the Mediterranean Water at \( \gamma = 27.65 \text{ kg m}^{-3} \) where we find \( D \approx 2 \times 10^{-4} \text{ m}^2 \text{ s}^{-1} \) towards deeper layers where the diapycnal diffusivities are seen to be about \( 0.5 \times 10^{-4} \text{ m}^2 \text{ s}^{-1} \). The epineutral diffusivities in this region of about \( 200 \text{ m}^2 \text{ s}^{-1} \) are much less than values nearer the sea surface, but may be appropriate for this depth and region of the ocean. The orderly progression of both \( D \) and \( K \) with density are probably due to our use of the rather smooth hydrographic atlas data. Nevertheless, the small error bars on these diffusivities gives us some hope that the idea of carefully choosing the reference temperatures and salinities and the ratios in order to achieve the three properties described in section 2.3 may add skill to the determination of diffusivities in a more general inverse problem. There have been few observational studies of diffusivity in this region. In the NATRE experimental region (25°N, 332°E) which is further to the south and west of the area we are considering, Ferrari and Polzin (2005) found \( D \approx 10^{-5} \text{ m}^2 \text{ s}^{-1} \) and \( K \approx 300 \text{ m}^2 \text{ s}^{-1} \). Our diapycnal diffusivity is a factor of about ten greater than this and our epineutral diffusivity of the same order.

The relative importance of vertical and epineutral diffusion in explaining the effective lateral advection of Mediterranean Water through this part of the North Atlantic is indicated in Figure 2.8. For each density layer we have used the least-squares determined values of \( D \) and \( K \) to evaluate the
fraction of the right-hand sides of the five equations (2.20) that are due to vertical and epineutral diffusion. The regular progression of the lines in this figure is probably indicative of the smoothed nature of the hydrographic atlas data and the fact that the areas that we consider are cumulative so that the properties of the area might be expected to vary slowly. The dashed curve with the smallest fractional influence is the curve for the largest horizontal area (the westernmost contour) and the successive dashed lines are the contours proceeding to the east in a monotonic fashion. The opposite occurs for the full lines where the curve for the smallest fractional influence is the easternmost contour with the smallest horizontal area. Perhaps this regular progression indicates that there may be more information that can be extracted using this technique, such as trying to determine a spatial variation of the diffusivities inside each isopycnal layer. We have not attempted to do so in this exploratory study but we note that if we took the diapycnal diffusivity to decrease and the lateral diffusivity to increase away from the Gulf of Cadiz, this would be consistent with the differences in \( D \) and \( K \) between the NATRE site and our study region as well as tending to have the five solid lines and five dashed lines in Figure 2.8 tending to collapse onto single full and dashed lines.

We have checked the robustness of the technique presented in this paper by varying the values of the vertical difference in salinity and conservative temperature, \( \Delta S \) and \( \Delta \Theta \) separately by \( \pm 10\% \) and then redoing the analysis. These four separate changes typically resulted in changes to the diffusivities of less that 1\%, confirming that the terms (2.18) that we neglected in arriving at our conservation equation (2.20) are indeed small. Another assumption that we have built into our results is the vertical structure of the volume flux of Mediterranean Water entering the North Atlantic. Since we examined only
Figure 2.8: Using the values of $D$ and $K$ for each layer (from Table 2.3 and Figure 2.6 and Figure 2.7), this figure shows the fractional contribution to the right hand side of (2.20) from the term in $D$ and the term in $K$ for each of the five equations for each layer.
the central range of densities between neutral densities of 27.50 and 27.80, we effectively chose the volume flux per unit density interval to be nearly constant (see Fig. 2.4). Nevertheless, this is an assumption and our results in each density layer are directly proportional to the volume flume entering the layer. Another key assumption we have made is to use the same diapycnal turbulent diffusivity for heat as for salt. This is equivalent to assuming that the diapycnal mixing is caused by isotropic turbulent mixing. St.Laurent and Schmitt (1999) found that this was not the case in the upper ocean near our study region. Investigation of this effect, in the deeper Mediterranean outflow waters, is beyond the scope of this thesis. Another assumption would be to take the ratio of the diapycnal fluxes of heat and salt to be that in salt fingers, but we have not attempted this here.

2.5 Diffusivities from Hallberg Isopycnal Model data

Model output from the same region of the North Atlantic as in Section 2.4 has been gained from the Hallberg Isopycnal Model (HIM) (Hallberg (2000)). There are three main advantages of using HIM as a means of testing the accuracy of the method already applied to the atlas data: First, HIM is one of the few global circulation models to impose a volume transport through the Straits of Gibraltar. If only temperature and salinity fluxes are allowed through the Straits the overall dynamics of the region are changed (Griffies et al., 2005). Second, by utilizing a density coordinate system HIM avoids the unwanted mixing effects caused by the Veronis effect in z-coordinate GCMs (Veronis, 1975). Third, as HIM is already on density surfaces, we avoid any error implicit in interpolating to density surfaces from a Cartesian grid of
Θ, S and p values. The HIM output used is the time average of the final 20 years of a 100 year coupled experiment. The model uses a \( \sigma_2 \)-coordinate system with a total of 49 layers in the vertical and with a horizontal resolution of one degree of longitude and latitude. The model is not eddy resolving, has an isopycnal diffusivity \( (K_{model}) \) of 600 m² s⁻¹ and employs a Richardson number (Ri) dependent vertical diffusion scheme which results in high vertical diffusivities \( (D_{model} \sim O(10^{-4} \text{ m}^2 \text{ s}^{-1})) \) in regions of high Ri and relaxes to a background profile \( [\text{depth dependent } O(10^{-5} \text{ m}^2 \text{ s}^{-1})] \) in the deep ocean. \( D_{model} \) is close to \( 1.8 \times 10^{-5} \text{ m}^2 \text{ s}^{-1} \) (i.e., within 10%) for the region explored by this study.

The model stores velocity, advective flux, and diffusive fluxes of heat and salt in the meridional, zonal and vertical directions. From the advective fluxes close to the Straits of Gibraltar we are able to assign a specific \( Q \) value to each layer in the vertical. Layers \( \sigma_2 = 36.0 \text{ kg m}^{-3}, 36.1 \text{ kg m}^{-3}, 36.2 \text{ kg m}^{-3} \) and \( 36.3 \text{ kg m}^{-3} \) are chosen as they represent the majority of the volume of the salt tongue and are bounded by 5 surfaces equally spaced in density coordinates. Contours are chosen such that they pass through a latitude of \( 35.5^\circ \text{N} \) and longitudes \( 336^\circ \text{E}, 338^\circ \text{E}, 340^\circ \text{E}, 342^\circ \text{E}, \) and \( 344^\circ \text{E} \). One downside of using model output is the unsmoothed nature of the data close to the continents. Along both the Portuguese and Moroccan coasts the pressures on density surfaces is quite variable and this data is removed by defining an Eastern boundary to the region that is 1-2 degrees from the coast. The inflow \( Q \) may now be approximated as the volume flux through that boundary rather than simply the flux out of the Gulf of Cadiz. The region in question is shown in Figure 2.9 along with the position of all five contours on the \( 36.2 \text{ kg m}^{-3} \) layer.

Once again we have five sets of areas, now on each of four density layers,
Figure 2.9: Contours of $\theta_0$ on the $\sigma_z=36.2 \text{ kg m}^{-3}$ density layer. These five contours define the western-most edge of five areas. The eastern most edge is also shown where the flux into a given area is the sum of flux values through that boundary. The colour bar and contours are of potential temperature (°C).
giving 20 equations of the form (2.20). Our initial over-determined least
squares solution is \( K = 586 \pm 131 \text{ m}^2 \text{ s}^{-1} \) and \( D = 3.42 \times 10^{-5} \pm 1.79 \times 10^{-5} \text{ m}^2 \text{ s}^{-1} \). It was found in the previous section that when \( K \) and \( D \) where
solved for each layer, much greater precision resulted and it is thus pertinent
to look to the intersection of lines in \( K - D \) space formed by each equation. A
comparison between the lines formed by areas on both the \( \sigma_2 = 36.2 \text{ kg m}^{-3} \)
and \( \sigma_2 = 36.3 \text{ kg m}^{-3} \) layers is shown in Figure 2.10. While the lines formed
by the 36.2 kg m\(^{-3}\) layer compare well to those of the hydrographic atlas, each
having varied slopes and intersecting within a small range, the lines formed
on the 36.3 kg m\(^{-3}\) layer are close to parallel. Clearly the ratio of coefficients
in front of \( K \) and \( D \) in equation (2.20), vary little within each extra area on
the 36.3 kg m\(^{-3}\) layer and thus not enough information is present to solve
for \( K \) and \( D \) within that layer individually. Possible explanations for this
may lie in the averaging problem mentioned earlier and the fact there may
be something implicit in HIM that maintains a constant ratio between the \( K \)
and \( D \) coefficients in (2.20) which is not observed in the real Mediterranean
Outflow.

We chose to solve for \( K \) and \( D \) for each set of equivalently sized areas
in the vertical making the assumption that \( K \) and \( D \) are constant on each
set. The values obtained along with the average values used in HIM for these
regions are displayed in Table 2.4 and Figures 2.11 and 2.12 (for \( F=1 \)) and
these results show that the diagnosed \( K \) and \( D \) values vary little throughout
this region. The model values also vary little in space but differ from those
diagnosed by around a factor of two in the \( D \) case and 20%–50% in the \( K \)
case.

In the above calculations the assumption that \( FQ = Q \) has been made
as was the case in Section 2.4 where an estimate for \( Q \) was all that was
Figure 2.10: Ten examples of the linear equation (2.20) relating $K$ and $D$ (for the known values of $FQ$). Unlike the hydrographic atlas data the lines representing (2.20) for the $\sigma_2 = 36.3 \text{ kg m}^{-3}$ layer (grey lines) are close to parallel suggesting $K$ and $D$ should not be determined for each layer independently.
<table>
<thead>
<tr>
<th>Longitude contours at 35.5°N</th>
<th>$D$ (m$^2$ s$^{-1}$)</th>
<th>$\sigma_D$ (m$^2$ s$^{-1}$)</th>
<th>$K$ (m$^2$ s$^{-1}$)</th>
<th>$\sigma_K$ (m$^2$ s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>336°E</td>
<td>3.34×10$^{-5}$</td>
<td>1.29×10$^{-5}$</td>
<td>612</td>
<td>153</td>
</tr>
<tr>
<td>338°E</td>
<td>3.52×10$^{-5}$</td>
<td>1.12×10$^{-5}$</td>
<td>569</td>
<td>131</td>
</tr>
<tr>
<td>340°E</td>
<td>3.25×10$^{-5}$</td>
<td>1.48×10$^{-5}$</td>
<td>582</td>
<td>135</td>
</tr>
<tr>
<td>342°E</td>
<td>3.13×10$^{-5}$</td>
<td>2.18×10$^{-5}$</td>
<td>588</td>
<td>146</td>
</tr>
<tr>
<td>344°E</td>
<td>3.65×10$^{-5}$</td>
<td>2.52×10$^{-5}$</td>
<td>573</td>
<td>167</td>
</tr>
</tbody>
</table>

Table 2.4: The diffusivities determined from HIM output and their standard errors, with both $D$ and $K$ being treated as constant for a set of areas in the vertical. The five contour locations are treated independently, and in each contour $D$ and $K$ are determined as the only two unknowns in an overdetermined set of four equations corresponding to the four different layers.
Figure 2.11: The diurnal diffusivity $D$ for each set of areas in the vertical, determined assuming that both $D$ and $K$ are constant within all the areas in the vertical, assuming $F = 1$ (grey and dashed), $F \neq 1$ (black) and the explicit value in HIM (small dashed).
Figure 2.12: The epineutral diffusivity $K$ for each set of areas in the vertical, determined assuming that both $D$ and $K$ are constant within all the areas in the vertical, assuming $F = 1$ (grey and dashed), $F \neq 1$ (black) and the explicit value in HIM (small dashed).
known. Since time averaged volume fluxes are known at every point in space of the HIM data, we examine whether the inclusion of a better approximate for $F$ will improve our ability to estimate $K$ and $D$. In Section 2.3, it was shown that $FQ$ can be thought of as the average of the flow into the layer from the Mediterranean and the flow out through the contour which bounds it. Making this modification to the left-hand side of equation (2.20) the overall least squares estimates become $K = 625 \pm 109 \text{ m}^2 \text{s}^{-1}$ and $D = 2.52 \times 10^{-5} \pm 1.87 \times 10^{-5} \text{ m}^2 \text{s}^{-1}$ and $K$ and $D$ for each set of areas in the vertical are shown in Figures 2.11 and 2.12. Here $K$ adjusts slightly upward while $D$ reduces by 30% to 50%, bringing the model $D$ value well within the standard deviation of that diagnosed by this inverse technique. With $F = 1$, the gain or loss of volume within each layer is not accounted for. In the above case, diapycnal fluxes create a reduction in the outflow volume of each layer and by constructing an inverse model without taking this into account gives an overestimate of the vertical diffusivities. When advective fluxes are incorporated into the inverse model through the variable $F$, a better estimate for $D$ results with little change to our estimate for the lateral diffusivity $K$.

2.6 Summary

By taking a carefully chosen linear combination of the conservation statements of volume, salinity and conservative temperature, we have arrived at (2.20) which is effectively a conservation equation for the variable $\Delta S(\Theta - \Theta_1) - \Delta \Theta(S - S_1)$ where the offset values and are chosen to be the volume averaged conservative temperature and salinity of the control volume, and the vertical differences $\Delta S$ and $\Delta \Theta$ are the area averaged differences between the values at the upper and lower interfaces bounding a density layer. Because
of its careful construction, the vertical gradient of this variable is on average zero and hence dianeutral advection makes a negligible contribution to (2.20) as does the vertical gradient of the dianeutral diffusivity. Rather, the process of dianeutral mixing enters (2.20) only as the average of the dianeutral diffusivity at the upper and lower interfaces. In addition, this dianeutral diffusion term is proportional to the vertical curvature of the $S - \Theta$ diagram which is much less subject to numerical noise than is traditionally involved with estimating second derivatives such as $S_{zz}$ and $\Theta_{zz}$. These advantages were the motivation for forming this rather careful linear combination of the conservation equations.

Needler and Heath (1975) analysed the Mediterranean Water signature in the North Atlantic using a salinity anomaly measured at fixed values of potential temperature with respect to a linear $S - \theta$ relation. Diapycnal advection of this salinity anomaly was ignored in their study. This can be justified at the central density of the Mediterranean Water because there this type of salinity anomaly shares the same important property as our (2.21), namely that its vertical gradient is zero and so the effect of diapycnal advection can be neglected. Above and below the central density, the vertical gradient of the salinity anomaly is non-zero and our technique which has a different ratio $\Delta \Theta/\Delta S$ for each density layer is preferred.

Before using this technique in a more general inverse model of ocean hydrography we have applied it to the hydrography in the eastern North Atlantic which has two very important advantages for our purposes. First, the volume flux entering the North Atlantic from the Mediterranean Sea can be taken as known and regarded as steady, and second, the hydrographic data has a very obvious Mediterranean Water signature. The extra terms (A1) - (A3) due to dianeutral advection and the vertical variation of the
The method seems able to distinguish between the effects of epineutral and dianeutral diffusion on the dilution of the Mediterranean Water signature in the eastern North Atlantic. Also the over-determined least-squares “inversion” gives dianeutral and epineutral diffusivities that vary smoothly in the vertical. The data we have examined here is from an ocean atlas Gouretski and Koltermann (2004), and it remains to be seen how the method performs on unaveraged ocean hydrography. In this region of the ocean meddies are known to transport a significant amount of heat and salt laterally Armi and Zenk (1984), and it is clear that this mechanism will not be well parameterized as diffusion. The analysis of the HIM model data provided a further test of the ideas. It is not clear why the coefficients of the two diffusivities were so closely proportional along one of the model layers, but in any case, the technique recognised this and attached large uncertainties to the diffusion coefficients. We are encouraged by the results of this technique in this rather special region and we plan to apply it as a more general inverse technique in the ocean where one does not have an a priori estimate of the volume flux entering one side of the box.

**Acknowledgments.** We wish to thank Drs Susan Wijffels and Nathan Bindoff for insightful and helpful comments on a draft of this paper. This work contributes to the CSIRO Climate Change Research Program and has been partially supported by the CSIRO Wealth from Oceans Flagship.
Chapter 3

Diagnosing the Southern Ocean Overturning from Tracer Fields

Published in:
3.1 Abstract

The strength and structure of the Southern Hemisphere Meridional Overturning Circulation (SMOC) is related to the along-isopycnal and vertical mixing coefficients by analyzing tracer and density fields from a hydrographic climatology. The meridional transport of Upper Circumpolar Deep Water (UCDW) across the Antarctic Circumpolar Current is expressed in terms of the along-isopycnal ($K$) and diapycnal ($D$) tracer diffusivities and in terms of the along-isopycnal potential vorticity mixing coefficient ($K_{PV}$). Uniform along-isopycnal ($<600$ m$^2$s$^{-1}$) and low vertical mixing ($10^{-5}$ m$^2$s$^{-1}$) can maintain a southward transport of less than 60 Sv (Sv=10$^6$ m$^2$s$^{-1}$) of UCDW across the Antarctic Circumpolar Current (ACC), which is distributed largely across the South Pacific and East Indian Ocean basins. For vertical mixing rates of $O(10^{-4}$ m$^2$s$^{-1}$) or greater the inferred transport is significantly enhanced. The transports inferred from both tracer and density distributions suggest a ratio of $K$ to $D$ of $O(2\times10^6)$ particularly on deeper layers of UCDW. Given the range of observed southward transports of UCDW, it is found that $K=300\pm150$ m$^2$s$^{-1}$ and $D=10^{-4}\pm0.5\times10^{-4}$ m$^2$s$^{-1}$ in the Southern Ocean interior. A view of the SMOC is revealed where dense waters are converted to lighter waters not only at the Ocean surface but also on depths below that of the mixed layer with vertical mixing playing an important role.
3.2 Introduction

Oceanographers debate how dense waters, formed at high latitudes, are returned as lighter waters to the ocean surface, completing the Meridional Overturning Circulation. Many argue that dense waters are upwelled through density layers across the abyssal ocean, requiring small scale mixing due to processes such as energy dissipation over rough topography. Others argue that dense waters are transported along sloping isopycnals to the outcropping regions of the Antarctic Circumpolar Current (ACC), where vigorous winds of the Southern Hemisphere provide the energy required to convert dense water to light (see Kuhlbrodt et al., 2007, for a comprehensive review).

The Southern Ocean links the three major ocean basins and it is there that many water masses are either formed or modified (Sverdrup et al., 1942). The ACC is a zonal current, circulating around Antarctica, with a transport of $134 \pm 13$ Sv ($Sv=10^6 \text{ m}^3 \text{ s}^{-1}$), as measured through Drake Passage (Whitworth, 1983; Whitworth and Peterson, 1985). The ACC is the dominant dynamical feature of the Southern Ocean. In contrast, the Southern Hemisphere Meridional Overturning Circulation (SMOC) is estimated to involve between 20 and 50 Sv of exchange between density classes over the entire circumpolar extent of the Southern Ocean. This exchange is thought to involve an upper and lower branch. In the upper branch, UCDW is converted to northward flowing Subantarctic Mode and Antarctic Intermediate waters. In the lower branch, UCDW and Lower Circumpolar Deep Water are converted to northward flowing bottom waters (Sloyan and Rintoul, 2001).

Although it is not integral to the analysis, we choose to equate the overturning, SMOC, to the southward transport of UCDW. It is UCDW that feeds both the upper and lower limbs of the SMOC. We provide evidence that a southward transport of 20 to 50 Sv of UCDW into ACC (Sloyan and Rintoul, 2001; Lumpkin and Speer, 2007) is consistent with observed mixing rates. Both transformation
above the mixed layer and vertical mixing in the ocean interior play important roles in determining the strength of the UCDW transport and hence the SMOC.

The absence of land barrier(s) at latitudes and depths of Drake Passage (around 55°S-60°S, and 0-1800 m respectively) deny the possibility of a mean geostrophic velocity across the ACC, in depth or pressure coordinates. The SMOC in this region consists of a northward Ekman transport at the surface due to the strong eastward wind stress and an eddy flux due to correlations between the thickness of isopycnal layers and the geostrophic flow. Only below 1800 m can the mean geostrophic flow contribute to the SMOC. Considering the overturning in density space, geostrophic flow across the ACC can contribute through both temporal and spatial correlations of the geostrophic velocity with the thickness of isopycnal layers. That is, in density space, both transient and standing eddies contribute to the overturning circulation. It is shown in section 3.5 that the effect of standing eddies can be neglected only in a small range of densities where a contour of constant potential vorticity $PV$, on an isopycnal, runs along the entirety of the ACC ($PV = f/h$; $f$ is Coriolis frequency and $h$ is thickness).

Unlike the ACC transport, which can be measured directly, the transport of UCDW can only be estimated using inverse methods and other indirect approaches. Inverse modeling has been used to estimate the southward transport of UCDW across 30°S -40°S by Lumpkin and Speer (2007) and Sloyan and Rintoul (2001). They infer 20 Sv and 52 Sv of UCDW, respectively, and find that it feeds both the upper and lower branches of the SMOC. The difference in the estimates lies in the $a priori$ constraints and mixing representations used in the inverse models.

Karsten and Marshall (2002) and Speer et al. (2000) estimate the rate of upwelling across the ACC by determining the surface Ekman, buoyancy and eddy flux components in a residual mean framework. They infer a surface divergence and hence a rate of upwelling of water masses into the mixed layer. Karsten and Marshall (2002) project the inferred upwelling down to depth using a simple vertical advective-diffusive balance (assuming a certain vertical diffusivity $D$). This
method is applied to the Antarctic Intermediate Water layers only (i.e. the upper branch of the SMO). Assumptions must be made about the upwelling at a particular mean streamline corresponding to a particular density layer as contours of sea surface density do not follow streamlines along which the divergence is computed. Olbers and Visbeck (2005) investigate the relationship between Ekman transport, eddy fluxes and vertical mixing in the Southern Ocean. They apply an \textit{a priori} estimate of the meridional transport of UCDW and Antarctic Intermediate Water and infer a thickness diffusivity. The thickness diffusivity diagnosed accounts for both eddy variability and large scale standing eddies and their solution is likely to be sensitive to their description of the Ekman velocity and their zonal averaging.

The transport of UCDW can be related to the along-isopycnal and vertical mixing coefficients through the temperature and salinity fields. Along the ACC there exist strong meridional temperature and salinity gradients on isopycnals. More precisely density layers are cooler and fresher at the outcropping regions to the south and become warmer and saltier to the north. For these gradients to exist in steady state there must be a balance between advection, transporting heat and salt up or down the tracer gradient on isopycnals, and the effects of both along-isopycnal and vertical mixing. Along-isopycnal mixing $K$ acts to mix tracer anomalies on the isopycnal, while vertical mixing destroys or enhances anomalies by transferring temperature and salinity across isopycnals. This advective diffusive balance is evident from observed tracer distributions.

In this study we determine the transport and spatial structure of UCDW as a function of the vertical and along-isopycynal mixing coefficients using the advective diffusive balance described above (section 3.4). As in Zika and McDougall (2008), the advective diffusive balance is applied by integrating along temperature contours on isopycnal layers.

Using established parameterizations for the bolus flux (i.e. the difference between the mean and thickness-weighted average flow in isopycnal coordinates), we show the dependence of the UCDW transport on the along-isopycnal thickness or
potential vorticity mixing coefficient (section 3.5). The upwelling across isopycnals along the ACC in terms of a vertical advective-diffusive balance is also considered (section 3.6).

It is shown that below approximately 500 m the ratio of the mean along-isopycnal mixing coefficient \( K \) to the mean vertical mixing coefficient \( D \) is \( O(2 \times 10^6) \) (section 3.7). In section 3.7 the ratio of \( K \) to \( D \) is also derived by applying conservation of volume to each layer, reaffirming a value of \( O(2 \times 10^6) \).

Section 3.8 contains a comparison and discussion of these results with previous theoretical and numerical studies of the SMOC. The consistency of a low along-isopycnal to vertical diffusivity ratio in the Southern Ocean is discussed in the context of coarse-resolution numerical models.

Here conservative temperature (\( \Theta \)) is used and is proportional to potential enthalpy and represents the ‘heat content’ per unit mass of seawater (McDougall, 2003). That is, where potential temperature (\( \theta \)) would commonly be used as a conservative variable for heat, we use \( \Theta \), as it is equivalent to \( \theta \) while being far more conservative. Note that the distinction between conservative temperature and potential temperature, and neutral and potential density is not central to this paper. We will frequently refer to \( \Theta \) as temperature and neutral density layers as isopycnals.
3.3 Water Mass Equation and Cross-Contour Flow

Consider the idealized scenario where no mixing or diffusive processes occur in the ocean. In such an ocean the path taken by a parcel of water with salinity $S$, conservative temperature $\Theta$ and neutral density $\gamma$ is simply the path where $S$, $\Theta$ and $\gamma$ are constant. Currents in such an ocean would closely follow contours of constant temperature and salinity on isopycnals. It is clear that the amount by which seawater chooses not to follow such a path, that is, the flow across temperature contours on isopycnals ($\overline{vh} \cdot \mathbf{n}_\Theta$), and vertically through density surfaces ($w^\gamma$), is determined purely by the magnitude of vertical and along-isopycnal mixing processes. Here $v$ is the absolute 2-D velocity, $\nabla \gamma$ is the gradient on the isopycnal, $\mathbf{n}_\Theta$ is the unit vector down the along-isopycnal temperature gradient ($\mathbf{n}_\Theta = \nabla \gamma / |\nabla \gamma|$) and $h$ is the vertical distance between closely spaced neutral density $\gamma$ surfaces. The equation describing the balance between the cross-contour flow $\overline{vh} \cdot \mathbf{n}_\Theta$ and mixing process in a steady ocean is

$$\frac{\overline{vh} \cdot \mathbf{n}_\Theta}{h} = K \left( \frac{\nabla \gamma \cdot (\overline{h} \nabla \gamma S) \Theta_z - \nabla \gamma \cdot (\overline{h} \nabla \gamma \Theta) S_z}{\overline{h} |\nabla \gamma S| \Theta_z - |\nabla \gamma \Theta| S_z} \right) + D \left( \frac{S_{zz} \Theta_z - \Theta_{zz} S_z}{|\nabla \gamma S| \Theta_z - |\nabla \gamma \Theta| S_z} \right)$$

$$= \frac{K}{\lambda^\perp} + \frac{D}{\lambda^\gamma} \tag{3.1}$$

McDougall (1984) first derived (3.1) (in a slightly different form) and described it as the ‘Water Mass Transformation’ equation (see Appendix B.1 for a detailed derivation). Cases where the along-isopycnal gradient of $K$ is a significant term in (3.1) are not considered in this study.

Here (3.1) represents a balance between the advection down a temperature gradient on an isopycnal and both along-isopycnal and vertical mixing. This down-gradient advection can be thought of as the ‘non-adiabatic’ component of the along-isopycnal flow. It is important to recognize that (3.1) does not involve the diapycnal velocity component $w^\gamma$, vertical differences in $D$ or individual second
derivatives of tracers in $z$. Instead, vertical mixing appears in (1) through the $\Theta - S$ curvature (see Appendix B.1), a quantity less sensitive to noise in hydrographic data than $S_{zz}$ and $\Theta_{zz}$ individually. In (3.1) $\lambda^\perp$ and $\lambda^\gamma$ are diffusive 'scale lengths'. Note that, although a singularity exists in (3.1) when $\nabla_\gamma \Theta = 0$, no contour exists either.

Ignoring, for a moment, the consequences of the nonlinear equation of state and ignoring the thickness gradient, the first term on the right hand side of (3.1) represents the ratio of along-isopycnal curvature of temperature to the along-isopycnal temperature gradient $\nabla_\gamma^2 \Theta / |\nabla_\gamma \Theta|$ (Appendix B.1). As in the 1-D vertical balance of ‘Abyssal recipes’ (Munk, 1966) where the ratio of the vertical gradient of temperature to the vertical curvature of temperature (and more accurately density) dictates the ratio of diapycnal advection to vertical mixing, here the similar ratio for tracers $\Theta$ and $S$ along-isopycnals dictates the ratio of cross-contour flow $\mathbf{v}_h \cdot \mathbf{n}_\Theta$ to along-isopycnal mixing $K$. One way of understanding this balance is to consider an isopycnal with an along-isopycnal temperature gradient $\nabla_\gamma \Theta$ and curvature $\nabla_\gamma^2 \Theta$ (Fig.3.1). Along-isopycnal mixing acts to smooth out the curvature of temperature and if the curvature is to remain in steady state there must be either up or down-gradient advection to maintain it.

The second term on the right hand side of (3.1) is proportional to the vertical curvature of temperature and salinity $d^2 S / d\Theta^2$. It is not simply $\Theta_{zz}$ or $S_{zz}$ that affects the balance on the isopycnal but the vertical curvature that involves both $\Theta_{zz}$ and $S_{zz}$. Vertical mixing ($D$) acts to smooth out the $\Theta - S$ curvature. In order for it to be maintained there must be cross-contour advection $(\mathbf{v}_h \cdot \mathbf{n}_\Theta)$ or along-isopycnal mixing ($K$) (Fig.3.2). Equation (3.1) allows each of these effects to be quantified. It also includes the effect of a thickness gradient as well as nonlinear effects due to cabbeling and thermobaricity.
3.4 The Southern Ocean Overturning

By integrating along contours of constant temperature and salinity in layers bounded by density surfaces, we can relate the total isopycnal transport across the ACC to both $K$ and $D$. The total thickness-weighted volume flux across such a contour between a pair of density surfaces provides an estimate of the meridional transport. Integrating (3.1) yields

$$\oint (\nabla_{\gamma} \Theta) \cdot \hat{n}_{\Theta} d\Theta = \oint (K/\lambda^\perp) h d\Theta + \oint (D/\lambda^\gamma) h d\Theta$$

(3.2)

where $x_{\Theta}$ is oriented along a contour of constant $\Theta$ (which is also a contour of constant salinity as it is on an isopycnal).

In order to apply (3.2) to the Southern Ocean we define circumpolar tracer contours of constant temperature from the WOCE Hydrographic Atlas (Orsi and Whitworth, 2004) compiled as a gridded climatology on neutral density layers.
Figure 3.2: A $\Theta - S$ curvature exists down the water column (solid line). Vertical mixing (curved arrows) acts to smooth this curvature. Temperature and salinity must be advected by $\mathbf{v}_h \cdot \mathbf{n}_\Theta$ (solid arrows) along-isopycnals to maintain the curvature in steady state.
Figure 3.3: (a) Colormap of conservative temperature (°C) along the ACC on $\gamma_n = 27.7$ kg m$^{-3}$ with positions of the northern, central and southern contours shown (dashed lines). (b) Temperature and salinity of northern (red), central (green) and southern contours (blue) whose extent is fully circumpolar between neutral densities $\gamma_n = 27.2$ kg m$^{-3}$ and $\gamma_n = 28$ kg m$^{-3}$.

(Gouretski and Koltermann, 2004; Jackett and McDougall, 1997). Each layer represents an interval of $\gamma_n = 0.1$ kg m$^3$ (i.e. the $\gamma_n = 27.6$ kg m$^3$ layer is between neutral density surfaces $\gamma_n = 27.55$ kg m$^3$ and $\gamma_n = 27.65$ kg m$^3$). In each layer between neutral densities of $\gamma_n = 27$ kg m$^{-3}$ and $\gamma_n = 28$ kg m$^{-3}$ three contours are chosen corresponding to a northern, central and southern contour of the ACC (Fig.3.3). Isopycnals above $\gamma_n = 27$ kg m$^{-3}$ outcrop and temperature contours on isopycnals below $\gamma_n = 28$ kg m$^{-3}$ are interrupted by topography.

UCDW, which is characterized by low oxygen concentration, is sandwiched between overlying fresher and higher oxygen concentrated Antarctic Surface Water (AASW) and, the underlying salinity maximum and higher oxygen concentration of Lower Circumpolar Deep Water (LCDW). Reviewing maps of oxygen and salinity from the WOCE Atlas we define UCDW to be between $\gamma_n = 27.4$ kg m$^{-3}$ and $\gamma_n = 28$ kg m$^{-3}$. Both Lumpkin and Speer (2007) and Sloyan and Rintoul (2001) also use this range to define UCDW.

Fields of the vertical and epineutral tracer gradients and curvatures are de-
termined from the WOCE climatology. The along-isopycnal mixing and vertical mixing terms in (3.1) are linearly dependent on $K$ and $D$ respectively. Using (3.1), we estimate the total meridional transport on particular density layers for various values of the along-isopycnal and vertical tracer diffusivities. We consider the case where $K=200 \, \text{m}^2 \, \text{s}^{-1}$ and $D=2 \times 10^{-4} \, \text{m}^2 \, \text{s}^{-1}$. Fluxes across the northernmost temperature contours of the ACC are northward for Antarctic Surface Water $\gamma_n < 27.4 \, \text{kg m}^{-3}$ and mostly southward for UCDW $\gamma_n > 27.4 \, \text{kg m}^{-3}$ (Fig.3.4). The level of zero cross-contour flow (i.e. the level where the flow is neither to the south nor to the north) is at approximately $\gamma_n = 27.5 \, \text{kg m}^{-3}$. Transports closer to the centre of the ACC show a similar structure to the northern contour albeit the level of zero cross-contour flow moves to denser layers. Both the vertical mixing and along-isopycnal mixing terms change sign from southward to northward on $\gamma_n = 27.6 \, \text{kg m}^{-3}$ across the ACC. This convergence suggests that UCDW feeds the upper and lower limbs of the SMOC between these contours.

The cumulative integral of (3.2) along a circumpolar path of each temperature contour, summed over layers from $\gamma_n = 27.4 \, \text{kg m}^{-3}$ to $\gamma_n = 28 \, \text{kg m}^{-3}$, gives the spatial variation in the meridional transport of UCDW (Fig.3.5). Both the along-isopycnal mixing and diapycnal mixing components of the overturning circulation vary smoothly, giving confidence that the use of second derivatives of the tracer fields is not particularly noisy, however, this may also relate to the smoothing applied to hydrographic data in order to produce the Atlas. For the northernmost contour the two components are mostly negative (southward) and vary in a similar way along the contour, suggesting that warm anomalies are advected southward and both vertical and along-isopycnal mixing act to mix them across and along-isopycynals respectively. At the southernmost contours the magnitude of the vertical mixing term is much larger than the along-isopycnal mixing term, suggesting that either vertical mixing dominates the balance or $K$ is large relative to $D$.

We sum the vertical and along-isopycnal mixing terms in (3.2), again from
Figure 3.4: Contributions to layer cross-contour transport from the along-isopycnal mixing term (taking $K$ to be $200 \, m^2 \, s^{-1}$; black bars) and the vertical mixing terms (taking $D$ to be $2 \times 10^{-4} \, m^2 \, s^{-1}$; white bars). Transports are across the southern (a), central (b) and the northern contours (c) of the ACC. The temperature and salinity of each contour is marked with a circle in of Fig.3.3b. Positive values are with increasing temperature (northward for the layers shown).
Figure 3.5: Terms contributing to the cumulative cross-contour transport between $\gamma_n = 27.4 \text{ kg m}^{-3}$ and $\gamma_n = 28 \text{ kg m}^{-3}$ due to the along-isopycnal mixing term ($K = 200 \text{ m}^2 \text{ s}^{-1}$; solid line) and the vertical mixing term ($D = 2 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$; dot-dashed line). Transport across the (a) southern, (b) central and (c) northern contours of the ACC. Positive values are with increasing temperature (northward for the layers shown)
\( \gamma_n = 27.4 \) to \( \gamma_n = 28 \, \text{kg m}^{-3} \). This results in the total flux of UCDW in \( K - D \) space (Fig.3.6). We now review observational estimates of the vertical mixing coefficient \( D \), the along-isopycnal or lateral mixing coefficient \( K \) and the transport of UCDW in the Southern Ocean (presented graphically in Fig.3.6).

Munk (1966) estimates \( D \) to be \( O(10^{-4} \, \text{m}^2 \, \text{s}^{-1}) \) in the deep ocean by considering the mixing necessary to close the global overturning circulation. However, Ledwell et al. (1993) observes a diffusivity of \( O(10^{-5} \, \text{m}^2 \, \text{s}^{-1}) \) by releasing a tracer across the pycnocline of the North East Atlantic. Recent observational estimates in the Southern Ocean suggest that mixing is at the upper end of this range and higher, close to rough topography and in the core of the ACC (Naveira-Garabato et al., 2004; Sloyan, 2005; Kunze et al., 2006).

In the Southern Ocean, estimates exist for a surface eddy diffusivity from satellite observations (Marshall et al., 2006) and float measurements have been used to calculate eddy kinetic energy and eddy diffusivity. Reconciling the various estimates that range from less than \( O(100 \, \text{m}^2 \, \text{s}^{-1}) \) to \( O(10,000 \, \text{m}^2 \, \text{s}^{-1}) \) is difficult, as there are likely to be large differences between buoyancy diffusivities, and tracer or potential vorticity diffusivities (see Smith and Marshall (2008)). In addition, the grid spacing of inverse models and coarse-resolution ocean models can play a large role in determining the estimated or required diffusivity. Phillips and Rintoul (2000) estimate the lateral diffusivity \( (K_{xy} = \overline{\nabla \theta} \left|_{z} \right/ \nabla_{z} \theta; \, z \text{ being a constant depth surface}) \) of temperature from a mooring array time series of velocity and temperature placed within the ACC near 50°S, 143°E. Their estimates are in the broad range 100-1000 \( \text{m}^2 \, \text{s}^{-1} \) for this one geographical location (500-1000 \( \text{m}^2 \, \text{s}^{-1} \) above 500m depth and 100-500 \( \text{m}^2 \, \text{s}^{-1} \) below; Dr Helen Phillips, private communication). Gille (2003) estimated eddy heat fluxes in the Southern Ocean using Autonomous Lagrangian Circulation Explorer (ALACE) floats and found the lateral mixing coefficient to be between 300 \( \text{m}^2 \, \text{s}^{-1} \) and 600 \( \text{m}^2 \, \text{s}^{-1} \) (at around 900m depth).

Estimates from McKeague et al. (2005), based on an inverse model of the ocean
Figure 3.6: Black contours show the northward transport (Sv) of UCDW between $\gamma_n = 27.4$ kg m$^{-3}$ and $\gamma_n = 28$ kg m$^{-3}$ into the northern side of the ACC from the sum of terms in (3.2) for various values of $K$ and $D$. Grey shading across the centre of the figure represents abyssal estimates of $D$ from both Munk (1966) and Ledwell et al. (1993). Below the x-axis, colored bars show relevant estimates of lateral or along-isopycnal diffusivities from McKee et al. (2005) (black bar-$K^y$; grey bar-$K^x$), Phillips and Rintoul (2000) (red bar), Gille (2003) (green bar) and Naveira-Garabato et al. (2007) (blue bar). UCDW transport estimates across hydrographic sections at 30-40°S are shown from Lumpkin and Speer (2007) (20 Sv; green contour) and Sloyan and Rintoul (2001) (52 Sv; red contour). Blue lines represent estimates of the ratio $K/D$ below 27.7$\gamma_n$, made in section 3.7, plus or minus one standard deviation. Taking the spread between the green and red contours to be a reasonable error range for the UCDW transport, the predicted range for the circumpolarly averaged mixing coefficients is cross hatched in light blue.
circulation on $\gamma_n = 28$ kg m$^{-3}$ in the South Atlantic, are relevant to our study as they considered the along-isopycnal mixing of tracers, including temperature and salinity, assuming steady state. They find a meridional diffusivity $K_x = 100 \pm 50$ m$^2$ s$^{-1}$ and zonal $K_y = 750 \pm 100$ m$^2$ s$^{-1}$. As temperature contours are close to lines of constant latitude, the meridional diffusivity is perhaps the most relevant here. However, as the authors suggest, the difference in magnitude may relate to the difference in grid sizing, which again makes interpretation of the eddy diffusivity difficult. Naveira-Garabato et al. (2007) were able to estimate the along-isopycnal mixing coefficient for a passive tracer in the South East Pacific and South West Atlantic Oceans along $\gamma_n = 27.98$ kg m$^{-3}$. They estimate an along-isopycnal diffusivity of $360 \pm 330$ m$^2$ s$^{-1}$ in the frontal regions of the ACC and an area average of $1,860 \pm 440$ m$^2$ s$^{-1}$, which is thought to be associated with intensification of eddy-driven mixing in the Scotia Sea relative to ACC-mean conditions. The range $100$ m$^2$ s$^{-1}$ - $1000$ m$^2$ s$^{-1}$ is consistent with that used by coarse-resolution models, higher diffusivities leading to unrealistic ACC transports.

In this study, estimates of vertical and along-isopycnal mixing may be used to infer the southward transport of UCDW. Lumpkin and Speer (2007) and Sloyan and Rintoul (2001) both determine the southward transport of UCDW with an inverse model, inferring 20 and 52 Sv respectively. Direct comparison of our estimates with those of Lumpkin and Speer (2007) and Sloyan and Rintoul (2001) is not exact as the transport diagnosed from (3.2) is across the northern flank of the ACC meandering close to 52.5$^\circ$S whereas the inverse estimates are calculated for hydrographic sections between 30$^\circ$S and 40$^\circ$S.

At the limit where vertical mixing $D$ is zero, an overturning circulation of O(20-50 Sv) would require an along-isopycnal diffusivity of about $200$ m$^2$ s$^{-1}$ to $500$ m$^2$ s$^{-1}$. At this limit the overturning circulation is driven by Ekman and eddy transport close to the surface. That is, UCDW flows to the south in the presence of along-isopycnal mixing only until it reaches the mixed layer. It is worth noting, however, that the zero vertical mixing case is only possible for the upper
branch of the SMOC where UCDW is transformed into Antarctic Intermediate, Subantarctic Mode and Surface waters. The lower branch of the SMOC involves conversion of UCDW and Lower Circumpolar Deep Water (LCDW) to Antarctic Bottom Waters (AABW). The lower branch requires abyssal diapycnal mixing in the Southern Ocean or other ocean basins to close the overturning circulation.

For an UCDW transport of O(20-50 Sv) there must be either strong vertical or strong along-isopycnal mixing or some combination thereof. An overturning circulation of less than 5 Sv would require small along-isopycnal diffusivities ($K < 50 \text{ m}^2 \text{s}^{-1}$) and vertical diffusivities between 0 m$^2$ s$^{-1}$ and $10^{-4}$ m$^2$ s$^{-1}$ would make little difference to the size of the overturning circulation (Fig.3.6). If a limit of 60 Sv where placed on the transport of UCDW at the northern side of the ACC this would imply an upper bound on $K$ of $600 \text{ m}^2 \text{s}^{-1}$ and on $D$ of $10^{-3}$ m$^2$ s$^{-1}$, as both have a positive contribution to the southward transport.

Any distribution of $K$ and $D$ can be applied to (3.2) to diagnose a transport of UCDW. Various potential distributions of mixing strengths may be considered by reviewing Fig.3.4 and Fig.3.5. Here if a lateral diffusivity of 400 m$^2$ s$^{-1}$ is assumed on a specific layer, the strength of the transport on that layer due to the $K$ term would be double that shown in Fig.3.4. The longitudinal variation in the transport can be considered in the same way for various distributions of $K$ and $D$ (Fig.3.5). Cases such as stronger along-isopycnal mixing due to added kinetic energy provided by the winds close to the surface or a steering level of baroclinically unstable waves (Smith and Marshall, 2008) or stronger vertical mixing on deeper layers perhaps due to interaction with topography could all be considered.
3.5 The Residual Mean Overturning and Bou-lus Velocity

In depth coordinates and at constant latitude above the depth of the shallowest topographic feature, the circumpolar integral of the geostrophic velocity is zero. Here we consider the circumpolar integral in isopycnal rather than depth coordinates and integrate along a contour of constant potential vorticity $PV = f/h$ rather than latitude. In this case also, the geostrophic component is zero and only an ageostrophic flow remains.

To a good approximation, the mean circulation on an isopycnal satisfies geostrophy and so the mean velocity can be represented by a geostrophic streamfunction $\Psi$ and an Ekman velocity $v^{\text{Ek}}$ such that

$$\mathbf{v} = -\frac{1}{f} \nabla\gamma \Psi \times \mathbf{k} + \mathbf{v}^{\text{Ek}}$$  \hspace{1cm} (3.3)

where $\mathbf{v}$ is the lateral velocity vector (overbars represent temporal averages), $f$ is the Coriolis frequency, and $\mathbf{k}$ is the unit vector in the vertical direction. The actual nature of the streamfunction is not relevant here, merely that one approximately exists. Following from (3.3) we define the transport within a density layer across a contour of constant potential vorticity $PV$ and decompose the thickness flux (velocity times layer thickness correlated) into a temporal mean and perturbation component

$$\bar{v} h = \mathbf{v} \cdot \mathbf{n} + \overline{v' h'}$$  \hspace{1cm} (3.4)

and considering the components down the $PV$ gradient for $\mathbf{n}_{PV} = \nabla\gamma PV/|\nabla\gamma PV|$ we have

$$\bar{v} h \cdot \mathbf{n}_{PV} = \mathbf{v} \cdot \mathbf{n}_{PV} \bar{h} + \overline{v' h'} \cdot \mathbf{n}_{PV}.$$  \hspace{1cm} (3.5)

We wish to find a relationship between the total transport across the ACC
and mixing. Integrating $\overline{\nabla h \cdot \mathbf{n}}_{PV}$ circumpolarly along a $PV$ contour in an isopycnal layer allows the mean geostrophic component to be eliminated. Hence the meridional transport is purely an ageostrophic one involving eddy and Ekman transports. Here the $PV$ contours need not remain within the latitude band of Drake Passage. The $PV$ contours only need to be fully circumpolar.

$$\oint \overline{nh} dx_{PV} = \oint \overline{v_{Ek}} h dx_{PV} + \oint \overline{v'} h' dx_{PV}. \quad (3.6)$$

Equation (3.6) shows that in isopycnal layers, for $PV$ contours which run along the entirety of the ACC, there is a barrier to the mean geostrophic transport (Fig.3.7). The transport in these layers can only come from the Ekman and transient eddy components. In layers where $PV$ contours do not run along the entire ACC there can be a net southward transport due to the mean geostrophic current (i.e. $\oint \overline{nh} dx \neq 0$ where $dx$ runs along the ACC). The mean geostrophic flow may add significantly to a net along-isopycnal transport across the ACC even if the net transport is zero on any given pressure level (Fig.3.7). The SMOC is defined in density space (Hallberg and Gnanadesikan, 2006), in which case mean geostrophic flows can contribute to the overturning in layers where $PV$ contours do not follow the entirety of the ACC, i.e. large scale standing eddies. Olbers and Visbeck (2005) consider the circumpolarly integrated transport of AAIW and UCDW in terms of an Ekman and eddy flux. For layers where $PV$ contours do not sufficiently coincide with the stream-wise averaging they use, the along-isopycnal mixing coefficient they impose must account for both eddy variability and large scale correlations between thickness and the geostrophic velocity.

The eddy transport $\overline{\nabla h'}$ may be parameterized as a down-gradient flux of thickness or bolus flux (with a corresponding bolus velocity $v^*$) such that (following McDougall (1991a))

$$\overline{\nabla h'} / h \equiv v^* \equiv -K_{PV} \nabla_{\gamma} \log(h) + K_{PV} (\beta/f) j \quad (3.7)$$

70
Figure 3.7: (a) Potential vorticity \( PV = f/h \) for \( h = 0.1 \text{kg m}^{-3}/\rho_z \) on \( \gamma_n = 27.7 \text{ kg m}^{-3} \) overlayed with the northern \( PV \) contour along the ACC. (b) The depth of layers between \( \gamma_n = 27.5 \text{ kg m}^{-3} \) and \( \gamma_n = 27.8 \text{ kg m}^{-3} \) (grey shaded area) along the path of the contour in (a). In the shaded area the mean geostrophic flow does not contribute to transport across the ACC \( (\oint v h \text{d}x_{PV} = 0) \) in isopycnal coordinates. Above and below the shaded area, the net geostrophic transport due to the mean flow may be present in each isopycnal layer. In depth coordinates there is no contribution to the overturning from the mean geostrophic flow above the shallowest topographic feature and at latitudes of Drake Passage \( (\oint \nabla h \text{d}x = 0 \text{ where } x \text{ is the zonal coordinate}) \). Below topography a zonal pressure gradient may exist allowing for meridional transport due to the mean geostrophic flow. Note: (b) is a cross section of isopycnals at latitudes and longitudes of the contour shown in (a) not the depths of the individual PV contours on each isopycnal.
\[ \int \nabla h \cdot dx_{PV} \equiv \int K_{PV} \nabla (-\nabla_{\gamma} \log(h) + \beta j) \cdot n \, dx_{PV}. \]  

(3.8)

In (3.8), \( K_{PV} \) is the along-isopycnal potential vorticity mixing coefficient, \( v^* \) is the bolus velocity, \( \beta = \partial f / \partial y \) and \( j \) is the unit vector in the meridional direction. Here along-isopycnal mixing \( (K_{PV}) \) acts to evenly distribute \( PV \). Cast in terms of a velocity, the first component represents the effect of gradients of \( h \) and the second is due to the meridional gradient of \( f \). A different approach is taken by Gent and McWilliams (1990); Gent et al. (1995) to parameterize the quasi-stokes velocity \( v^+ \) (the counterpart to the bolus velocity in Eulerian coordinates).

\[ v^+ \equiv -K_{PV} \nabla_{\gamma} \log \left( \frac{\partial z}{\partial \rho} \right). \]  

(3.9)

This parameterization ignores the \( \beta \) effect and is commonly used in Ocean Circulation Models and is the same as the first term on the right hand side of (3.7) for small \( \Delta \rho \).

Given (3.8) we quantify the meridional transport of UCDW in terms of the along-isopycnal potential vorticity diffusivity \( (K_{PV}) \). This is done in layers where contours of constant \( PV \) are continuous around the circumpolar path of the ACC and are sufficiently distant, in the vertical, from the Ekman layer. We define contours of constant \( PV \) in \( \gamma_n \) layers from the WOCE climatology where \( h \) is the thickness of the layer. Contours may only be defined for northern, central and southern pathways of the ACC in layers \( \gamma_n = 27.5 \text{ kg m}^{-3} \) to \( \gamma_n = 27.8 \text{ kg m}^{-3} \) \((\gamma_n = 27.5 \text{ kg m}^{-3} \text{ to } \gamma_n = 27.7 \text{ kg m}^{-3} \) in the case of the northern contour). We assume a uniform \( K_{PV} \) of 200 m\(^2\) s\(^{-1}\) and quantify both the thickness and \( \beta \) terms in (3.8).

If \( K_{PV} \) is assumed to be the same as the along-isopycnal mixing coefficient for tracers, \( K \), it is expected that the meridional transports derived from (3.2) and (3.8) should be similar for contours with similar paths and depths (away from Ekman effects). This is the case for the deepest of layers (Fig.3.4 and Fig.3.8). Northward transports suggested from (3.2) for \( \gamma_n < 27.7 \text{ kg m}^{-3} \) may be explained
Figure 3.8: Contributions to bolus transport for $K = 200 \text{ m}^2 \text{ s}^{-1}$ from the thickness gradient term (black bars) and the beta gradient term (white bars) in (3.8). (a) Southern contour, (b) central contour and (c) northern contours of the ACC. Positive values are with increasing $PV$ (northward for the layers shown).
by Ekman effects as isopycnals shoal in the southward direction and perhaps a strong gradient of \( K \) at the bottom of the mixed layer.

### 3.6 Diapycnal flow and the 1-D balance

Since the work of Munk (1966), it has been common to consider a 1-D balance where a vertical mixing 'scale length' relates a rate of upwelling to a vertical mixing coefficient.

Following a similar approach as that which led to (3.1) an equation may be derived for the diapycnal velocity \( w^\gamma \)

\[
\begin{align*}
w^\gamma &= K \left( \frac{\nabla^2 \Theta |\nabla \gamma S| - \nabla^2 \gamma |\nabla \Theta|}{|\nabla \gamma S| |\nabla \Theta|} \right) + D \left( \frac{\Theta_{zz} |\nabla \gamma S| - S_{zz} |\nabla \gamma \Theta|}{|\nabla \gamma S| |\nabla \Theta|} \right) \\
&= \frac{K}{\eta^\perp} + \frac{D}{\eta^\gamma} \tag{3.10}
\end{align*}
\]

where \( \eta^\gamma \) and \( \eta^\perp \) are diffusive 'scale heights' (see Appendix B.2). Cases where \( D_z \) is a significant term in (3.10) are not considered in this study. If a linear equation of state is assumed the along-isopycnal mixing term is zero and (3.10) reduces to \( w^\gamma = D \gamma_{zz}/\gamma_z \) (Appendix B.2). This simplification of (3.10) is commonly used in studies attempting to infer large scale dynamics in the Southern Ocean from observations (Karsten and Marshall, 2002; Naveira-Garabato et al., 2007). Here we shall retain both the linear and nonlinear components and determine their relative role in cross isopycnal transport.

The diapycnal velocity is computed for neutral density surfaces from tracer fields of the WOCE climatology and integrated circumpolarly between temperature contours corresponding to the southern and northern sides of the ACC. With a vertical diffusivity of \( 2 \times 10^{-4} \, \text{m}^2 \, \text{s}^{-1} \) the vertical velocity through \( \gamma_n = 27.7 \, \text{kg} \, \text{m}^3 \) due to the \( D \) term in (3.10) is determined (Fig.3.9). The velocity is of order \( (10^{-7} \, \text{m} \, \text{s}^{-1}) \) to the north of the ACC and is of order \( (10^{-6} \, \text{m} \, \text{s}^{-1}) \) to the south (Fig.3.9). The accumulated diapycnal transport between the temperature contours is ap-
Figure 3.9: (a) Diapycnal velocity \((\log_{10}, \text{m s}^{-1})\) due to the diapycnal diffusivity (second term on right hand side of (3.10)) for \(D = 2 \times 10^{-4} \text{m}^2 \text{s}^{-1}\) on \(\gamma_n = 27.7 \text{ kg m}^{-3}\). Overlayed are the northern and southern temperature contours of the ACC. (b) The accumulated transport through the \(\gamma_n = 27.7 \text{ kg m}^{-3}\) surface, between the two temperature contours due to both the first and second terms in (3.10) for \(K = 200 \text{ m}^2 \text{s}^{-1}\) and \(D = 2 \times 10^{-4} \text{ m}^2 \text{s}^{-1}\) (dot dashed and solid, respectively).

approximately 8 Sv for the \(D\) term in (3.10). The diapycnal transport is reasonably evenly spread across the entirety of the ACC with the net effect being an upwelling. When a value for \(K\) of 200 m\(^2\) s\(^{-1}\) is used the downwelling due to this term is negligible (\(\leq 1\) Sv). If \(K\) where \(O(1000 \text{ m}^2 \text{s}^{-1})\) the downwelling would be first order (see Iudicone et al., 2008, for a comprehensive discussion). The lateral mixing term is often neglected, even when large along-isopycnal or lateral diffusivities are observed (Karsten and Marshall, 2002; Naveira-Garabato et al., 2007).

The UCDW which downwells to form Antarctic Bottom Water is likely to do so in concentrated areas around the Antarctic Continental Shelf due to entrainment. Given the lake of observations around the shelf, our analysis is restricted to the ACC region only and regions where UCDW downwells are not considered. This does not imply that the southward flowing UCDW is all upwelled.
3.7 The relative role of vertical and along-isopycnal mixing

In this section we present evidence that the ratio of the along-isopycnal and vertical mixing coefficients is $O(2 \times 10^6)$ along the ACC in the density layers of UCDW. This is done by combining the concept of a cross-contour flow $\mathbf{v} \cdot \mathbf{n}_\Theta$ discussed in section 3.3 with the $PV$ parameterizations of section 3.5. This combination yields a balance of vertical and along-isopycnal mixing, independent of the mean velocity. Secondly we combine the cross-contour transports derived in section 3.4 with the diapycnal transports in section 3.6. We then apply continuity to finite volumes around the entire ACC, arriving at a balance of along-isopycnal and vertical mixing.

From (3.4) we have

$$\mathbf{v} = \mathbf{v}_h/h - \mathbf{v}'h'/h.$$  \hfill (3.11)

Taking the component of the mean geostrophic velocity in the direction of the along-isopycnal temperature gradient ($\mathbf{n}_\Theta$) and substituting (3.1) we have

$$\mathbf{v} \cdot \mathbf{n}_\Theta = K/\lambda^\perp + D/\lambda^\gamma + K_{PV} \left( \left( \beta/f \right) \mathbf{j} - \nabla_\gamma \log(h) \cdot \mathbf{n}_\Theta \right).$$  \hfill (3.12)

Above $\mathbf{v} \cdot \mathbf{n}_\Theta$ is the mean velocity down the along-isopycnal temperature gradient. Equation (3.12) differs from (3.1) as the left hand side is only the mean velocity and not the residual one. Away from the Ekman layer the mean velocity is approximately geostrophic, hence

$$\mathbf{v} \cdot \mathbf{n}_\Theta = -\frac{1}{f} \nabla_\gamma \Psi \times \mathbf{k} \cdot \mathbf{n}_\Theta.$$  \hfill (3.13)

From (3.13), the circumpolar integral of $f \mathbf{v} \cdot \mathbf{n}_\Theta$ along circumpolar temperature contours is zero. Hence, integrating $f$ times equation (3.12) gives
\[ 0 = \oint \frac{f K}{\lambda} dx_\Theta - \oint f K_{PV} \left( \frac{\beta j}{f} - \nabla \log(h) \right) \cdot n dx_\Theta + \oint f D/\lambda^\perp dx_\Theta. \quad (3.14) \]

Equation (3.14) relates the along-isopycnal, vertical and potential vorticity mixing coefficients and is independent of the mean velocity. It is thought that the along-isopycnal diffusivity is approximately the same for both PV and passive tracers (Smith and Marshall, 2008). On isopycnals, \( S \) and \( \Theta \) are approximately passive tracers as the act of stirring them locally does not change the stratification (although there may be some effects due to the nonlinear equation of state). We assume \( K = K_{PV} \). From (3.14) the ratio of \( K \) to \( D \) is

\[ \frac{K}{D} = \oint \frac{f/\lambda^\perp dx_\Theta}{\oint (\beta j - f \nabla \log(h)) \cdot n dx_\Theta - \oint f/\lambda^\perp dx_\Theta}. \quad (3.15) \]

The above, (3.15), holds for any enclosed temperature contour on an isopycnal that does not interact with the Ekman layer. Above average depths of 500m the ratio of \( K \) to \( D \) is zero or negative suggesting these contours interact with the Ekman layer or lateral gradients of \( K \) become important, somewhere along their circumpolar path. Below 500m the ratio is consistently positive and of \( O(2 \times 10^6) \) (Fig.3.10).

We have established equations to determine both the along-isopycnal transport across temperature contours (3.2) and the transport through isopycnals (3.10) in terms of the vertical and along-isopycnal mixing coefficients \( K \) and \( D \). We may apply continuity of volume in order to ascertain the ratio of these two coefficients. Writing the steady continuity equation for a particular volume on a density layer, bound by contours of constant temperature to the south and north (\( \Theta_1 \) and \( \Theta_2 \) respectively), we have

\[ \left[ \oint \frac{K}{\lambda^\perp + D/\eta^\perp} h dx_\Theta \right]^{\Theta_2}_{\Theta_1} + \left[ \oint \left( \frac{K}{\eta^\perp + D/\eta^\perp} \right) dA \right]_l^u = 0 \quad (3.16) \]
where \( \oint dA \) are integrals over the upper \( u \) and lower \( l \) bounding neutral density surfaces between temperature contours (\( \Theta_1 \) and \( \Theta_2 \) respectively). Assuming again that the mixing coefficients are constant in space we may write

\[
\frac{K}{D} = -\frac{\left[ \oint \frac{1}{\lambda} hdx_{\Theta_1}\big|_{\Theta_2} + \oint \frac{1}{\eta} dA\big|_{u} \right]_{l}}{\left[ \oint \frac{1}{\lambda} hdx_{\Theta_1}\big|_{\Theta_1} + \oint \frac{1}{\eta} dA\big|_{l} \right]_{u}}.
\]

Equation (3.17) is applied to volumes between contours on the northern side of the ACC below an average depth of 500m (Fig.3.10). We find that the ratio of \( K \) to \( D \) is, again, of \( O(2 \times 10^6) \) there.

### 3.8 Discussion and conclusions

This study has investigated the relationship of along-isopycnal mixing (\( K \)) and diapycnal mixing (\( D \)) to the strength of the Southern Ocean Meridional Overturning Circulation (SMOC) as quantified by the southward transport of Upper Circumpolar Deep Water (UCDW). The total transport of UCDW, within isopycnal layers, has been diagnosed from tracer distributions. The transports are inferred directly from observations through a linear relationship with the mixing coefficients. The sensitivity of the overturning transport and spatial characteristics to a range of possible diffusivities is a direct result of the analysis (section 3.4).

An important aspect of this study is the discussion of cross-contour transport in density-temperature space. The thickness weighted velocity down the temperature gradient on an isopycnal \( \bar{v} h \cdot \mathbf{n} \), discussed in section 3.3, is dependent on \( K \) and \( D \) locally and thus the spatial structure of the transport of specific water masses can be analyzed as well as their circumpolar integral.

The diffusive scale lengths, \( \lambda^\perp \) and \( \lambda^\gamma \) integrated circumpolarly, show that UCDW is transported southward where it feeds the upper and lower cells of the SMOC. Careful comparison of the fluxes inferred from tracer gradients, the bolus transport and conservation considerations for along-isopycnal and vertical transports suggest a ratio of \( K \) to \( D \) of \( O(2 \times 10^6) \). The implications of such a ratio for
Figure 3.10: (a) Along contour average position of the volumes defined between circumpolar contours. (b) Ratio of $K$ to $D$ determined using (3.17) for layers bounded by contours below 500 m along the ACC. Estimates represented by open circles are from equation (3.17) for the entire volumes shown in (a). Open squares and filled circles in (b) define values using volumes between open squares and filled circles respectively in (a). Estimates represented in (b) by open triangles are from equation (3.15). The solid line in (b) is the mean of all the estimates on each layer (equally weighted) and the shaded region represents ± one standard deviation, $\sigma$. The outlier on $\gamma_n = 27.9\text{kg m}^{-3}$ is removed as it is more than $3\sigma$ from the mean.
the overturning, the possible range of mixing coefficients and in turn for numerical modeling of the Southern Ocean, are discussed below.

The southward transport of UCDW inferred from inverse studies (Sloyan and Rintoul, 2001; Lumpkin and Speer, 2007) suggest a range of 20-52 Sv. We have derived a linear relationship between the southward transport of UCDW and $K$ and $D$ (3.2) and we have estimated the ratio of $K$ to $D$ in the UCDW layers to be $2 \times 10^6 \pm 10^6$ (Fig.3.10). Thus we estimate $K$ and $D$ individually and find $K = 300 \pm 150 \text{ m}^2 \text{s}^{-1}$ and $D=10^{-4} \pm 0.5 \times 10^{-4} \text{ m}^2$ (blue cross hatching in Fig.3.6). Such rates of diapycnal mixing are considered to be large in the mid-latitude oceans but are supported by observations of diapycnal mixing ($D$) in the ACC such as Sloyan (2005), Naveira-Garabato et al. (2004) and Kunze et al. (2006). Our results suggest $D$ is $O(10^{-4} \text{ m}^2 \text{s}^{-1})$ beneath the mixed layer in the Southern Ocean, supporting the hypothesis that vertical mixing in the ocean interior, along the ACC, makes a significant contribution to water mass conversion. The diagnosed along-isopycnal mixing coefficient ($K$) is also within the range estimated by both Phillips and Rintoul (2000) and McKeague et al. (2005).

For along-isopycnal mixing coefficients of $O(200 \text{ m}^2 \text{s}^{-1})$, the nonlinear terms contributing to the diapycnal transport below the mixed layer in the Southern Ocean are small (Section.3.6). However, for values of $O(10^3 \text{ m}^2 \text{s}^{-1})$, as used in Naveira-Garabato et al. (2007), downwelling due to cabbeling and thermobaricity is significant. This study has not considered cases where $\nabla \gamma K$ is a significant term in (3.1). Although this term is not necessarily negligible in the Southern Ocean, sufficient observations of $K$ do not exist to reasonably quantify it.

Although there is mounting evidence that there is intense vertical mixing along the ACC in the Southern Ocean, global climate simulations are yet to include spatially and vertically varying diffusivities. This study presents further evidence that in the Southern Ocean $D$ is indeed of $O(10^{-4} \text{ m}^2 \text{s}^{-1})$ or greater and that a significant fraction of upwelling UCDW is transported across isopycnals below the mixed layer. This study also suggests that the along-isopycnal mixing coefficient of
$O(10^2 \text{ m}^2 \text{ s}^{-1})$, used in the majority of such simulations, are appropriate. Isopycnal mixing of $O(10^3-10^4 \text{ m}^2 \text{ s}^{-1})$ as measured at the surface (Karsten and Marshall, 2002; Sallée et al., 2008) are inappropriate for deep layers as they would give extremely large values for the southward transport of UCDW (Fig.3.6).

Along-isopycnal and vertical mixing coefficients inferred from observations are often difficult to compare with those required by coarse-resolution ocean models. Observational studies using moorings, Lagrangian tracers and satellite altimetry diagnose an effective diffusivity while numerical models require a diffusivity to represent the velocity to property correlations (i.e. $\overline{\mathbf{v}'\theta'}$) not accounted for by the temporal and spatial resolution of the model. Numerical models also require a diffusivity to remain numerically stable. The conservation equations used to derive (3.1) involve a vertical and along-isopycnal diffusivity, which represent the long term effect of temporal correlations between perturbations of the mean velocity and tracer. It is expected that the major contribution to $K$ is due to mesoscale eddies and hence the $K$ discussed in this paper is equivalent to that desired in a coarse-resolution ocean model. In continuing work we have developed an inverse technique that diagnoses the down-gradient transport and the mixing coefficients $K$ and $D$ (Zika et al., 2010). This is done by using the integral of (3.1) along a tracer contour and a form of the thermal wind equation cast in terms of differences in geostrophic streamfunction $\Psi^7$ along such a contour. The technique is validated against the output of a 20 year average of a 100 year climate simulation of the Hallberg Isopycnal Model at $1^\circ \times 1^\circ$ resolution. In so doing we show that the $K$ and $D$ used in (3.1) is close to that explicitly applied to a coarse-resolution ocean model.

**Acknowledgments.** We thank Drs Jean-Baptiste Sallée and Steve Rintoul for insightful and helpful comments on a draft of this paper. We also thank two anonymous reviewers for their helpful remarks. This work contributes to the CSIRO Climate Change Research Program and has been partially supported by the CSIRO.
Wealth from Oceans Flagship and the Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems.
Chapter 4

A Tracer-Contour Inverse Method for Estimating Ocean Circulation and Mixing

In press as:
4.1 Abstract

A method is developed for estimating the along-isopycnal and vertical mixing coefficients ($K$ and $D$) and the absolute velocity from time-averaged hydrographic data. The method focuses directly on transports down tracer gradients on isopycnals. When the tracer considered is salinity or an appropriate variable for heat, this down-gradient transport constitutes the along-isopycnal component of the thermohaline overturning circulation. In the method, a geostrophic streamfunction is defined that is related on isopycnals by tracer-contours and by the thermal wind relationship in the vertical. Volume and tracer conservation constraints are also included. The method is overdetermined and avoids much of the signal to noise error associated with differentiating hydrographic data in conventional inverse methods. The method is validated against output of a layered model. It is shown to resolve both $K$ and $D$, the down-gradient isopycnal transport and the mean flow on isopycnals in the North Pacific and the South Atlantic.

Importantly, an understanding is established of both the physics underlying the method and the circumstances necessary for an inverse method to determine the mixing rates and the absolute velocity. If mixing is neglected the method is the Bernoulli inverse method. At the limit of zero weight on the tracer-contour equations the method is a conventional box inverse method. Comparisons are drawn between each method and their relative merits are discussed. A new closed expression for the absolute velocity is also presented.
4.2 Introduction

How abyssal water masses are returned to shallower regions of the ocean is not fully understood. The two mechanisms identified as possible sources of this ‘pull’ of water back toward the ocean surface are (i) diapycnal upwelling due to vertical mixing; either throughout the deep ocean or at particular hot spots around the globe and (ii) wind driven upwelling along sloping isopycnals in the Southern Ocean. Both are elements of ocean dynamics for which our observations are sparse and inconclusive (see Kuhlbrodt et al., 2007, for a comprehensive review). A detailed global understanding of the diffusive and advective processes that contribute to the modification of water masses and the overturning circulation is needed.

The most established method for diagnosing the general circulation, from hydrographic data, is the box inverse method (Wunsch, 1978). Although it is able to accurately resolve major components of the circulation, box inversions of hydrographic data have seldom incorporated vertical mixing and never incorporate along-isopycnal mixing coefficients as unknowns. The few that do have difficulty resolving such values for areas of the Southern Ocean where isopycnals outcrop either because of the added uncertainty of air-sea fluxes or because interior and mixed layer processes cannot be separated (Ganachaud et al., 2000; Sloyan and Rintoul, 2000). The Bernoulli or streamfunction inverse method of Killworth (1986) and the beta-spiral method of Stommel and Schott (1977) rely on the isopycnal gradient of potential vorticity ($PV$), or other tracers, changing direction in the vertical and, in their classical form, assume the flow is non-diffusive. Although these methods are applied in different ways, it can be shown that they are based on the same physical principles, particularly the beta-spiral and box inverse methods (Davis, 1978). The tracer-contour inverse method presented here draws on aspects of each of the box, Bernoulli and beta-spiral inverse methods, but is designed so that mixing processes are of leading order importance.

In this article we establish an inverse method that allows us, to accurately
quantify both the magnitude and distribution of turbulent vertical mixing, $D$, along-isopycnal mixing, $K$, and the geostrophic flow, with particular emphasis on the velocity component down the tracer-gradient (usually $\Theta$ and $S$ on isopycnals).

This article is structured as follows: Section 4.3 establishes the concept of a tracer-contour and presents the central ‘tracer-contour equations’, which follow from simple combinations of conservation statements. Results from the application of the tracer-contour equations to output of the Hallberg Isopycnal Model (HIM; Hallberg, 2000) in a region of both the North Pacific and the South Atlantic are given in Section 4.4. We discuss how the tracer-contour equations may be used independently in Section 4.4.2 and the results are compared with a box inverse method in Section 4.4.3. The two approaches are combined to form the tracer-contour inverse method in 4.4.3.

Comparison with the Bernoulli method is made in Section 4.4.4. The effect of random error for different relative weights on the box and tracer-contour equations is discussed in Section 4.5. Criteria needed to diagnose the mixing coefficients and theoretical implications are discussed in Section 4.6, including a new closed expression for the absolute velocity. Concluding remarks are made in Section 4.7.

The tracer-contour inverse method is formulated using neutral density, $\gamma^n$, as a vertical coordinate and conservative temperature, $\Theta$, as a variable for ‘heat’. Here neutral density is considered as it is along neutral tangent planes that parcels of water may be exchanged adiabatically, without restoring buoyancy forces (McDougall, 1987a). Conservative temperature, $\Theta$, is used as it is the most conservative temperature variable available. Conservative temperature is proportional to potential enthalpy, which represents the ‘heat content’ per unit mass of seawater (McDougall, 2003).

Where this study discusses output of the Hallberg Isopycnal Model, HIM, potential density, $\sigma_2$ (i.e. potential density referenced to a depth of 2000m) and potential temperature, $\theta$, are used. In HIM, unlike in the real ocean, properties are exchanged adiabatically along $\sigma_2$ surfaces and potential temperature is a per-
fectly conserved quantity. Note however, that the distinction between conservative temperature and potential temperature, and neutral and potential density is not central to this study. We will frequently refer to $\Theta$ or $\theta$ as temperature, and neutral or potential density surfaces as isopycnals.

4.3 Tracer-contours

Consider the idealized scenario where no mixing or diffusive processes occur in the ocean. In such an ocean the path taken by a parcel of water of a given tracer concentration, $C$ and neutral density, is simply the path where the $C$ and $\gamma^h$ are constant (ignoring subtle effects on $\gamma^h$ due to the changes in pressure and longitude and latitude along the path). Thus currents in such an ocean would closely follow contours of constant tracer on isopycnals. It is clear then that the amount by which trajectories do not follow such a path, that is, the amount of flow across $C$ contours on isopycnals and vertically through isopycnals, is determined purely by the magnitude of vertical and along-isopycnal mixing processes. Diagnosing the magnitude of such processes is thus best undertaken in a $\gamma^h$ vs. $C$ reference frame.

The tracer-contour inverse method, as presented here, considers contours of constant temperature and salinity on isopycnals, as these are the most well observed tracers in the ocean. Both temperature and salinity form the same contours on isopycnals. They are effectively passive tracers on isopycnals as mixing them along neutral tangent planes does not affect the stratification. The isopycnal flow across these contours (the cross-contour flow) is the along-isopycnal component of the thermohaline overturning circulation. In principle the method can be generalized to consider contours of any conservative tracer on isopycnals (Appendix C.3).

Here, $h$ is the temporally averaged ‘thickness’ of a density layer ($h = \Delta\rho/\rho_z$, for some small $\Delta\rho$). In principle the thickness, $h$, can be infinitely small. The tracers $C$ (e.g. $\Theta$ and $S$) are thickness weighted temporally averaged quantities.
The thickness weighted velocity is written \( \overline{v h}/h \), where \( v \) is the lateral velocity vector. The overbar denotes the *temporal average*. We will later separate the thickness weighted velocity into an isopycnal-mean component, \( \mathbf{v} \) (not thickness weighted), and a perturbation component, \( \overline{\mathbf{v} h}/h \) (see 3.5).

The thickness weighted velocity component, directed down the along-isopycnal temperature and salinity gradient is \( (\overline{v h}/h) \cdot \mathbf{n} \). The cross contour direction is defined by \( \mathbf{n} = \nabla \gamma \Theta /|\nabla \gamma \Theta| \) where \( \nabla \gamma \) is the epineutral gradient.

Along-isopycnal and vertical mixing act down-gradient with a coefficient \( K \) and \( D \) respectively. The equation describing the balance between this flow and mixing process in steady state and assuming no along-isopycnal gradient of \( K \) is

\[
(\overline{v h}/h) \cdot \mathbf{n} = K \left( \frac{\nabla \gamma \cdot (h \nabla \gamma S) \Theta_z - \nabla \gamma \cdot (h \nabla \gamma \Theta) S_z}{h(|\nabla \gamma S| \Theta_z - |\nabla \gamma \Theta| S_z)} \right) + D \left( \frac{S_{zz} \Theta_z - \Theta_{zz} S_z}{|\nabla \gamma S| \Theta_z - |\nabla \gamma \Theta| S_z} \right) + \epsilon_\perp
\]

\[
= K/\lambda_\perp + D/\lambda^\gamma + \epsilon_\perp
\]  

(4.1)

Equation (4.1) was first derived by McDougall (1984) in a slightly different form and described as the ‘Water Mass Transformation’ equation. In the current form, (4.1) was used by Zika et al. (2009) to relate the strength of the Southern Ocean Overturning circulation to the diffusivities \( K \) and \( D \).

Equation (4.1) represents a balance between down-gradient advection and diffusion through the along-isopycnal and vertical mixing lengths \( \lambda_\perp \) and \( \lambda^\gamma \), respectively. See Appendix C.1 for a full derivation. Along-isopycnal mixing enters largely through the along-isopycnal curvature of either temperature or salinity. More precisely, in the case of a linear equation of state and no thickness gradient (i.e. if the thermal expansion and haline contraction coefficients \( \alpha \) and \( \beta \) are constant and \( \nabla h = 0 \)) the along-isopycnal mixing length simplifies so that

\[
1/\lambda_{\mathrm{linear}}^\perp = \nabla^2_\gamma \Theta /|\nabla \gamma \Theta|.
\]

Both linear and nonlinear terms are retained here.

Notably, (4.1) does not involve the diapycnal velocity component \( w^\gamma \), vertical differences in \( D \) nor second derivatives of tracers in \( z \). Instead vertical diffusion appears through the \( \Theta - S \) curvature (noting that \( S_z^3 \frac{d^2 \Theta}{dz^2} = S_z \Theta_{zz} - \Theta_z S_{zz} \)). The \( \Theta - S \) curvature is less sensitive to noise in hydrographic data due to heave,
than $S_{zz}$ and $\Theta_{zz}$. Assuming a linear equation of state, the $\Theta - S$ curvature is proportional to the curvature of $\Theta$ or $S$ with respect to $\rho$.

In Appendix C.3 we derive a general from of (4.1) for advection of the tracer $C$ (C.21). Again, assuming a linear equation of state, vertical mixing influences (C.21) through the curvature of $C$ with respect to $\rho$.

The error term, $\epsilon_{\perp}$, represents the effect of an unsteady or ‘trend’ term, the effect of the gradient of the along-isopycnal mixing coefficient and any other form of error due to measurement or sampling ($\epsilon_{noise}$). That is

$$\epsilon_{\perp} = \nabla_\gamma K \cdot n - \Theta_t|_\gamma / |\nabla_\gamma \Theta| + \epsilon_{noise}. \tag{4.2}$$

In (4.2), $\Theta_t|_\gamma$ represents the trend in temperature at constant density and not the variability that is encompassed in the eddy coefficient $K$. It is straightforward to retain the $\nabla_\gamma K$ term in (4.1) if along-isopycnal variations in $K$ are to be resolved.

We split the thickness weighted velocity into an isopycnal-mean and perturbation component

$$\overline{\nabla h}/h = \overline{v} + \overline{\nabla h'}/h. \tag{4.3}$$

It is understood that to a good approximation, the mean velocity (not thickness weighted) on an isopycnal satisfies geostrophy and that the mean velocity can be represented by a geostrophic streamfunction $\Psi^\gamma$ (McDougall, 1988) such that

$$\overline{v} = -\frac{1}{f} \nabla_\gamma \Psi^\gamma \times k. \tag{4.4}$$

In (4.4), $f$ is the Coriolis frequency and $k$ is the vertical unit vector normal to geopotential surfaces. The perturbation component of (4.3) may be parameterized as a down-gradient flux of thickness (or bolus flux) such that

$$\overline{\nabla h'}/h \equiv v^* \equiv -K_{PV} \nabla_\gamma \log(h), \tag{4.5}$$

where $v^*$ is the bolus velocity and $K_{PV}$ is the epineutral thickness or potential vorticity diffusion coefficient. Following McDougall (1991a), a beta term, $K\beta/f$, where $\beta$ is the meridional gradient of $f$, may be included on the right hand side.
of (4.5). However, as HIM has only a thickness diffusion term, the beta term is omitted for the time being. Another way of writing such a parameterization is in terms of the vertical density gradient as in Gent et al. (1995).

The component of the mean geostrophic velocity that crosses temperature and salinity contours on isopycnals is the cross-contour velocity

$$\mathbf{v} \cdot \mathbf{n} = \frac{\nabla \gamma \Psi \times \mathbf{k} \cdot \mathbf{n}}{h}. \quad (4.6)$$

Substituting, (4.1) and (4.5) into (4.6) and defining a third mixing length for thickness $\lambda^h$, where $1/\lambda^h = \nabla \gamma \log(h) \cdot \mathbf{n}$, the cross-contour geostrophic velocity is then purely related to the mixing coefficients $K$, $D$, and $K_{PV}$

$$\mathbf{v} \cdot \mathbf{n} = K/\lambda^\perp + D/\lambda^\gamma + K_{PV}/\lambda^h + \epsilon_{\perp}. \quad (4.7)$$

[See Appendix C.1 for the complete form of (4.7)]. Multiplying (4.7) by $f$, yields the cross-contour velocity, in terms of both along-isopycnal and vertical mixing coefficients and the gradient of the geostrophic streamfunction

$$-(\nabla \gamma \Psi \times \mathbf{k}) \cdot \mathbf{n} = K f (1/\lambda^h + 1/\lambda^\perp) + D f/\lambda^\gamma + f\epsilon_{\perp}. \quad (4.8)$$

In (4.8) we have assumed that $K_{PV} = K$. However, if desired, $K_{PV}$ could be retained as a third mixing variable.

Integrating along a contour of constant temperature and salinity (or any tracer) on an isopycnal from point $(x_1, y_1)$ to $(x_2, y_2)$ and assuming the mixing coefficients $K$ and $D$ are constant along the contour we have an equation that relates the difference in streamfunction between $(x_1, y_1)$ and $(x_2, y_2)$ to the along-isopycnal and vertical mixing coefficients

$$\Psi^\gamma(x_2, y_2) - \Psi^\gamma(x_1, y_1) = \Delta \Psi^\gamma = K \int_{x_1, y_1}^{x_2, y_2} f(1/\lambda^h + 1/\lambda^\perp) dx + D \int_{x_1, y_1}^{x_2, y_2} f/\lambda^\gamma dx + f\epsilon_{\perp}. \quad (4.9)$$

Equation (4.9) states that the total geostrophic flow crossing a $\Theta$ contour on an isopycnal, between points at either end of the contour, is the integral of the reciprocal of the scale lengths, $\lambda_{\perp}$, $\lambda_h$, and $\lambda_{\gamma}$, multiplied by the mixing coefficients.
and the Coriolis frequency. Moving between points \((x_1, y_1)\) and \((x_2, y_2)\) on the isopycnal, the gradient of temperature points to the left.

The error term, \(\epsilon_{\Psi^\gamma}\), may be largely attributed to the integral of \(\epsilon_{\perp}\). \(\epsilon_{\Psi^\gamma}\) also represents the correlations along the contour between coefficients \(K\) and \(D\) and the scale lengths \(\lambda^\perp\) and \(\lambda^\gamma\) and the inadequacies of the steady geostrophic assumption implicit in using \(\Psi^\gamma\).

Both Zika and McDougall (2008) and Zika et al. (2009) exploit temperature contours on isopycnals, relating down-gradient advection to mixing. In this study, we consider the entire water column simultaneously and explicitly solve for a geostrophic streamfunction on one reference level.

Equation (4.9) may be used along with the following relationship between streamfunctions in the vertical from Zhang and Hogg (1992)

\[
\Psi^\gamma(x, y) - \Psi^{\gamma_0}(x, y) = (p - \tilde{p})\delta - (p_0 - \tilde{p}_0)\delta_0 - \int_{p_0}^{p} \delta dp' + \epsilon_c
\]

with specific volume anomaly \(\delta = 1/\rho(S, \Theta, p) - 1/\rho(S_{\text{const}}, \Theta_{\text{const}}, p)\) (conventionally \(\Theta_{\text{const}} = 0^\circ C\) and \(S_{\text{const}} = 35\)), \(p_0\) and \(\delta_0\) are the pressure and specific volume anomaly on the reference level \(\gamma_0\) and, \(\tilde{p}\) and \(\tilde{p}_0\) are the pressures averaged over isopycnals \(\gamma\) and \(\gamma_0\), respectively.

This formulation may be extended to specifically describe the difference in \(\Psi^\gamma\) along the tracer-contour on the isopycnal \(\gamma\) between points \((x_1, y_1)\) and \((x_2, y_2)\) to the difference for the same \((x, y)\) points on the reference level \(\gamma_0\) (Fig. 4.1) such that

\[
\Delta \Psi^{\gamma_0} - \Delta \Psi^\gamma = - \left[ (p - \tilde{p})\delta - (p_0 - \tilde{p}_0)\delta_0 - \int_{p_0}^{p} \delta dp' \right]_{(x_1, y_1)}^{(x_2, y_2)} = \mathcal{C}
\]

We may now write a single equation for \(\Psi^{\gamma_0}\), \(K\) and \(D\) for the tracer-contour connecting \((x_1, y_1)\) to \((x_2, y_2)\) on a particular isopycnal

\[
\Delta \Psi^{\gamma_0} - K \int_{x_1}^{x_2} f(1/\lambda^h + 1/\lambda^\perp)dx - D \int_{x_1}^{x_2} f/\lambda^\gamma dx = \mathcal{C} + \epsilon.
\]

Equation (4.12) may be written for a large number of tracer contours to solve for the reference level streamfunction and individual diffusivities (on each isopycnal
Figure 4.1: The difference in streamfunction $\Delta \Psi^\gamma$ on an isopycnal $\gamma$ may be related to the along-isopycnal and vertical mixing coefficients $K$ and $D$ using (4.9). Just as thermal wind relates velocities in the vertical, the difference $\Delta \Psi^\gamma$ may be related to $\Delta \Psi^{\gamma_0}$ on the reference level $\gamma_0$ using (4.11). Incorporating many contours on many isopycnals into the inversion makes the system overdetermined and allows $K$, $D$ and $\Psi^{\gamma_0}(x, y)$ to be estimated or some given spatial function). There are likely benefits in referencing to a fixed pressure and these are detailed in Appendix C.4. This study retains an isopycnal as the reference level so that the method can be compared to the box inverse method and allows a reference level minimizations to be imposed. McDougall and Klocker (Submitted) discuss different streamfunctions considered on different isopycnals and their corresponding errors. Here we consider isopycnals where the variation in pressure on each isopycnal is small and hence difference between the methods has a negligible effect on the result.
The system of equations is represented as

\[ \mathbf{Ax} = \mathbf{b} + \epsilon \]  

(4.13)

The matrix \( \mathbf{A} \) is made up of the coefficients on the left hand side of (4.12) and \( \mathbf{b} \) is simply each \( \ell \) value for each tracer-contour. The vector of unknowns, \( \mathbf{x} \), is

\[ \mathbf{x} = [K_1, \ldots, K_N, D_1, \ldots, D_N, \Psi_{\gamma_0}^1, \ldots, \Psi_{\gamma_0}^L] \]  

(4.14)

In the case of \( L \) boundary points, with \( N \) isopycnals, a solution is sought for \( \Psi_{\gamma_0} \) at each \( L \) and for \( K \) and \( D \) on each \( N \). The error \( \epsilon \) is minimized using an Singular Value Decomposition solver.

It is likely that for \( L \) streamfunction points, the effective number of tracer-contours which add orthogonal information is \( L/2 \) (see Appendix C.5 for a complete discussion). Thus the approximate number of effective equations is \( N \times L/2 \), resulting in an overdetermined system when \( N \times L/2 > 2N + L \).

### 4.4 Application of the tracer-contour equations

The inverse method is applied to two regions of the ocean using output of HIM. This allows direct comparisons between the inverse method solution and the HIM diffusivities and transports. The inverse method is applied in several ways: using only the tracer-contour equations (4.12); using box inverse method equations with a reference level minimization and; using a combination of both box and tracer-contour equations without any reference level minimization. The final case forms the tracer-contour inverse method. A discussion of row and column weighting and the effect of random error on the solution is given in Section 4.5.
4.4.1 The Hallberg Isopycnal Model and Study Regions

There are two main advantages of using HIM as a means of testing the accuracy of the method: First, by utilizing a density coordinate system HIM avoids the unwanted mixing effects that can arise from the Veronis effect in $z$-coordinate GCMs (Veronis, 1975). Second, as HIM is already in an isopycnal coordinate, we avoid any error implicit in interpolating on to isopycnals from a Cartesian grid of $\Theta$, $S$ and $p$ values. The HIM output used is the time average of the final 40 years of a 100 year coupled experiment at $1^0 \times 1^0$ resolution. The model has a fixed diffusivity for thickness and tracer along isopycnals ($K$) of $600 m^2 s^{-1}$. The model employs a Richardson number dependent vertical diffusion scheme, which results in high vertical diffusivities $[O(10^{-4} m^2 s^{-1})]$ in the mixed and bottom boundary layers and relaxes to a background profile in the ocean interior. Trends in temperature, salinity and thickness, spurious or otherwise, are corrected such that the steady state conservation and geostrophic equations hold exactly.

The inverse method is applied to two regions, both with relatively uncomplicated topography that is everywhere deeper than 2000m. The first is in the North East Pacific between 210$^\circ$E -230$^\circ$E and 10$^\circ$N -30$^\circ$N. This region is characterized by a low velocity at depth (i.e. $|v| << 1 cm s^{-1}$ at around 2000m). The second region is in the South West Atlantic between 310$^\circ$E -330$^\circ$E and 48$^\circ$S -36$^\circ$S where the ACC steers northward and interacts with the Malvinas (Falkland) Current in the Scotia Sea. It can be characterized by strong depth integrated velocity (i.e. $|v| > 1 cm s^{-1}$ at around 2000m). No attempt will be made to compare the model simulation to observations. For simplicity the experiments presented here do not involve tracer-contours which reach the ocean surface or topography, that is, the method is applied to regions distant from continental boundaries, and above the sea floor. In principle tracer-contours can be defined for such regions.
4.4.2 Contours Only

To focus on diffusive processes in the ocean interior we define tracer-contours on isopycnals between 500m and two model grid points above any topographic feature (around 2600m), the second lowest complete isopycnal being taken as the reference level (in the case of the North Pacific region this isopycnal has the lowest RMS velocity). For the North Pacific region there are 56 boundary grid points \((L = 56)\) and 15 isopycnals \((N = 15)\). For the South Atlantic \(L = 31\) and \(N = 15\).

The geostrophic component of the mean velocity crossing \(\Theta\) contours on isopycnals, \(\mathbf{v} \cdot \mathbf{n}\), is \(-\nabla_{\gamma}\,\Psi \times \mathbf{k} \cdot \mathbf{n}/f\). So, the average value of \(\mathbf{v} \cdot \mathbf{n}\), along the tracer-contour, is approximately \(-\Delta \Psi / f \Delta x\), where \(f\) is the average \(f\) along the contour and \(\Delta x\) is the length of the contour. An important aspect of the tracer-contour method is its ability to determine this down-gradient advection. It is instructive to present all the terms in (4.12), divided by \(f \Delta x\), in \(\Theta - S\) space, so that the production of individual water masses can be identified since each contour has a unique temperature and salinity and thus represents a specific location on such a diagram (Figs.4.2 and 4.3; b, c and d). In the North Pacific region the strongest cross-contour flow (with respect to the reference level) occurs in the lightest, freshest waters, where the subtropical gyre is oriented almost directly southward bringing cool North Pacific waters into the equatorial zone. This pattern is evident from the terms on the right hand side of (4.12) which are largest in the lightest waters. This suggests that a weak reference level is a reasonable approximation in this region. In the South Atlantic region, the terms on the right hand side of (4.12) are large throughout the whole water column. The patterns in Figs. 4.3b and 4.3c do not coincide with Fig.4.3d, suggesting a weak reference level velocity is not a good approximation there.

In the possible case that the coefficients \(1/\lambda_{\perp} + 1/\lambda_h\) and \(1/\lambda_{\gamma}\) are linearly dependent on each isopycnal, \(K\) and \(D\) cannot be solved for independently and other information must be included, for example the ratio of \(K\) to \(D\) may be set. It is instructive then to plot the mixing terms on the left hand side of (4.12) against
Figure 4.2: a: Start points (green crosses) and end points (red circles) of tracer-contours on the deepest and shallowest isopycnals defined in North Pacific region, joined by blue lines. The reference level is shown with an open gray mesh and the sea floor with a filled gray mesh. b, c and d: Terms in (4.12), scaled by the length, $\Delta x$, and the average Coriolis frequency of the contour, $\tilde{f}$, to cast each term in units of velocity. b: $K_0 \int_{x_1}^{x_2} f(1/\lambda^h + 1/\lambda^\perp)dx/(\Delta x \tilde{f})$ using $K_0 = 500m^2s^{-1}$, c: $D_0 \int_{x_1}^{x_2} (f/\lambda^h)dx/(\Delta x \tilde{f})$ using $D_0 = 2 \times 10^{-5}m^2s^{-1}$ and d: $\mathcal{C}/(\Delta x \tilde{f})$, which is zero on the reference level at $\sigma_2 = 36.49$. 

96
Figure 4.3: Same as Fig. 4.2 but for South Atlantic with the reference level at $\sigma_2 = 36.95$ and using $D_0 = 1 \times 10^{-4} m^2 s^{-1}$. 
That is, \( K_0 \int_{x_1}^{x_2} f(1/\lambda^b + 1/\lambda^\perp) dx/(\Delta x \hat{f}) \) vs. \( D_0 \int_{x_1}^{x_2} (f/\lambda^\gamma) dx/(\Delta x \hat{f}) \) using \( K_0 = 500 \text{m}^2\text{s}^{-1} \) and \( D_0 = 2 \times 10^{-5} \text{m}^2\text{s}^{-1} \). Gray shading represents the potential density, \( \sigma_2 \), of the tracer-contours.

Figure 4.4: Scatter plot of mixing terms in (4.12) for North Pacific region. One can see that \( K_0 \) and \( D_0 \) are equivalent to typical \( K \) and \( D \) values. These plots show that in the two regions considered here the coefficients are not linearly dependent (Figs. 4.4 and 4.5).

In the case of the North Pacific the coefficients vary smoothly with respect to one another while they are much more scattered in the South Atlantic region. For the lightest isopycnals in the South Atlantic region, the coefficient of \( D \) (i.e. \( 1/\lambda^\gamma \)) is close to zero because the \( \Theta - S \) curvature is small relative to the curvature along the isopycnal (Fig. 4.5). The \( D \) diagnosed on these isopycnals for this region is especially sensitive when random error is added to \( b \), in the system \( Ax = b \) (Section 4.5.2).
We define a streamfunction variable at each point around the boundary of the domain on the reference level with the reference level velocity \( v_{\text{ref}} \) \( (v_{\text{ref}} = \nabla \gamma \Psi_0 \times \mathbf{k} / f) \). The streamfunction at the ends of tracer-contours that are not at grid points are linearly interpolated to the closest boundary grid points. A simple matrix inversion is applied to solve for the reference level streamfunction on the reference level and the mixing coefficients \( K \) and \( D \) on each isopycnal and the added constraint that the mean value of \( \Psi_0 \) be minimized. This does not minimize the reference level velocity but helps the inversion find a solution for \( \Psi_0 \).

The resulting solution for the mixing coefficients is shown in Figs. 4.6 and 4.7 (dotted line) and is in agreement with the HIM values (grey solid line). The component of the reference level velocity directed into the domain, \( v_{\text{ref}} \), for \( v_{\text{ref}} = v_{\text{ref}} \cdot \mathbf{m} \), where \( \mathbf{m} \) is the unit vector perpendicular to the domain boundaries, is also shown (Figs. 4.8 and 4.9; dotted line). The reference level velocity is the same in HIM (grey line in Figs. 4.8 and 4.9) to within interpolation error and the
4.4.3 The thickness-weighted mean box inverse method

The principal equations employed in a box inverse method are thermal wind, and conservation of volume, temperature and salinity. These are commonly represented in a ‘layered’ framework (neutral or otherwise), assuming steady state and allowing for vertical diffusion, $D$, and vertical advection, $w^*$, or the net diapycnal flux, $F_T$, of property $T$. Such methods have been shown to accurately describe the circulation when the row and column weights of the underdetermined problem and the reference level are chosen appropriately (McIntosh and Rintoul, 1997). This was done by taking a snapshot of an eddy-permitting ocean model. Model drift or ‘storage’ terms were corrected for in McIntosh and Rintoul (1997) as is done here. We formulate the box equations such that the full thickness-weighted mean isopycnal flow is considered. That is, we include the sum of the mean velocity, $v$, plus the bolus velocity, $v^*$ [parameterized as in (4.5)].

Sloyan and Rintoul (2000) showed that while net vertical transports may be resolved accurately, diagnosing a diapycnal mixing coefficient is problematic. This is perhaps due to correlations between the vertical velocity and tracer values on the isopycnal faces $w'\Theta'$. In McIntosh and Rintoul (1997) and Sloyan and Rintoul (2000), snapshots of numerical models were used in which case the advection rate of the isopycnals in the vertical may have contributed as an error term. The inferred vertical velocity from the inversion, $w^*$, has both a diapycnal component $w^\gamma$ (i.e. through the isopycnals) and a component due the movement of isopycnals $\partial p/\partial t|_{\gamma}$. If a mixing coefficient is inferred from the combined velocity $w^*$, an error results. As the HIM data used here are 20 year means and the steady advective-diffusive equations are constrained to hold exactly, no such error may appear.

We reformulate the box inverse equations somewhat, to describe a statistically steady state assuming the use of temporally averaged hydrographic data (either from an atlas or repeat hydrographic sections). The equations representing the
conservation of volume and the tracer $C$ are

$$\oint \mathbf{v} \cdot m dl - \oint K \nabla \gamma h \cdot m dl + \left[ \int w^{\gamma} dA \right]_l = 0$$

(4.15)

$$\oint \mathbf{v} \cdot mhC dl - \oint K \nabla \gamma h \cdot mC dl + \left[ \int w^{\gamma} C dA \right]_l = \oint K \nabla \gamma C \cdot mh dl + \left[ \int DC_z dA \right]_l.$$  

(4.16)

Here $\int ...dA$ is the integral over the isopycnal interface, again $m$ is directed out of the box and $\oint ...hdl$ is the integral over the vertical face of the perimeter of the density layer with thickness $h$. The conservation equation, (4.16), is written for the general tracer $C$. Here we will use anomalies to the tracers $\Theta$ and $S$ such that $C = \Theta - \tilde{\Theta}$ and $C = S - \tilde{S}$. Where the tilde represents the mean over the layer.

In 4.15 we are considering a layer of finite thickness, $h$, between isopycnals. To be precise the layers defined in (4.15) and (4.16) are finite volumes ‘about’ the isopycnals on which tracer contours are defined in (4.9). That is, the tracer contours are written in advective form, while the box model equations are in divergence form. In practice however, $h$ is the same in (4.1) as in (4.15) and (4.16). That is the ‘layers’ considered in this section are essentially the same as the ‘isopycnals’ considered in Section 4.4.2 and we will continue to describe them as isopycnals.

Following Zika et al. (2009), we relate the diapycnal velocity to the vertical and along-isopycnal mixing coefficients by

$$w^{\gamma} = \frac{K}{\eta^\perp} + \frac{D}{\eta^\gamma} + D_z$$

(4.17)

where

$$1/\eta^\perp = \frac{\nabla^2 \Theta |\nabla \gamma S| - \nabla^2 S |\nabla \gamma \Theta|}{|\nabla \gamma S| \Theta_z - |\nabla \gamma \Theta| S_z}$$

and

$$1/\eta^\gamma = \frac{\Theta_{zz} |\nabla \gamma S| - S_{zz} |\nabla \gamma \Theta|}{|\nabla \gamma S| \Theta_z - |\nabla \gamma \Theta| S_z}.$$  

(4.18)

If a linear equation of state is assumed the along-isopycnal diffusion term, $1/\eta^\perp$, is zero and neglecting vertical variations in $D$, (4.17) reduces to $w^{\gamma} = D \gamma_{zz}/\gamma_z$ as
commonly used in studies attempting to infer large scale dynamics from observations (Karsten and Marshall, 2002; Naveira-Garabato et al., 2007). Here we shall retain both the linear and nonlinear components and the vertical gradient of $D$ [See Appendix C.2 for a full derivation of (4.17)].

As in Section 4.4.2 we solve for a geostrophic streamfunction on the reference level and a single value for $K$ and $D$ on each isopycnal including one $D$ value above and below the top and bottom isopycnals, respectively. This is so that $D_{zz}$ may be represented on each isopycnal where the box equations are applied. Later, when tracer-contour and box equations are combined, tracer-contours are defined on these upper and lower isopycnals as $D_z$ and $D_{zz}$ do not enter the tracer-contour equations (4.12).

As the system is underdetermined an infinite number of solutions exist. To find a unique solution, we impose a weak ‘level of no motion’ constraint. This is done by adding a set of equations that define the reference level velocity to be zero and impose a very small weight on these equations such that a large error is tolerated on them relative to (4.15 and 4.16). In practice the relative weighting used is $10^{-5}$ [i.e. when all the terms in $A$ and $x$ are weighted to be $O(1)$ the reference level minimization equations are weighted to be $O(10^{-5})$]. Changing the relative weight by plus or minus an order of magnitude does not change the resulting solution (not shown).

In the North Pacific region the reference level is chosen as $\sigma_2 = 36.89 kg m^{-3}$ at a depth of around 2000m. Given that the reference level velocity in this region is very small (order $10^{-5} m s^{-1}$) the box inverse method gives a good representation of the vertical and along-isopycnal mixing coefficients (Fig.4.6), diverging in the case of $K$ by $\pm 10\%$ or so, on the shallowest isopycnals (Fig.4.6). The box inverse method reveals some information about the reference level velocity (Fig.4.8).

In the South Atlantic region the reference level is $\sigma_2 = 36.96 kg m^{-3}$ at a depth of around 2600m. In this region depth integrated velocities are large. Using the box inverse method, the reference level velocity and mixing coefficients are much less
accurately described by the box model (Fig. 4.7). The solution would be improved, given a more appropriate a priori value for the reference level velocity.

It is likely that if more isopycnals were included in the box method the solution for the reference level velocity would improve. McIntosh and Rintoul (1997) were able to resolve the reference level velocity using approximately 25 levels. They showed that given a sufficient number of levels, no equation error and known or zero diapycnal fluxes, in principle, a box inversion can become overdetermined and an exact solution can be found. However they found that increasing the number of isopycnals makes the solution far more sensitive to errors.

Here, boxes are bound by ‘sections’, which start and end within an ocean basin, away from continental boundaries. Both McIntosh and Rintoul (1997), Sloyan and Rintoul (2000) and Ganachaud (2003) test the box inverse method on boxes bounded by transcontinental sections, which may constrain their solution better. Incorporating the tracer-contour equations into basin scale inversions is left to future work. Also, box inversions are seldom used as stand-alone tools without a priori oceanographic knowledge included as added constraints. Such constraints may include: the net transport across a transcontinental section, velocity from floats or moorings, a reference level minimization based on observations of water masses or an assumption of the magnitude diapycnal mixing. The box inverse method offers a practical way of including such varied information in a statistically consistent way while constraining the large-scale flow to be mass, heat and salt conserving. These are all good reasons for combining the box and tracer-contour
Figure 4.6: Mixing coefficient solutions for the North Pacific region using: (i) the tracer-contour equations (dotted), (ii) only the box equations and a reference level minimization (dashed) and (iii) combining tracer-contours and box equations with no reference level minimization (solid). The implicit coefficient in HIM is also shown (grey). a: Vertical mixing coefficient $D$. b: Along-isopycnal mixing coefficient $K$. For $D$ all the inverse solutions are virtually identical, while for $K$ only the box inversion solution is visibly different from the HIM value of 600 m$^3$ s$^{-1}$.
Figure 4.7: Same as Fig.4.6, but for South Atlantic region. For both $D$ and $K$ all the inverse solutions overlay the model value except for the box inversion (dashed line)
Figure 4.8: Solution for reference level velocity component directed into the region ($\mathbf{v}_{ref} \cdot \mathbf{m}$) on the reference level ($\sigma_2 = 36.9 \text{kgm}^{-3}$) for the North Pacific region, using: (i) only the tracer-contour equations (dotted), (ii) only the box equations (dashed), (iii) combining tracer-contours and boxes (solid), and (iv) assuming the flow is non-diffusive (Bernoulli inversion; dot-dashed). The velocity in HIM is also shown (grey). Positive values are directed into the region.

Figure 4.9: Same as Fig.4.8, but for South Atlantic region.
equations in a flexible manner.

Given that both the box equations and tracer-contour equations are written in terms of the same unknowns it is trivial to combine the two methods. Solving the full set of equations (without a reference level minimization) the solution is again well defined to within interpolation error (Figs. 4.6 and 4.7). When the system is overdetermined, the reference level minimization is not needed.

4.4.4 The Bernoulli streamfunction inverse method

First proposed by Killworth (1986) the Bernoulli inverse method involves following contours of constant potential vorticity, $PV$, on isopycnals. In the method it is assumed that such a contour is a streamline of the geostrophic flow and a reference level streamfunction is diagnosed using thermal wind. In a more recent application of this method, Cunningham (2000) assumes that streamlines follow contours of constant temperature and salinity. The method is then identical to using the tracer-contour equations for the case where $K$ and $D$ are assumed to be zero. If the assumption of zero along-isopycnal and vertical mixing is correct, the Bernoulli method should resolve the reference level velocity with the same accuracy as the tracer-contour equations (Section 4.4.2). It is clear however that the method will give no information about the mixing coefficients, $K$ and $D$, nor about the down-gradient fluxes of tracers, such as temperature and salinity, as these are explicitly minimized in the Bernoulli inverse method.

Given output of HIM, we apply a form of the Bernoulli method such that [from (4.11)]

$$
\Delta \Psi^{70} = \left[ (p - \tilde{p}) \delta - (p_0 - \tilde{p}_0) \delta_0 - \int_{p_0}^{p} \delta dp \right]^{(x_2,y_2)}_{(x_1,y_1)} + \epsilon. \quad (4.19)
$$

In (4.19) the cross-contour flow $\Delta \Psi^7$ is minimized. The only unknown is the reference level streamfunction $\Psi^{70}$.

We apply (4.19) to the North Pacific and South Atlantic regions, as described in the previous sections. The reference level velocity, $v_{ref}$, diagnosed using this
method is not in good agreement for the North Pacific region. This is because in
the North Pacific region, the velocity on the reference level, the unknown, is small
$O(0.1 \text{mm s}^{-1})$, while the cross-contour velocity on shallower isopycnals [assumed
to be zero in (4.19)] is large $O(5 \text{ mm s}^{-1})$. The Bernoulli streamfunction method
fares much better in the South Atlantic region where the reference level velocity
is large $O(1 \text{cms}^{-1})$ and the flow is largely along the path of the temperature
contours of the ACC.

The Bernoulli method may be an appropriate method for diagnosing the along
contour component of the absolute velocity $\mathbf{v} \cdot (\mathbf{n} \times \mathbf{k})$. A recent proposal by
Alderson and Killworth (2005), to use Argo data in real time, may be one such
example. However no such method can be applied to time-mean hydrographic
data to infer any information about the down-gradient transports of tracers as
these are explicitly minimized.

4.5 Effect of random error

The results presented in Section 4.4 leave open some critical questions: What is the
sensitivity of the new equations to random error and how should such equations
be weighted relative to one and other?

4.5.1 Weighting the different equations and unknowns

The condition number of a matrix is $||A||_{\infty}||A^{-1}||_{\infty}$, where $||A||_{\infty}$ is the infinity
norm of the matrix $A$. The condition number is a key indicator of a matrix’s sen-
sitivity to error (Turing, 1948; Tarantola, 1987). Following McIntosh and Rintoul
(1997) weights are chosen in the tracer-contour method such that the condition
number of the matrix $A$ is minimized. McIntosh and Rintoul (1997) adjust col-
umn weights such that the inversion is penalized for error in either the vertical
velocity or the reference velocity. As the inclusion of tracer-contours makes the
inversion overdetermined, column weights have no effect and thus only row weights
are important.

We must consider the relative weights of the 2 types of equations: box property conservation equations (4.15) and (4.16) (as well as reference level minimization) and contour equations (4.12). In practice the reference level minimization only has a significant effect when its relative weight is equivalent to the weights on the tracer-contour equations. The ratio of box to tracer-contour weight is adjusted to give a minimum condition number. Weights are found to be within an order of magnitude of the initial scaling ($K$ and $D$ coefficients are initially scaled by $10^3 \text{ m}^2 \text{s}^{-1}$ and $10^{-5} \text{ m}^2 \text{s}^{-1}$ respectively and each row is scaled by its mean value).

In the analysis of this Section, there is no reference level minimization. Inclusion of a reference level minimization, as in Section 4.4.3, makes little or no difference to the result, as the tracer-contour equations are always included and make the inversion overdetermined.

### 4.5.2 Propagation of random error

McIntosh and Rintoul (1997) find that in cases where interfacial fluxes are known or not present, adding a sufficient number of isopycnals to a box inversion can make the inversion overdetermined and an exact solution can be found. However, they find that adding random errors of $\pm 10\%$ to each equation gives a noisier solution than in the case of less isopycnals.

In order to test whether adding tracer-contours has a similar effect to adding more isopycnals to the inversion, we add 10% randomly distributed errors to $b$ ($\sigma = \pm 0.1b$). This is repeated 100 times, each time with new errors, to find the ensemble average solution and the normal distribution of the solution. We use the full tracer contour method, with box and tracer-contour equations, optimally weighted to give the lowest condition number. The solution for $D$ and $K$ varies by only 10%-20% in both the North Pacific and South Atlantic regions (Figs.4.10 and 4.11). In the North Pacific region the ensemble mean solution for the reference level is in good agreement with the HIM velocity. However, the magnitude of the
error is large, relative the magnitude of the HIM value (Fig.4.12). It should be noted however that these are small velocities \( O(0.1 \, \text{mm s}^{-1}) \). In the South Atlantic region the full inversion with 10% errors fares far better than the conventional box inverse method and the reference level velocities are well resolved (Fig.4.13).

Here we have added only random independent errors to \( b \). It is possible that the use of a gridded climatology from limited hydrographic observations, may introduce correlated errors. It is uncertain what effect such errors would have on the results shown here. The sensitivity of the method to the way data are gridded and averaged, will be the subject of future work.

The terms in (4.2) are also possible sources of correlated errors. The effect of the unsteady term, \( \Theta_t/\nabla_{\gamma} \Theta \), may be tested using a linear trend, over a 50 year period say. Preliminary work has suggested that this term is small, especially in the ocean below the thermocline, where changes on isopycnals occur very slowly. The \( \nabla_{\gamma} K \) term in (4.2) is likely to be small, distant from the boundary between different mixing regimes. That is, as long as the region of the inverse method is not either side of a strong and permanent mixing barrier, such as a strong front. The effect of a strong gradient of \( K \) could be taken into account, given knowledge of its spatial variability. For example, consider the unknown, \( K \), on a given isopycnal. One may assume that \( K \) has the same spatial variations as a lateral mixing coefficient, observed at the surface, but have a different magnitude. One could then assume \( K(\gamma, x, y) = K_1(\gamma)K_{\text{obs}}(x, y) \), where \( K_1(\gamma) \) is an unknown function of density (or depth) and \( K_{\text{obs}}(x, y) \) is an assumed function of latitude and longitude. The sensitivity of this method to such choices will be the subject of future work.

4.5.3 Value of the box equations

Using the conventional box equations it is not possible to make the inversion overdetermined when interfacial fluxes are included. We have demonstrated in Section 4.4.2 that using the tracer-contour equation (4.12) we can develop an
Figure 4.10: Mixing coefficient solutions for the tracer-contour inverse method for the North Pacific region with ±10% error (solid) and the implicit coefficient in HIM (dot dashed), shaded areas represent the standard deviation of the 100 ensemble results. a: Vertical mixing coefficient $D$. b: Along-isopycnal mixing coefficient $K$. 
Figure 4.11: As in Fig.4.10 but for the South Atlantic region.
Figure 4.12: Solution for reference level velocity component directed into the region ($\mathbf{v}_{\text{ref}} \cdot \mathbf{m}$) on the reference level ($\sigma_2 = 36.9\text{kgm}^{-3}$) for the North Pacific region ($\sigma_2 = 36.9\text{kgm}^{-3}$) and the HIM velocity (dashed) and diagnosed from the streamfunction solution of the tracer-contour inverse method (solid). Shaded areas represent the standard deviation of the 100 ensemble results. Positive values are directed into the region.

Figure 4.13: As in Fig.4.12, but for South Atlantic region
overdetermined system that solves for a geostrophic streamfunction $\Psi$ and, the along-isopycnal and vertical mixing coefficients. Here we demonstrate that there is some skill gained by combining both the box equations (4.15 and 4.16) with the tracer-contour equation (4.12) to form the tracer-contour inverse method, as is done in Section 4.4.3.

We may plot the condition number as a function of the relative row weight, i.e. the relative weight of the tracer-contour equations to box equations (Fig.4.14). For low box to contour weight (i.e. strongest influence of the tracer-contour equations) the condition number plateaus at order $10^3$. At the other extreme (strongest influence of the box equations) the condition number rises linearly. Notably near a relative contour to box weighting of $10^{-1}$ there exists a dip in the condition number, suggesting the box equations add conditionality to the system. Of course, the condition number does not necessarily correspond to the error in the inverse estimate.

In order to see what effect the relative weight of each equation type has on the error in the solution for $K$, $D$ and $v_{ref}$, we run an ensemble of inversions as in the previous Section for each relative weighting. We determine the average error for $K$, $D$ and $v_{ref}$ by subtracting the inverse result from the HIM value for each ensemble run and taking the average over the 100 inversions (Fig.4.14). The results clearly show an increase in error for greater weighting on the box inversion equations. For greater weight on the tracer-contour equations the error plateaus 1-3 orders of magnitude lower than at the other extreme. There is a slight dip in the error at close to a relative weighting of $10^{-1}$. This is especially the case for $K$. The only case where there is no change in the error with weight is for $D$ in the North Pacific region as the solution is close to perfect at both limits. The minimum error corresponds to the minimum condition number to within an order of magnitude.

The addition of box equations does improve the result of the inversion. The box equations constrain the flow to conserve volume, heat and salt on isopycnals

114
Figure 4.14: Condition number and standard deviation of error in diffusivities $K$ and $D$ and velocity $v_{\text{ref}} \cdot m$ on the reference level versus the relative box to tracer-contour weighting, $W_{\text{Box}}/W_{\text{Contour}}$ (Logarithmic scale). At each relative weighting, 100 inversions are conducted, each with 10% random error added to the each equation. There is a weak reference level minimization included in the box inverse method equations. Left (a): North Pacific region. Right (b): South Atlantic region

while the tracer-contour equation does not (although it is derived from those conservation equations). Further reasons for retaining the box equations are given in Section 4.7.

### 4.6 How does the tracer-contour method work?

This section shows how the tracer-contour method is able to solve for the mixing coefficients and the absolute velocity. It is shown that in order to solve for the
mixing coefficients $K$ and $D$ and the absolute velocity, all inverse methods require either non-zero neutral helicity ($H^n$) or that the direction of the along-isopycnal tracer gradient spirals in the vertical. By neutral helicity, we are not referring to diapycnal upwelling associated nonlinear process but simply

$$H^n = |\nabla_{\gamma} \Theta| T_b (N^2/g) \langle k \times \nabla_{\gamma} p \rangle \cdot n \quad (4.20)$$

which is a property of the mean hydrography (McDougall and Jackett, 2007), diapycnal upwelling being one consequence of non-zero $H^n$. In (4.20), $T_b$ is the thermobaric coefficient, a quantity which varies little from $2.7 \times 10^{-12}$ K$^{-1}$ Pa$^{-1}$.

### 4.6.1 Mixing, spiraling and neutral helicity

For convenience we split the lateral velocity, $v$, into a component perpendicular to tracer-contours on isopycnals, $v^\perp$ (i.e. the cross-contour velocity), and a component parallel to tracer-contours on isopycnals, $v^\parallel$ (i.e. the along-contour velocity) such that

$$v = v^\perp n + v^\parallel k \times n \quad (4.21)$$

where

$$v^\perp = \nabla \cdot n \quad \text{and} \quad v^\parallel = \nabla \cdot (k \times n). \quad (4.22)$$

Here we consider the along-isopycnal temperature and salinity gradient, $\nabla_{\gamma} \Theta$ and $\nabla_{\gamma} S$, such that $n = \nabla_{\gamma} \Theta/|\nabla_{\gamma} \Theta| \equiv \nabla_{\gamma} S/|\nabla_{\gamma} S|$. The analysis here may however be generalized for any tracer by replacing $\Theta$ with $C$ [see Appendix C.3 for a derivation of the analogous form of (4.7)]. In this section we consider $\Theta$ (which is equivalent to $S$ on an isopycnal) as a connection may be drawn to neutral helicity.

The tracer-contour method considers the component of the lateral velocity, down the along-isopycnal tracer gradient, $v^\perp$, given by (4.7). In steady state, $v^\perp$ is directly related to mixing (4.7), while $v^\parallel$ has no direct implication for property conservation or mixing. In this sense $v^\parallel$ is truly an ‘adiabatic’ velocity component.
In a steady ocean without along-isopycnal or vertical mixing, flow would exactly follow contours of constant tracer on isopycnals.

In the tracer-contour method, velocity is related in the vertical through (4.11), which, when reduced to a point (laterally and vertically) is simply thermal wind. Here we write the thermal wind equation, at a point, in terms of the along-isopycnal gradient of pressure, \( p \), such that

\[
\frac{N^2}{f \rho g} \mathbf{k} \times \nabla \gamma p = \mathbf{v}_z. \tag{4.23}
\]

As we are interested in the down-gradient component of the flow, \( v^\perp \), we consider (4.23), in the down-gradient direction, \( \mathbf{n} \), such that

\[
\frac{N^2}{f \rho g} (\mathbf{k} \times \nabla \gamma p) \cdot \mathbf{n} = \mathbf{v}_z \cdot \mathbf{n}. \tag{4.24}
\]

The right hand side of (4.24) may be expanded, using the chain-rule, into a component involving the vertical derivative of \( v^\perp \) and another involving the vertical derivative of the unit vector \( \mathbf{n} \). Hence

\[
\frac{N^2}{f \rho g} (\mathbf{k} \times \nabla \gamma p) \cdot \mathbf{n} = (\mathbf{v} \cdot \mathbf{n})_z - \mathbf{v} \cdot \mathbf{n}_z. \tag{4.25}
\]

Considering each term in (4.25), we will show that it describes a balance between neutral helicity, mixing, and spiraling of the along-isopycnal temperature gradient. We will then discuss the physical meaning of each term.

We note that \( H^n \) is proportional to the gradient of pressure along a temperature contour on an isopycnal \( (4.20) \). Neutral helicity is thus proportional to the component of thermal wind in the cross-contour direction. The left hand side of (4.25) is then

\[
\frac{N^2}{f \rho g} (\mathbf{k} \times \nabla \gamma p) \cdot \mathbf{n} = \frac{H^n}{f \rho |\nabla \gamma \Theta| T_b}. \tag{4.26}
\]

The second term on the right hand side of (4.25) represents the mean velocity, \( \mathbf{v} \), dotted with the vertical gradient of the direction of the along-isopycnal tracer gradient, \( \mathbf{n}_z \). McDougall (1995) notes that

\[
\mathbf{n}_z = \mathbf{k} \times \mathbf{n} \phi_z \tag{4.27}
\]
where $\phi$ is the direction of $\mathbf{n}$, expressed in radians and $\phi_z$ is the rate of spiraling of the isopycnal temperature gradient in the vertical. So the second term on the right hand side of (4.25) becomes

$$\mathbf{v} \cdot \mathbf{n} \_z = \mathbf{v} \cdot \mathbf{k} \times \mathbf{n} \phi_z = v^{||} \phi_z.$$  

Hence, substituting (4.26), (4.22) and (4.28) into (4.25) we have

$$ \frac{H_{n}}{f \rho | \nabla_\gamma \Theta | T_b} = \begin{pmatrix} \langle v^\bot \rangle_z - v^{||} \phi_z \end{pmatrix}. $$

Neutral Helicity

Mixing  Spiralling

(4.29)

From (4.29), we infer that in order to diagnose mixing, $v^\bot$, and the along-contour flow, $v^{||}$, the technique requires that there be some component of the thermal wind vector down the along-isopycnal temperature gradient or that there be spiraling of this gradient. In the first case, there must be a pressure change along a tracer-contour on an isopycnal and hence that neutral helicity must be non-zero. Neutral helicity and spiraling give information on the vertical derivative of the cross-contour flow, $v^\bot$. Equation (4.29) suggests that the tracer-contour inverse method will be least sensitive to error in determining $v^\bot$, and hence $K$ and $D$, in regions where neutral helicity and spiraling are strong.

### 4.6.2 The absolute velocity and the beta spiral and Bernoulli methods

Both Needler (1985) and McDougall (1995) follow a similar approach to that presented here, but in terms of potential vorticity $PV$, rather than temperature and salinity gradients. McDougall (1995) shows that the beta-spiral method can be collapsed to a point and the absolute velocity can be presented in terms of mixing and the spiraling of the along-isopycnal $PV$ gradient. The Bernoulli method, considered in terms of $PV$, requires the same spiraling and along-contour pressure gradient, as in the beta-spiral method. Using $\Theta$, both the beta-spiral and Bernoulli methods are described by equation 4.29.
Determination of the absolute velocity vector has been the motivation of a long history of work (Stommel and Schott, 1977; Needler, 1985; Killworth, 1986; McDougall, 1995). We now develop a new expression for the absolute velocity, dependent on neutral helicity, mixing and spiraling.

Dividing (4.29) by $\phi_z$ (for $\phi_z \neq 0$) and substituting into (4.21) the absolute velocity is

$$v = \left[ \frac{H_n}{\rho \gamma \Theta} + \frac{(v^\perp)_z}{\phi_z} \right] k \times n + v^\perp n. \quad (4.30)$$

As some degree of spiraling of temperature contours is ubiquitous in the global ocean, (4.30) is a closed expression for the absolute velocity. The velocity is expressed in terms of $H_n$ and mixing, $v^\perp$.

Assuming both steady-state and zero mixing (i.e. $v^\perp = 0$), the 2nd and 3rd terms in (4.30) vanish, in which case

$$v = \left[ \frac{H_n}{\rho \gamma \Theta} \right] k \times n. \quad (4.31)$$

Equation (4.31) is a closed expression for the absolute velocity in a mixing-free ocean where $\nabla \gamma \Theta$ and $\phi_z$ are non-zero; that is, where an along-isopycnal temperature gradient exists and its direction is not uniform in the vertical. The mean velocity would be solely in the along-contour direction $k \times n$ with magnitude $|\mathbf{v}| = \left| (N^2/\rho_0 g \phi_z) \nabla \gamma p \times n \right|$. Expression (4.31) is far less differentiated than those recognized by Needler (1985) or McDougall (1995).

The Bernoulli method and the beta-spiral method use thermal wind (4.23). In their classical formulation, these methods assume no mixing and that the geostrophic flow is along tracer-contours ($PV$ or $\Theta$) on isopycnals. This is imposed explicitly in the Bernoulli method and implicitly in the beta-spiral method. Then, for each method, determining the absolute velocity depends on two things: the component of thermal wind down the tracer gradient (neutral helicity) and the rate of spiraling of the tracer gradient ($\phi_z$).

The box model is a special case, in that it uses thermal wind and tracer conservation, but also has an explicit reference level minimization (i.e. the velocity at
some deep level is minimized to an a priori value. This reference level minimization sets the magnitude of $v$ throughout the water column. In this sense, even in the absence of neutral helicity, the along-contour velocity, $v^\parallel$, and the cross-contour velocity, $v^\perp$, may be determined [i.e. the right hand side of (4.29)]. With knowledge of $v$, the mixing coefficients may also be inferred, through property conservation on each isopycnal (4.15 and 4.16).

The dynamical importance of neutral helicity has been associated with the nonlinearity of the equation of state (McDougall and Jackett, 1988, 2007), and has implications for diapycnal upwelling (McDougall and Jackett, 1988; Klocker and McDougall, 2009a). Here we have shown its critical role in determining the mean circulation in the ocean (4.29). A connection between these two findings has not yet been identified and is left to future work.

### 4.7 Discussion and Ongoing Work

An inverse method has been presented for diagnosing rates of along-isopycnal and vertical mixing and the mean geostrophic velocity, from time-averaged hydrographic data. The method involves integrating the ‘curvatures’ of hydrographic data along tracer-contours on isopycnals. These curvatures form the coefficients of the diffusivities $D$ and $K$, and the sum of these are related to differences in a geostrophic streamfunction along tracer-contours. The streamfunction on that isopycnal is then related to a reference level streamfunction, on either a reference isopycnal or on a pressure surface. Assuming that the flow is in steady state and that the imposition of a spatial structure on the mixing coefficients is reasonable (e.g. constant on each isopycnal) an overdetermined system is developed, which accurately resolves the mixing coefficients and the mean flow.

The tracer-contour inverse method combines physically consistent equations including conservation of mass, heat and salt and geostrophic balance. Importantly it does not require a reference level velocity minimization. Mixing coefficients are
leading order in the inversion as is the component of the large scale residual flow down the along-isopycnal tracer gradient. Where the tracer is conservative temperature (equivalently salinity) on an isopycnal, this component of the velocity is effectively the along-isopycnal component of the thermohaline circulation. Tracer-contours are combined with the large scale property budgets of the box inverse method, a framework in which \textit{a priori} information such as section transports and observed mixing parameters may be included to constrain the results. The influence of the box equations versus the tracer-contours can be easily adjusted. Results presented here suggest the inclusion of tracer-contours in large scale inversions will allow a significant improvement in our understanding of diffusive processes and the general circulation. The method is presently being developed for more general application to time averaged hydrographic data including the treatment of coastlines, the mixed and Ekman layers, and multiple boxes.

\textbf{Acknowledgments.} We thank Dr Peter McIntosh for his helpful feedback on this work and thank Drs Carl Wunsch and Andreas Thurnherr for their insightful comments on an earlier version of the manuscript. The authors thank Dr Robert Hallberg for providing the numerical model output. This work contributes to the CSIRO Climate Change Research Program and has been partially supported by the CSIRO Wealth from Oceans Flagship and the Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems.
Chapter 5

Weak mixing in the eastern North Atlantic: An application of the tracer-contour inverse method

Submitted for publication in:
5.1 Abstract

The tracer-contour inverse method is used to infer mixing and circulation in the eastern North Atlantic. Solutions for the vertical mixing coefficient $D$, the along-isopycnal mixing coefficient, $K$, and a geostrophic streamfunction, $\psi$, are all direct outputs of the method. The method predicts a vertical mixing coefficient of order $10^{-5}$ m$^2$ s$^{-1}$ in the upper 1000 m of the water column consistent with in situ observations. The method predicts a depth dependent along-isopycnal mixing coefficient which reduces from order 1000 m$^2$ s$^{-1}$ close to the mixed layer to order 100 m$^2$ s$^{-1}$ in the interior, consistent also with observations and previous hypotheses. The robustness of the results are tested with a rigorous sensitivity analysis including the use of two independently constructed data sets.

This study confirms the utility of the tracer-contour inverse method. The results presented support the hypothesis that vertical mixing is small in the thermocline of the subtropical Atlantic Ocean. A strong depth dependence of the along-isopycnal mixing coefficient is also demonstrated, supporting recent parameterizations for coarse resolution ocean models.

5.2 Introduction

The vertical mixing coefficient, $D$, controls the diapycnal component of the meridional overturning circulation (Munk and Wunsch, 1998). The lack of knowledge of vertical mixing, its magnitude and spatial variation, leads to large uncertainties in our estimates of the volume, heat, freshwater, nutrient and tracer transport, and the role of the ocean in climate sensitivity and climate feedbacks (Manabe and Stouffer, 1988). The along-isopycnal mixing coefficient, $K$, is that which mixes tracers such as temperature and salinity or potential vorticity along isopycnal surfaces. The value of $K$, like $D$, is thought to strongly control the overturning circulation and climate sensitivity (Gnanadesikan, 1999; Sijp et al., 2006).
Munk (1966) pointed out the role of vertical mixing in the global overturning circulation using simple theoretical arguments. He discusses the balance between upwelling and vertical mixing in the thermocline. Given 25 Sv (Sv=10^6 m^3 s^{-1}) of diapycnal upwelling, Munk showed that the globally averaged vertical mixing coefficient, $D$, must be approximately $10^{-4}$ m^2 s^{-1}.

Zika et al. (2009), similar to the analysis of Munk (1966), explore both the along-isopycnal and diapycnal components of the Southern Ocean meridional overturning circulation. The along-isopycnal component of the overturning is that which is “upwelled” along isopycnal layers to the surface boundary layer of the Southern Ocean. Zika et al. (2009) found that in order to upwell 20 - 50 Sv of dense water, the along-isopycnal mixing coefficient, $K$, must be 150 - 450 m^2 s^{-1} and the vertical mixing coefficient must be 0.5 - 1.5 $10^{-4}$ m^2 s^{-1} below the thermocline at latitudes of the ACC. What fraction of the meridional overturning circulation is upwelled through isopycnals below the thermocline, or upwelled along isopycnals in the Southern Ocean, is unknown.

Most observations of $D$, have been much smaller than those predicted by Munk (1966). From the distribution of the tracer SF$_6$, released below the thermocline in the eastern North Atlantic, Ledwell et al. (1993) estimates $D$ to be $O(10^{-5}$ m^2 s^{-1}). Many studies have used microstructure to estimate vertical mixing (e.g. Gregg, 1987). These have tended to reveal weak mixing, $O(10^{-5}$ m^2 s^{-1}), in most regions. Other in situ measurement approaches, while confirming weak mixing over much of the world ocean, have identified stronger mixing, $O(10^{-4}, 10^{-3}$ m^2 s^{-1}), close to rough topography and in energetic regions such as the Southern Ocean (Naveira-Garabato et al., 2004; Kunze et al., 2006; Sloyan, 2005). As direct estimates of diapycnal mixing are sparse and infrequent, it is not clear whether vertical mixing in these energetic regions is sufficient to induce global diapycnal upwelling of order 25 Sv.

Few direct estimates exist for $K$ in the oceans. Ledwell et al. (1998) estimated that the SF$_6$ tracer, released in the eastern North Atlantic at an approximate depth
of 300m, mixed horizontally with a rate of order 1000 m$^2$ s$^{-1}$ over the 18 months of their experiment. Horizontal mixing at the sea surface has been estimated to be equal to or greater than that determined by Ledwell (i.e. 1000-10,000 m$^2$ s$^{-1}$; Zhurbas and Oh, 2004; Marshall et al., 2006; Sallée et al., 2008).

A depth dependence of the mixing coefficient, particularly which mixes potential vorticity or interface height (commonly referred to as $\kappa$ in the Eulerian coordinate parameterization of Gent et al., 1995), is reinforced by adjoint inversions (Ferraira et al., 2005) and eddy resolving models (Eden and Greatbatch, 2008). It should be noted that calculations of $K$, in eddy resolving simulations, are not trivial, even with complete knowledge of a model’s full velocity and density fields (Eden et al., 2007).

Inverse methods are tools used to diagnose the ocean circulation from hydrographic data. Such methods seldom give insight into mixing processes. The most well known method is the box inverse method (Wunsch, 1978). In the box method oceanic sections bound regions of the oceans, or entire ocean basins. In each region, mass, heat, salt and other properties are conserved on isopycnal layers. Inverse box models simultaneously solve a set of equations for various unknowns including reference level velocities and diapycnal fluxes of properties or the vertical mixing coefficient, $D$. Box inversions are always underdetermined and the final solution depends largely on the choice of model and model variance. This sensitivity is particularly strong when mixing coefficients are considered as unknowns in inverse studies (Tziperman, 1988). Inverse models have tended to perform better when individual diapycnal property fluxes are considered (Sloyan and Rintoul, 2000) or when observed mixing coefficients are included as ‘knowns’ (St.Laurent et al., 2001). Recent adjoint and gridded methods (Wunsch and Heimbach, 2007; Herbei et al., 2008) have shown promise but have so far been unable to resolve the spatial structure of $D$ and $K$.

Zika et al. (2010), hereafter ZMS09, present a new inverse method called the tracer-contour inverse method. This inverse method is less sensitive to error than
The $\gamma^{rf} = 26.525 \text{ kg m}^{-3}$ surface from the climatology of DW09. Also marked is the region considered in this study (solid black rectangle), the site of mooring C from the Subduction Experiment (black cross; Joyce et al., 1998) and the NATRE study regions showing the approximate SF$_6$ tracer extent during the fall of 1992 (dark grey rectangle), the spring of 1993 (medium grey rectangle) and November of 1994 (light grey rectangle; Ledwell et al., 1998).

existing methods and has been validated against the output of a numerical model. With these advances and the presence of new observations which give a better representation of the upper 2000m of the Ocean, it may now be possible to accurately infer the ocean circulation and rates of along-isopycnal and vertical mixing directly from hydrographic data. The purpose of this study is 2 fold: firstly to demonstrate the utility of this new inverse method by applying it to ocean observations in the North Atlantic Tracer Release Experiment region (NATRE; Ledwell et al., 1993, Fig.5.1), where direct estimates of vertical and along-isopycnal mixing exist; and secondly, to reveal the depth dependence of the mixing coefficients in that region.

This article is structured as follows: In section 5.3 we briefly describe the
tracer-contour inverse method and the ocean climatology used. In section 5.4 we present the results of the tracer-contour inverse method, as applied to the NATRE region of the North Atlantic. Conclusions are given in section 5.5. In Appendix A we test the sensitivity of the method to the adjustment of various parameters and differing data sources.

5.3 The tracer-contour inverse method and hydrographic data

Here we briefly describe the tracer-contour inverse method of ZMS09 and the data used in this study. We then explain how the inverse method is applied in the eastern North Atlantic.

5.3.1 The tracer-contour inverse method

The components of the mean velocity across tracer contours on isopycnals and through isopycnals are related to mixing of temperature, salinity and potential vorticity through the following advective-diffusive balances

\[ \mathbf{v} \cdot \mathbf{n} = \frac{K}{\lambda^\perp} + \nabla_\gamma K \cdot \mathbf{n} + \frac{D}{\lambda^\gamma} + \frac{K_{PV}}{\lambda^h} - \Theta_t |\gamma|/|\nabla_\gamma \Theta| \]  
\[ w^\gamma = \frac{D}{\eta^\gamma} + D_z + \frac{K}{\eta^\perp} \]  

(5.1)  

(5.2)

where \( \mathbf{v} \) is the mean lateral velocity ([u,v,0]), \( w^\gamma \) is the diapycnal velocity component, \( S \) is salinity and \( \Theta \) is conservative temperature. All variables are temporal and thickness-weighted means, averaged on neutral density surfaces \( (\gamma) \), except \( \mathbf{v} \) which is split into a temporal mean on isopycnals (LHS of 5.1) and a bolus term \( (K_{PV}/\lambda^h; \text{on the RHS of 5.1}) \). The direction of the along-isopycnal temperature (and salinity) gradient is \( \mathbf{n} \) \( (\mathbf{n} = \nabla_\gamma \Theta / |\nabla_\gamma \Theta|) \). See Appendix B for the scale lengths \( \lambda^h, \lambda^\perp, \lambda^\gamma \) and scale heights \( \eta^\gamma \) and \( \eta^\perp \). These scale lengths and heights represent the balance between advection and either vertical or along-isopycnal
mixing.

The density variable used is neutral density (McDougall, 1987a) and the temperature variable used is conservative temperature (McDougall, 2003). The previous two points are not necessarily critical to the analysis presented here and the variables potential temperature (θ) and potential density (σ₁ or σ₂) could be used instead.

Equations (5.1) and (5.2) are derived and discussed in Zika et al. (2009) and ZMS09. For completeness we briefly describe them here. In (5.1), if there is a component of the mean velocity, down the along-isopycnal temperature gradient (i.e. the LHS is nonzero), some combination of the following processes must be at play: along-isopycnal mixing works to compensate the anomaly introduced by the down-gradient advection \(1/\lambda_\perp \approx \nabla_\perp^2 \gamma \Theta / |\nabla_\perp \gamma \Theta|\) and \(\nabla_\perp K \cdot n\); vertical mixing acts to compensate the anomaly by mixing of the \(\Theta - S\) curvature \(1/\lambda_\gamma = \Theta_\gamma^2 \frac{R_\rho}{|\nabla_\gamma S| (R_\rho - 1)} d^2 S d^2 z\) where \(R_\rho = \frac{|\nabla_\gamma S_{\Theta \Theta}|}{|\nabla_\gamma S_{\Theta S}|}\); there is a compensating advection due to the bolus velocity \(1/\lambda_h = \nabla_\perp h \cdot n/h\); or the temperature contour simply moves in space \((\Theta_t/\gamma_{\perp}/|\nabla_\perp \Theta|))\). That is, some or all of the terms on the RHS of (5.1) must be nonzero to balance the LHS. Nonlinear effects due to cabling and thermobaricity are included in the \(1/\lambda_\perp\) term in (5.1). Equation (5.1) is used by Zika et al. (2009) to infer the Southern Ocean overturning circulation as a function of \(K\) and \(D\).

In (5.2), if there exists mean advection across a density surface \(w^\gamma\) there must be a compensating effect of vertical mixing \(1/\eta^\gamma \approx \gamma_{zz}/\gamma_z\) and/or \(D_z\) or nonlinear effects such as cabling and thermobaricity \(1/\eta^\gamma\). Equation (5.1) differs from (5.2) in that the vertical coordinate moves with the isopycnal surfaces, while the lateral coordinate is fixed in space, hence there is no \(\gamma_t/\gamma_z\) term in (5.1) as there would be if depth \((z)\) where the vertical coordinate. Equation (5.2) is similar to that used by Munk (1966), as it relates diapycnal upwelling to vertical mixing, but also includes nonlinear processes (i.e. the \(K\) term on the right hand side).

In the tracer-contour inverse method, the pointwise advective-diffusive balance
Equations are integrated along tracer contours on isopycnals using (5.1) and across isopycnals using (5.2). Here we consider contours of conservative temperature which are equivalent, on isopycnals, to contours of constant salinity. As such, mixing is related to the large scale circulation which is represented by a geostrophic streamfunction. The equations used in the tracer-contour inverse method are

\[
\psi \bigg|_{p_0} \bigg|_{x_2, y_2}^{x_1, y_1} - \int_{x_1, y_1, \gamma}^{x_2, y_2, \gamma} \left( \frac{K}{\lambda} + \nabla_\gamma K \cdot \mathbf{n} + D/\lambda \right) + \frac{K_{PV}}{\lambda h} - \Theta_{|_{\gamma}}/|\nabla_\gamma \Theta| \, dx_\Theta = \left[ \int_{p_0}^{p} \delta_{\text{contour}}(p') dp' \right]_{x_3, y_3}^{x_4, y_4} \tag{5.3}
\]

\[
\int (v_{p_0}) \cdot \mathbf{m} h d\mathbf{l} + \left[ \int w^\gamma C dA \right]_{I}^{U} - \int K_{PV} \nabla_\gamma h \cdot \mathbf{m} d\mathbf{l} = \int K \nabla_\gamma C \cdot \mathbf{m} d\mathbf{l} - \int D C z dA \left[ \int DC_z dA \right]_{I}^{U} = \int (v_{p_0} - v) \cdot \mathbf{m} h d\mathbf{l}. \tag{5.4}
\]

Equation (5.3) is the tracer-contour equation and relates mixing on an isopycnnal, \( \gamma \), to the flow on a reference pressure, \( p_0 \). In (5.3), \( \psi |_{p_0} \) is the geostrophic streamfunction on \( p_0 \) (Fig. 5.2). The Coriolis frequency is \( f \). The coordinate \( x_\Theta \) is that running perpendicular to \( \mathbf{n} \) on the isopycnal. That is, \( \int dx_\Theta \) is the integral along a temperature contour on an isopycnal. The tracer-contour specific volume anomaly is \( \delta_{\text{contour}}(p) = 1/\rho(S, \Theta, p) - 1/\rho(S_{\text{contour}}, \Theta_{\text{contour}}, p) \), where \( S_{\text{contour}} \) and \( \Theta_{\text{contour}} \) are the conservative temperature and salinity values of the contour between \((x_2, y_2)\) and \((x_1, y_1)\). That is, \( \delta_{\text{contour}} \) is redefined for each contour.

Equation (5.4) is the layer conservation equation for volume and any conservative tracer \( C \) and, like (5.3), relates mixing on \( \gamma \) to flow at \( p_0 \) (Fig. 5.2). For conservation of volume \( C = 1 \), for conservation of temperature anomaly \( C = \Theta - \overline{\Theta} \) and for conservative of Salinity anomaly \( C = S - \overline{S} \), where \( \overline{\Theta} \) and \( \overline{S} \) are layer averages. In (5.4), \( v_{p_0} \) is the lateral velocity on the reference pressure and \( \mathbf{m} \) is the direction normal to the lateral boundaries of the isopycnal layer. The thickness of the isopycnal layer is \( h = \Delta \gamma^\mathbf{H}/\gamma^\mathbf{H} \) for some arbitrary \( \Delta \gamma^\mathbf{H} \) about the surface \( \gamma \); where \( || \) \( \gamma \) is the difference between the upper and lower interfaces of that layer (Fig. 5.2). The diapycnal advection, \( w^\gamma \), is explicitly related to mixing using (5.2).
We explicitly relate the lateral advection terms in (5.4) (i.e. the first term on the LHS and the RHS) to the geostrophic streamfunction, $\Psi^0$, using a form of the depth integrated thermal wind equation, as shown in Appendix B.

The tracer-contour inverse method combines aspects of the three major inverse modeling concepts of modern oceanography: the box, beta spiral and Bernoulli methods, and the advective-diffusive balance concept is motivated by Munk (1966). Even with a small number of vertical layers, the tracer-contour inverse method is over-determined. There is a direct relationship between lateral advection and diffusion, and the diffusion terms are leading order. In the method there is a linear relationship between advection and vertical mixing through the $\Theta - S$ curvature ($d^2\Theta/dS^2 |_{x,y}$, a quantity less susceptible to noise due to heave than $\Theta_{zz}$ and $S_{zz}$ are individually). Down-temperature and down-salinity gradient transports along isopycnals are explicitly considered in the tracer-contour inverse method and it is these transports which constitute the along-isopycnal component of the thermohaline overturning circulation. In the tracer-contour inverse method, a streamfunction variable is solved for, along ‘sections’, allowing for direct integration with the box inverse method and the computation of section transports for comparison with, or constraint by, shipboard observations.

### 5.3.2 Data and set up of the inversion

The climatology of Durack and Wijffels (submitted), hereafter DW09, is used in this study. Hydrographic data from shipboard observations and Argo floats, up to and including 2008, have been used in their analysis. Annual and inter-annual cycles as well as a linear trend have been locally fit to the data, minimizing temporal aliasing. The data was averaged by DW09 on approximate neutral density surfaces ($\sigma^T_f$; Jackett and McDougall (2005)) spaced by 0.025 kg m$^{-3}$ and on a $1^\circ$ Latitude - $2^\circ$ Longitude grid. The tracer-contour inverse method has also been applied to the climatology of Gouretski and Koltermann (2004) and these results are discussed in Appendix A. The inversion is carried out for the NATRE region.
Figure 5.2: Schematic showing how the tracer-contour inverse method is implemented. Mixing at points along a tracer contour between \((x_1,y_1)\) and \((x_2,y_2)\) on the isopycnal surface \(\gamma\), is related to a geostrophic streamfunction on the reference pressure \(p_0\) using (5.3), shown in green. The unit vector \(\mathbf{n}\) is normal to the contour on the isopycnal. The tracer \(C\), typically volume, conservative temperature anomaly or salinity anomaly, is conserved on an isopycnal layer bounded by sections using (5.4), shown in blue. Properties are advected or mixed through the upper and lower bounding surfaces of the layer and across each section. The velocity across a section is related to the streamfunction on the reference surface using a form of the thermal wind equation (Appendix B). The unit vector \(\mathbf{m}\) is normal to the section. In this study \(\Psi^{p_0}\) is solved for each cast \((i)\) and the mixing coefficients, \(K\) and \(D\), are solved for on each layer.
in the North Atlantic (40°W–25°W, 20°N–30°N) (Ledwell et al., 1993, 1998, Fig. 5.1).

Regularly spaced points on the boundary of the domain (between 40°W and 25°W and 20°N and 30°N), given by the grid of the climatology, are referred to as ‘casts’ and the eastern, northern, western and southern boundaries of the domain are referred to as ‘sections’. There are a total of 34 casts. Tracer contours (contours of constant $\Theta$ and $S$ in this study) are defined on $\gamma_{RF}$ surfaces between 200m and 1800m. As in ZMS09, contours are defined, on each surface, by the temperature and salinity at the cast locations. That is, there is about one tracer-contour per cast per surface. At each point on a contour the flow across the contour, in the direction $n$, is related to along-isopycnal and diapycnal mixing through the scale lengths $\lambda^h$, $\lambda^\perp$ and $\lambda^\gamma$ (5.1). So integrating $f\lambda^h$, $f\lambda^\perp$ and $f\lambda^\gamma$ along each contour (green line in Figure 5.2) gives the difference in geostrophic streamfunction, between the ends of the contour, as a function of mixing (5.3).

Maps of the lateral mixing coefficient at the sea-surface from Zhurbas and Oh (2004) suggest gradients of the mixing coefficient are very small in the latitude bands considered here. For this reason we choose to find solutions where $K$ does not vary laterally and hence ignore the $\nabla_\gamma K \cdot n$ term in (5.1). It is likely that at the boundary between different mixing regimes this term will become important and the sensitivity of the method to this choice is left to future work. By default we assume a statistically steady state, hence assuming the unsteady term, $\Theta_t |_{\gamma} / |\nabla_\gamma \Theta|$ is negligible. This choice, and sensitivity to it, are discussed in Appendix A.

Volume, conservative temperature anomaly, and salinity anomaly are conserved on 28 isopycnal layers between 200m and 1800m (5.4). The density range of each layer is chosen such that at least 40 tracer contours are within each layer, allowing $K$ and $D$ to be well resolved within each layer. The diapycnal fluxes of properties due to advection, $w^\gamma$, are related to mixing through (5.2). The geostrophic advection across a ‘section’ is $v \cdot m$, where $m$ is the unit vector normal to the section (Figure 5.2). This advection is related to the geostrophic streamfunction along

132
the section. The difference in geostrophic streamfunction is related, at each pair of casts to the reference level streamfunction, $\Psi_{p0}$ on at 1000db (D.2). A streamfunction is solved for, on the reference pressure, at each of the 34 cast locations (D.3). The diffusivities, $K$ and $D$, are solved for on each density layer.

Contours with very small along isopycnal gradients ($\nabla_\gamma \Theta < 2 \times 10^{-8} \text{ K m}^{-1}$) are excluded. The results are generally insensitive to minor changes to the choice of reference level, depth range, minimum number of contours per layer and the minimum along-isopycnal temperature gradient. The inversion is more sensitive to the factors considered in Appendix A.

The 28 isopycnal layers give 28 $D$ variables and 28 $K$ variables. There are 34 reference level streamfunction variables and 1608 tracer-contour equations. All equations (contour and box equations) are scaled to be order 1. The contour equations are additionally scaled by the normalized inverse of the mean isopycnal gradient ($\frac{\text{max}(\int |\nabla_\gamma \Theta| dx \Theta)}{\int |\nabla_\gamma \Theta| dx \Theta / \int dx \Theta}$, where max() is over all tracer contours) as (5.1) becomes singular for very small $\nabla_\gamma \Theta$. As in ZMS09, the relative weighting of the tracer-contour and box equations [i.e. the weight given to (5.3) relative to (5.4)] is determined by minimizing the condition number of the matrix $A$ in the linear system $Ax = b$. Sensitivity analysis of the changes to the equation weighting are given in Appendix A.

5.4 Results

5.4.1 Vertical Mixing

The tracer-contour inverse method gives direct estimates of the mixing coefficients, $K$ and $D$ (Fig. 5.3 and Fig. 5.4). The vertical mixing coefficient, $D$, in the upper 200 m to 1000 m of the water column is $O(10^{-5} \text{ m}^2 \text{ s}^{-1})$. There is a modest increase in $D$ towards the deepest layers at 1800 m. The deepest value of $D$ is $2.6 \pm 0.8 \times 10^{-5}$ at 1800 m.
Figure 5.3: Vertical mixing coefficient, $D$, as determined using the tracer-contour inverse method (solid line) and the standard error of the inversion (light grey shading). Also shown are the estimates of $D$ from Ledwell et al. (1993), Ledwell et al. (1998), and Ferrari and Polzin (2005) and their reported uncertainties.
An SF$_6$ tracer has been released in the eastern North Atlantic at an approximate depth of 300 m and its dispersion used to infer a vertical mixing coefficient (Ledwell et al., 1993, 1998). Two observation of the effective mixing coefficient for that tracer were made: $1.2 \pm 0.1 \times 10^{-5} \text{m}^2 \text{s}^{-1}$ in the spring of 1992 and $1.7 \pm 0.1 \times 10^{-5} \text{m}^2 \text{s}^{-1}$ in the fall of 1993 (Fig. 5.3). As there are only two measurements, a suitable error range for the 'long term mean' $D$ ambiguous. Promisingly, the error range of $D$ in the upper 200 m - 400 m depth range, as estimated in this study, almost overlaps with the spring 1992 measurements of Ledwell et al. (1993).

Microstructure measurements of a vertical mixing coefficient in the NATRE region at depths of 200 m - 400 m are generally $O(10^{-5} \text{m}^2 \text{s}^{-1})$. For example St.Laurent and Schmitt (1999) observed mixing of $0.1 - 1.3 \times 10^{-5} \pm 10^{-5} \text{m}^2 \text{s}^{-1}$ considering diffusivities for density ($D_p = 0.1 \times 10^{-5} \pm 0.3 \times 10^{-5} \text{m}^2 \text{s}^{-1}$) potential temperature ($D_\theta = 0.8 \times 10^{-5} \pm 0.1 \times 10^{-5} \text{m}^2 \text{s}^{-1}$) and salinity ($D_S = 1.3 \times 10^{-5} \pm 0.1 \times 10^{-5} \text{m}^2 \text{s}^{-1}$). Unlike St.Laurent and Schmitt (1999) we do not consider double-diffusive convection since we have taken the vertical mixing coefficient, $D$ to be the same for both $\Theta$ and $S$. Ferrari and Polzin (2005) find an equally low turbulent diffusivity close to $0.7 \times 10^{-5} \text{m}^2 \text{s}^{-1}$. There are large uncertainties in microstructure estimates of $D$ due to the assumption of a turbulent mixing efficiency. Despite this uncertainty, the estimates of $D$ presented here, agree very strongly with the microstructure estimates of $D$ of Ferrari and Polzin (2005) and St.Laurent and Schmitt (1999). The increase with depth of $D$ toward 1800m, observed in this study, is also consistent with Ferrari and Polzin (2005) (Fig. 5.3).

Microstructure estimates of $D$ are near instantaneous, representing turbulent motions evolving over minutes and hours. The estimates of $D$ by Ledwell et al. (1993) and Ledwell et al. (1998) represent mixing of a tracer integrated over a 6 - 18 month time period. The estimate of $D$ presented here represent the long-term mean, effective mixing of heat and salt across isopycnals. That the three approaches (microstructure, tracer release and tracer-contour inverse method) each with their respective time scales, spatial scale and uncertainties, agree on a value of
$D$ of $O(10^{-5}$ m$^2$ s$^{-1}$) is encouraging. Our results support the hypothesis that below the thermocline in the eastern North Atlantic, the $D$ estimated by microstructure studies is equivalent to the long term mean $D$ and is thus the appropriate value for use in ocean circulation models. This canonical ‘background’ value is consistent with the hypothesis that mixing is low away from boundary layers and is unable to sustain the upwelling required in the balance of Munk (1966).

5.4.2 Along-Isopycnal Mixing

The along-isopycnal mixing coefficient, $K$, is estimated to be $O(1000$ m$^2$ s$^{-1}$) within the upper 200 m - 400 m of the water column, consistent with the analysis of Ledwell et al. (1998) for the same region. The mixing coefficient smoothly reduces with depth. Between 500 m and 1000 m depth, $K$ reduces to around 0 - 200 m$^2$ s$^{-1}$ (Fig. 5.4). These values are broadly consistent with nearby estimates from float trajectories (Joyce et al., 1998; Spall et al., 1993), $^3$He, $^3$H and tritium tracer studies (Jenkins, 1987, 1998) and other inverse studies (Armi and Stommel, 1983; Zika and McDougall, 2008, Fig. 5.4). Ferrari and Polzin (2005) are able to derive a vertical profile of the along-isopycnal mixing coefficient using a mixing length argument and combining CTD and mooring data from the NATRE and Subduction experiments (Joyce et al., 1998). Although there is uncertainty in the mixing efficiency that Ferrari and Polzin (2005) choose and the heuristic arguments that go into the approach, their results are very similar to those presented here.

The vertical profile of $K$ presented here is consistent with the findings of Ferraira et al. (2005) which suggested a depth dependence to the mixing coefficient $K$ which is mostly $O(1000$ m$^2$ s$^{-1}$) in the upper 200 m - 500 m of the water column and reduces to $O(100$ m$^2$ s$^{-1}$) towards 1000 m - 2000 m depth. However, in the NATRE region, the estimates of Ferraira et al. (2005) are mostly negative. Recent parameterization efforts have focused on the need for a depth-dependence of the along-isopycnal mixing coefficient (Eden and Greatbatch, 2008; Danabasoglu and Marshall, 2008), motivated largely by inferences from eddy resolving models and
Figure 5.4: Along-isopycnal mixing coefficient, $K$, as determined using the tracer-contour inverse method (solid line) and the standard error of the inversion (light grey shading). Also shown are the estimates of $K$ from Joyce et al. (1998), Ledwell et al. (1998), Jenkins (1998), Armi and Stommel (1983), Spall et al. (1993) and Zika and McDougall (2008) and their reported uncertainties.
theoretical arguments. This study represents one of the few demonstrations of a strongly depth dependent along-isopycnal mixing coefficient, inferred directly from observational data.

5.4.3 Geostrophic Flow

As the geostrophic streamfunction at the reference level is an output of the tracer-contour inverse method, the full depth geostrophic velocity and transport are easily inferred (Fig.5.5).

The mean geostrophic velocity is to the southwest through most of the study region. The velocity is intensified in the upper 400m of the water column with velocities of up to 1-2 cm s$^{-1}$. The flow is generally stronger in the zonal direction and intensified in the south-west corner of the domain. This pattern is consistent with the largely westward migration of the SF$^6$ tracer throughout the NATRE study at approximately 1.3m s$^{-1}$ (Ledwell et al., 1998; Table 1). The flow is also consistent with the south-eastward flowing limb of the wind-driven North Atlantic subtropical gyre in Sverdrup balance (Sverdrup, 1947). The typical columnar nature of the velocity field, seen in conventional inverse calculations using one time hydrographic surveys, is not seen here partly because individual eddies are not present in the climatology.

The deep flow is in no way minimized or assumed small in this analysis. Despite this the solution for the geostrophic velocity at 1800 m is small, $O(0.2 \text{ cm s}^{-1})$, relative to the upper 500m $O(2 \text{ cm s}^{-1})$. However, the inferred contribution to the section transport between 1800 m and the sea floor across each section is significant, $O(2 - 3 \text{ Sv})$, demonstrating the need for an appropriate estimate of the deep velocity. Some structure to deep currents is revealed including subsurface zonal currents.
Figure 5.5: Top row: Cross-section velocity, $\mathbf{v} \cdot \mathbf{m}$, on $\gamma_{rf} = 26.525 \text{ kg m}^{-3}$ ($\sigma_0 \approx 26.75 \text{ kg m}^{-3}$). Length of arrows represents magnitude of $\mathbf{v} \cdot \mathbf{m}$ while the direction indicates whether the flow is into or out of the study region. Each section has a corresponding velocity figure starting from western section (far left) moving clockwise and to the southern section (far right). The grey box indicates the study region. Ledwell et al. (1998) release an SF$_6$ tracer at $\sigma_0 = 26.75 \text{ kg m}^{-3}$ which is advected to the South-East, consistent with the flow field observed here. Bottom Row: Cross-section of velocity on neutral density surfaces $\mathbf{v} \cdot \mathbf{m}$ (positive values are directed out of the study region). As in the top row each section has a corresponding velocity figure.
5.5 Conclusions

The tracer-contour inverse method, has been applied to the mean hydrography of the eastern North Atlantic. Vertical profiles of the vertical mixing coefficient, \( D \), and along isopycnal mixing coefficient, \( K \) have been diagnosed, as well as the mean geostrophic circulation. The mixing coefficients and circulation patterns are consistent with in situ measurements.

Vertical mixing is found to be weak, \( O(10^{-5} \text{ m}^2 \text{ s}^{-1}) \), through most of the water column. These low mixing values are consistent with the hypothesis that abyssal mixing is small in the subtropical oceans and is insufficient, on its own, to close the meridional overturning circulation. It is likely that the methods demonstrated in this study will be able to identify regions of intense mixing and quantify the global spatial structure of the mixing coefficients.

In the NATRE region, along-isopycnal mixing is found to be strong, \( O(1000 \text{ m}^2 \text{ s}^{-1}) \), near the thermocline and reduces with depth to background values of order \( O(100 \text{ m}^2 \text{ s}^{-1}) \) below 500 m depth. Our results confirm that along-isopycnal mixing varies with depth as inferred by Ferrari and Polzin (2005) and Ferraira et al. (2005), further motivating parameterizations for this effect in coarse resolution ocean models (Eden and Greatbatch, 2008; Danabasoglu and Marshall, 2008). Uniquely, we reveal a depth dependence of the along-isopycnal mixing coefficient from mean hydrographic data alone.

The eastern North Atlantic is thought to be a region of low eddy kinetic energy and small diapycnal mixing and, as such, is a suitable region for determining the background \( K \) and \( D \). The tracer-contour inverse method accurately reproduces in situ observations there, suggesting that the method can be used to identify regions of both weak and intense vertical and along-isopycnal mixing in the world oceans.

Acknowledgments. We thank Paul Durack and Dr Susan Wijffels for the use of their climatology. This work contributes to the CSIRO Climate Change Research Program and has been partially supported by the CSIRO Wealth from
Oceans Flagship and the Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems.
Chapter 6

Concluding remarks

The relationship between ocean circulation and both along-isopycnal and vertical mixing has been explored. Specific regions have been studied where the known advection of given watermasses are used to directly infer mixing rates. Furthermore, an inverse method has been developed for estimating the general circulation and mixing from mean hydrographic data. The method is called the tracer-contour inverse method and is validated against both a numerical model and in-situ observations. Here the implications of this thesis and future research directions are discussed.

6.1 Theoretical implications

Equations have been developed, precisely relating along-isopycnal advection to along-isopycnal and vertical mixing. These relationships have been applied in divergence form (Chapter 2) and in advective form (Chapter 3). The explicit consideration of tracer-contours on isopycnals is a unique feature of this thesis. It is pointed out that around any closed contour of constant tracer, for example along the entirety of the ACC, the ratio of along-isopycnal to vertical mixing can
be derived from the mean hydrography alone.

Along-isopycnal and vertical mixing are related to changes in a geostrophic streamfunction along tracer-contours on isopycnals (Chapter 4). This relationship allows one to relate the difference in streamfunction, at two remote locations, to mixing process between them. In this sense tracer-contours become the most appropriate coordinate around which to develop a new inverse method for estimating mixing.

The inverse method developed, the tracer-contour inverse method, combines aspects of the three major inverse methods of physical oceanography: the box method (Wunsch, 1978), beta spiral method (Stommel and Schott, 1977) and the Bernoulli method (Killworth, 1986). The tracer-contour inverse method solves for geostrophic flow on ‘sections’ and constrains large scale budgets of properties much like the box inverse method (Wunsch, 1978). The streamfunction on one section is related to that on a distant section via tracer-contours on isopycnals much like the Bernoulli method (Killworth, 1986). The component of the mean circulation along tracer-contours, is inferred when the isopycnal tracer gradient spirals in the vertical, much like the beta spiral method (Stommel and Schott, 1977).

The tracer-contour inverse method accurately reproduces the circulation and mixing coefficients of a numerical model and in situ observations (Chapters 4 and 5). In the regions considered, the results are not particularly sensitive to changes to the set up of the inverse method or the data used.

The tracer-contour inverse method diagnoses mixing and circulation from the mean hydrography. Methods exist which infer mixing rates from the variability of the hydrography (Ferrari and Polzin, 2005; Thompson, 1977). As both approaches use independent information, the merging of the two could improve our understanding of ocean mixing even further.
6.2 Oceanographic implications

The outflow of relatively warm and salty Mediterranean Sea (Med) water, into the North Atlantic can be described as a balance of along-isopycnal advection to along-isopycnal and vertical mixing (Chapter 2). Given the observed transport of Med water past the Gulf of Cadiz (Baringer and Price, 1997), we infer vertical mixing, $D$, on the isopycnals of the Med water, of $O(10^{-4} \text{ m}^2 \text{s}^{-1})$. The regions we consider encompass both coastal and open ocean regions. As estimates of $D$ in the open ocean of the North Atlantic are $O(10^{-5} \text{ m}^2 \text{s}^{-1})$ (i.e. Ledwell et al., 1993, Chapter 5) our analysis supports the hypothesis that vertical mixing is elevated along continental boundaries. Here along-isopycnal mixing in the Med water layers of the North Atlantic is estimated to be $O(200 \text{ m}^2 \text{s}^{-1})$ suggesting a strong depth dependence to the along isopycnal mixing in this region, which is discussed below.

The Southern Ocean Meridional Overturning (SMOC) can be related to along-isopycnal and diapycnal mixing by following conservative temperature and salinity contours, on isopycnals, along the Antarctic Circumpolar Current (Chapter 3). Contours of constant $\Theta$ (equivalently $S$) on isopycnals, cover the entirety of the southward flowing Upper Circumpolar Deep Water (UCDW), which feeds the upper and lower limbs of the SMOC. Without along-isopycnal or diapycnal mixing, flow would not cross $\Theta$ contours on isopycnals, and a steady SMOC could not exist. The mixing ‘aspect-ratio’ beneath the mixed layer of the Southern Ocean, is estimated to be $10^6$. Assuming an UCDW transport of 20 - 50 Sv, $K$ is estimated to be 150 - 450 $\text{m}^2 \text{s}^{-1}$ and $D 0.5 - 1.5 \times 10^{-4} \text{ m}^2 \text{s}^{-1} \text{m}^2 \text{s}^{-1}$. Again $D$ is large, relative to other open ocean estimates, suggesting vertical mixing is elevated along the ACC. Lateral mixing is observed to be large in the surface layers of the Southern Ocean (Karsten and Marshall, 2002; Sallée et al., 2008) suggesting there is a strong depth dependence to the along isopycnal mixing coefficient or mixing is restricted along the fronts of the ACC at mid-depth.

Using the methods developed in Chapter 4, the magnitude and depth depen-
dence of along-isopycnal and diapycnal mixing in the open ocean is revealed in Chapter 5. Using the tracer-contour inverse method we reveal weak vertical mixing in the eastern North Atlantic, $O(10^{-5} \text{ m}^2 \text{s}^{-1})$. Our estimates of $D$, representing long term mean mixing processes, correspond well to the direct estimates of $D$ using a tracer release (Ledwell et al., 1993) and microstructure (Ferrari and Polzin, 2005). A strong depth dependence to the along-isopycnal mixing coefficient is revealed in the eastern North Atlantic. The mixing coefficient reduces from highs of $O(1000 \text{ m}^2 \text{s}^{-1})$ near the pycnocline to lows of order $O(100 \text{ m}^2 \text{s}^{-1})$ below depths of 500m.

6.3 Implications for ocean modelling

A new inverse method has been developed for estimating along-isopycnal and diapycnal mixing coefficients. These parameters are explicitly chosen by ocean modelers. A lack of observations of mixing and understanding of their uncertainty has made these decisions arbitrary, and has made developing consistent theories and parameterisations difficult. The methods developed here may go a long way to addressing this issue.

Contrasts in the magnitude of the along-isopycnal and vertical mixing coefficients, in different regions, are revealed in this study. Vertical mixing is seen to be large, $O(10^{-4} \text{ m}^2 \text{s}^{-1})$, in coastal areas and energetic regions such as the Southern Ocean and small, $O(10^{-5} \text{ m}^2 \text{s}^{-1})$, in more placid open ocean regions such as the eastern North Atlantic. Resolving this contrast should be an aim of parameterisations of vertical mixing.

Along-isopycnal mixing is seen to be large, $O(1000 \text{ m}^2 \text{s}^{-1})$, in the upper 500m of the ocean, reducing to $O(100 \text{ m}^2 \text{s}^{-1})$ below 500m. These results support recent parameterisation efforts which describe a depth dependence to along-isopycnal mixing (Eden and Greatbatch, 2008; Danabasoglu and Marshall, 2008).
6.4 Future research

Further research into the Southern Ocean Meridional Overturning Circulation is needed. It is likely that a complete understanding of the SMOC requires going beyond the ‘streamwise average’ views of the literature or the ‘contourwise average’ view presented in Chapter 3. Variations in along-isopycnal and vertical mixing along the ACC, are likely to strongly control the SMOC (Naveira-Garabato et al., Submitted), and in turn control the Southern Oceans role in climate variability.

Determining rates of along-isopycnal and vertical mixing and their spatial structure will remain a major path of future research for years to come. The tracer-contour inverse method presented here is ripe for application to areas of oceanographic interest. Furthermore, the tracer-contour inverse method may be implemented to the global ocean as a whole, similar to the inversions of Ganachaud (2003) and Lumpkin and Speer (2007). Further development of the inverse method is being undertaken, incorporating coastal areas, and the upper and lower ocean boundary layers.

Further theoretical development is necessary into the appropriate treatment of bolus transports in inverse methods and numerical ocean models, particularly in the Southern Ocean. The relationship between neutral helicity, mixing and the mean circulation is a topic requiring further work.
Appendix A

Appendix: Chapter 2

A.1 The three extra terms in equation (2.20)

There are three other terms that arise from the last three terms of (2.18) (and the corresponding equation for conservative temperature) that should appear in (2.20) but have been omitted because we can show that they are small. These three terms are the spatial integrals of

\[-(w^{\gamma} - w^{\gamma}) \left\{ \frac{1}{2}(\Theta^u + \Theta^l) - \Theta_1 \right\} \Delta S - \frac{1}{2}(S^u + S^l) \Delta \Theta \right\}, \quad (A.1)\]

\[-(w^{\gamma} + w^{\gamma})[(\Theta^u - \Theta^l)\Delta S - (S^u - S^l)\Delta \Theta] \quad (A.2)\]

and

\[(D^u - D^l) \left[ \frac{1}{2}(\Theta^u_z + \Theta^l_z)\Delta S - \frac{1}{2}(S^u_z + S^l_z)\Delta \Theta \right]. \quad (A.3)\]

Equation (A.1) can be written as

\[(w^{\gamma} - w^{\gamma}) \left[ \frac{1}{2}(S^u + S^l) - S_1 \right] \left[ \Delta \Theta - \frac{\alpha \Delta S}{\beta} \right] \quad (A.4)\]

where \(\beta\) and \(\alpha\) are the saline contraction and thermal expansion coefficients (because the variations of salinity and temperature along density surfaces are related by \(\beta \nabla_S S = \alpha \nabla \Theta \)). The last term in (A.4) does not vary much because the ratio
\( \beta/\alpha \) is a slowly varying function of space. Now consider varying \( S_1 \) between \( S_M \) and \( S_0 \). At one extreme the middle term in (A.4) is positive throughout the whole area of integration, while for the other extreme choice of \( S_1 \) the middle term in (A.4) is negative over the whole area. When area integrated, (A.1) and (A.4) will then very likely give either a positive or a negative integral for these extreme choices. It is likely that the value of the salinity offset that makes the area average of (A.1) zero will be close to the area average salinity. This is the value of \( S_1 \) that we seek since we wish to minimize the influence this term can have on (2.18).

Spatial correlations between \((w^u - w^d)\) and the middle term in (A.4) will cause the value of \( S_1 \) that zeros the spatial average of (A.1) to be different from the spatially average salinity, however, unless these spatial correlations are very strong, we feel justified in taking \( S_1 \) to be close to the area-averaged salinity, that is we assume \( S_1 \approx 0.5\langle S^u + S^l \rangle \) and \( \Theta_1 \approx 0.5\langle \Theta^u + \Theta^l \rangle \). In any case, we will assume that the lateral and vertical diffusivities do not vary along each layer, so in that sense it is consistent to make this assumption for \( S_1 \) as it is also effectively as assumption about minimum spatial variation, namely of \((w^u - w^d)\). Bearing in mind the geometry of the isohalines of the Mediterranean Water as it spreads into the North Atlantic, if \( S_1 \approx 0.5\langle S^u + S^l \rangle \) then we take the ratio of salinities in (2.16) as

\[
0.25 < \frac{|S_1 - S_0|}{|S_M - S_0|} < 0.5. \tag{A.5}
\]

With values of \( c \) between 0.7 and 1.0, the resulting range of \( F \) is approximately 0.85<\( F <1.0 \).

Notice that the choice of the constant salinity \( S_1 \) appears in the conservation equation (2.20) only through its influence on \( F \) and in one of the additional terms, namely term (A.1). Since we will ignore this term, it is important to ensure that it is small, and a suitable choice of \( S_1 \) will ensure that this term is zero. While we are unsure of this value of \( S_1 \), we believe that it lies in a range that implies that \( F \) is between 0.85 and 1.0. Hence we conclude that the price of ignoring the contribution of (A.1) to (2.20) is merely to suffer this relatively small uncertainty.
in the magnitude of the product $FQ$. Now we consider the influence of the ignored terms (A.2) and (A.3) on (2.20). Since $0.5(S^u_z + S^l_z) \approx (S^u - S^l)/h$, (A.2) and (A.3) can be combined as

$$- \left[ \frac{1}{2}(w^u + w^l) - (D^u - D^l)/h \right] \left[ (\Theta^u - \Theta^l)\Delta S - (S^u - S^l)\Delta \Theta \right] \quad (A.6)$$

The area integral of the second curly bracket in (A.6) is zero by definition, so that if the first curly bracket did not vary in space in this layer, then the area integral of (A.6) would be zero. Because of spatial correlations between the two curly brackets, the area average of (A.6) can be non-zero. However, if we compare the terms involving diapycnal diffusivities in our main equation (2.20) and in (A.6) we see that in (2.20) this term involves the sum of these diffusivities multiplied by the curvature of the $S - \Theta$ curve which is of one sign in this region, while in (A.6) we have the difference in the diffusivities multiplying a term whose area average is zero. Hence we do not expect the neglect of (A.6) to be important. If we leave one of $\Delta S$ or $\Delta \Theta$ as given by (2.19) and vary the other one then one could achieve a zero value for the area integral of (A.6). This is the value of $\Delta S/\Delta \Theta$ that we need to choose so as to ensure that (A.2) and (A.3) do not contribute to (2.20), although we have no way of knowing exactly this value of $\Delta S/\Delta \Theta$. We will perform our method with three different values of $\Delta S/\Delta \Theta$ which span a variation of 20% and will find that the results are not sensitive to this uncertainty.

In summary, we believe that the errors involved in ignoring the contributions of the three terms (A.1) - (A.3) to the equation (2.20) can be made to be small and will not materially affect the results. Ignoring (A.1) causes us to suffer some uncertainty in $F$ and hence in the effective lateral advection $FQ$ along the layer, while the effect of ignoring the terms in (A.2) and (A.3) will be shown to be small by empirically varying the ratio $\Delta S/\Delta \Theta$ that we use in (2.20) away from the value set by (2.19).
Appendix B

Appendix: Chapter 3

B.1 Derivation of the Water Mass Equation

The conservation equations for salinity $S$ and conservative temperature $\Theta$ in steady state are

\begin{align}
(\nabla h/\overline{h}) \cdot \nabla \gamma S + w^\gamma S_z &= \overline{h}^{-1} \nabla \gamma \cdot (\overline{h} K \nabla \gamma S) + (DS)_z, \\
(\nabla h/\overline{h}) \cdot \nabla \gamma \Theta + w^\gamma \Theta_z &= \overline{h}^{-1} \nabla \gamma \cdot (\overline{h} K \nabla \gamma \Theta) + (D\Theta)_z.
\end{align}

(B.1)  
(B.2)

These equations have been written in the advective form and with respect to neutral density ($\gamma_n$) layers (Jackett and McDougall, 1997) so that $w^\gamma$ is the vertical velocity through neutral density surfaces (i.e. the diapycnal velocity component of the vertical velocity) and $(\nabla h/\overline{h})$ is the thickness-weighted horizontal velocity obtained by temporally averaging the horizontal volume transport between closely-spaced neutral density surfaces. Similarly, the salinity $S$ and conservative temperature $\Theta$ in (B.1) and (B.2) are the thickness-weighted values obtained by averaging between closely spaced pairs of neutral density surfaces (McDougall and McIntosh, 2001). In these equations $\overline{h}$ is the mean thickness between two closely-spaced neutral density surfaces. The mixing processes that appear on the right-
hand sides are simply along-isopycnal mixing of passive tracers (with coefficient $K$) along the density layer and vertical small-scale turbulent mixing (with coefficient, $D$). We have not included double-diffusive convection or double-diffusive interleaving. We define the cross-contour direction as $\nabla_\gamma S / |\nabla_\gamma S| \equiv \nabla_\gamma \Theta / |\nabla_\gamma \Theta|$. Notably $n_\Theta = \nabla_\gamma S / |\nabla_\gamma S|$ as we are considering gradients on a neutral density layer. Multiplying (B.1) by $\Theta$ and (B.2) by $S$ gives

\[(\overline{v}/\overline{h}) \cdot n_\Theta |\nabla_\gamma S| \Theta_z + w^\gamma S_z \Theta_z = h^{-1} \nabla_\gamma S \cdot (\overline{h} K \nabla_\gamma S) \Theta_z + D S_z \Theta_z + D_z S_z \Theta_z,\]  
(B.3) 

\[(\overline{v}/\overline{h}) \cdot n_\Theta |\nabla_\gamma \Theta| S_z + w^\gamma \Theta_z S_z = h^{-1} \nabla_\gamma \Theta \cdot (\overline{h} K \nabla_\gamma \Theta) S_z + D \Theta_S S_z + D_z \Theta_z S_z.\]  
(B.4)

Subtracting (B.4) from (B.3), the $D_z$ and $w^\gamma$ terms are eliminated and we find

\[(\overline{v}/\overline{h}) \cdot n_\Theta = K/\lambda^\perp + D/\lambda^\gamma + \nabla_\gamma K \cdot n_\Theta\]  
(B.5)

where

\[
1/\lambda^\perp = \frac{\Theta_z \nabla_\gamma S \cdot (\overline{h} \nabla_\gamma S) - S_z \nabla_\gamma \Theta \cdot (\overline{h} \nabla_\gamma \Theta)}{\overline{h} (\Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta|)}
\]

\[
1/\lambda^\gamma = \frac{S_z \Theta_z - \Theta_z S_z}{\Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta|}
\]

The stability ratio is $R_\rho = \alpha \Theta_z / \beta S_z$. On a neutral density surface $\alpha/\beta = |\nabla_\gamma S|/|\nabla_\gamma \Theta|$, so $R_\rho \equiv |\nabla_\gamma S| \Theta_z / |\nabla_\gamma \Theta | S_z$. Hence neither $1/\lambda^\perp$ nor $1/\lambda^\gamma$ contain singularities if the water column is stably stratified (i.e. if $R_\rho \neq 1 \Rightarrow \Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta| \neq 0$)

We may separate the thickness gradient from $1/\lambda^\perp$ such that

\[
1/\lambda^\perp = \frac{\Theta_z \nabla_\gamma^2 S - S_z \nabla_\gamma^2 \Theta}{\Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta|} + \nabla_\gamma \log(h) \cdot n_\Theta\]  
(B.6)
In the case of a linear equation of state $\nabla_\gamma \alpha \equiv \nabla_\gamma \beta \equiv 0$, hence

\[
1/\lambda \perp = \frac{\nabla^2_\gamma \Theta(\Theta_z(\alpha/\beta) - S_z)}{|\nabla_\gamma \Theta|(\Theta_z(\alpha/\beta) - S_z)} + \nabla_\gamma \log(h) \cdot n_\Theta
\]

\[
= \frac{\nabla^2_\gamma \Theta}{|\nabla_\gamma \Theta|} + \nabla_\gamma \log(h) \cdot n_\Theta
\]

\[
\equiv \frac{\nabla^2_\gamma S}{|\nabla_\gamma S|} + \nabla_\gamma \log(h) \cdot n_\Theta
\]  

(B.7)

Given $\Theta^3 \frac{d^2 S}{d\Theta^2} = \Theta_z S_{zz} - S_z \Theta_{zz}$, $1/\lambda^\gamma$ may be written

\[
1/\lambda^\gamma = \frac{\Theta^2_z}{|\nabla_\gamma S|} \frac{R_\rho}{(R_\rho - 1)} \frac{d^2 S}{d\Theta^2}.
\]  

(B.8)
B.2 Derivation of the Density Equation

Multiplying (B.1) by $|\nabla_\gamma \Theta|$ and (B.2) by $|\nabla_\gamma S|$ and expanding the along-isopycnal mixing terms in both equations we have

\begin{align}
(\nabla_\gamma^h/\nabla_h) \cdot \mathbf{n}_\Theta |\nabla_\gamma S||\nabla_\gamma \Theta| + w^\gamma S_z |\nabla_\gamma S| &= h^{-1} \nabla_\gamma (\nabla_h K) \cdot \mathbf{n} |\nabla_\gamma S||\nabla_\gamma \Theta| + K \nabla^2_\gamma S |\nabla_\gamma \Theta| \\
&\quad + (DS_z)_z |\nabla_\gamma \Theta|, \quad \text{(B.9)} \\
(\nabla_\gamma^h/\nabla_h) \cdot \mathbf{n}_\Theta |\nabla_\gamma \Theta||\nabla_\gamma S| + w^\gamma \Theta_z |\nabla_\gamma S| &= h^{-1} \nabla_\gamma (\nabla_h K) \cdot \mathbf{n} |\nabla_\gamma \Theta||\nabla_\gamma S| + K \nabla^2_\gamma \Theta |\nabla_\gamma S| \\
&\quad + (D\Theta_z)_z |\nabla_\gamma S|. \quad \text{(B.10)}
\end{align}

Subtracting (B.9) from (B.10) the $\nabla_\gamma (hK)$ and $(\nabla_\gamma^h/\nabla_h) \cdot \mathbf{n}_\Theta$ terms are eliminated and we find

\begin{align}
w^\gamma &= K/\eta^\perp + D/\eta^\gamma + D_z \quad \text{(B.11)}
\end{align}

where

\begin{align}
1/\eta^\perp &= \frac{\nabla^2_\gamma \Theta |\nabla_\gamma S| - \nabla^2_\gamma S |\nabla_\gamma \Theta|}{\Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta|} \quad \text{(B.12)} \\
1/\eta^\gamma &= \frac{\Theta_{zz} |\nabla_\gamma S| - S_{zz} |\nabla_\gamma \Theta|}{\Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta|}. \quad \text{(B.13)}
\end{align}

Again, neither $1/\eta^\perp$ nor $1/\eta^\gamma$ are singular if the water column is stably stratified. If a linear equation of state is assumed such that $\rho_z \propto -\alpha \Theta_z + \beta S_z$, for $\alpha$ and $\beta$ constant, and again $\alpha/\beta = |\nabla_\gamma S|/|\nabla_\gamma \Theta|$, the vertical mixing term $1/\eta^\gamma$ becomes

\begin{align}
1/\eta^\gamma &= \frac{\alpha \Theta_{zz} - \beta S_{zz}}{\alpha \Theta_z - \beta S_z} \\
&= \rho_{zz}/\rho_z. \quad \text{(B.14)}
\end{align}

Also, as $\nabla_\gamma \alpha \equiv \nabla_\gamma \beta \equiv 0$, the along-isopycnal mixing term $1/\eta^\perp$ becomes
\begin{align*}
1/\eta^\perp &= \frac{\nabla^2_\gamma \Theta |\nabla_\gamma S| - (\alpha/\beta) \nabla^2_\gamma \Theta |\nabla_\gamma \Theta|}{\Theta_\perp |\nabla_\gamma S| - S_\perp |\nabla_\gamma \Theta|} \\
&= 0. \tag{B.15}
\end{align*}

Hence for a linear equation of state, the vertical velocity through isopycnal surfaces simplifies to \( w^\perp = D\rho_{zz}/\rho_z + D_z \).
Appendix C

Appendix: Chapter 4

C.1 Derivation of the water mass equation

The conservation equations for salinity $S$ and conservative temperature $\Theta$ are

$$
S_t|_\gamma + (\overline{v h}/h) \cdot \nabla_\gamma S + w^\gamma S_z = h^{-1} \nabla_\gamma \cdot (hK \nabla_\gamma S) + (DS_z)_z, \quad (C.1)
$$

$$
\Theta_t|_\gamma + (\overline{v h}/h) \cdot \nabla_\gamma \Theta + w^\gamma \Theta_z = h^{-1} \nabla_\gamma \cdot (hK \nabla_\gamma \Theta) + (D\Theta_z)_z. \quad (C.2)
$$

These equations have been written in the advective form and with respect to neutral density ($\gamma^h$) so that $w^\gamma$ is the vertical velocity through isopycnals (i.e. the diapycnal velocity), $\mathbf{v}$ is the horizontal velocity and $(\overline{v h}/h)$ is the thickness-weighted horizontal velocity. The thickness is $h = \Delta \rho / \rho_z$ for some small $\Delta \rho$. The conservative temperature $\Theta$ and salinity $S$ in (C.1) and (C.2) are the thickness-weighted values. The mixing processes that appear on the right-hand sides are simply along-isopycnal mixing of passive tracers (with coefficient $K$) along the density surfaces and vertical small-scale turbulent mixing (with coefficient, $D$). We have not included double-diffusive convection or double-diffusive interleaving. We define the cross-contour direction as $\mathbf{n} = \nabla_\gamma S/|\nabla_\gamma S|$. Notably $\nabla_\gamma S/|\nabla_\gamma S| \equiv \nabla_\gamma \Theta/|\nabla_\gamma \Theta|$ as we are considering gradients on a neutral density surface. Multiplying (C.1) by
Θ_{z} and (C.2) by \( S \) gives

\[
S_{t}|_{\gamma} \Theta_{z} + (\nabla \Theta / h) \cdot \mathbf{n} \frac{\nabla_{\gamma} S}{\Theta_{z}} + w^{7} S_{z} \Theta_{z} = \\
h^{-1} \nabla_{\gamma} \cdot (h K \nabla_{\gamma} S) \Theta_{z} + D S_{zz} \Theta_{z} + D_{z} S_{z} \Theta_{z}, \tag{C.3}
\]

\[
\Theta_{t}|_{\gamma} S_{z} + (\nabla / h) \cdot \mathbf{n} \frac{\nabla_{\gamma} \Theta}{S_{z}} + w^{7} \Theta_{z} S_{z} = \\
h^{-1} \nabla_{\gamma} \cdot (h K \nabla_{\gamma} \Theta) S_{z} + D \Theta_{zz} S_{z} + D_{z} \Theta_{z} S_{z}. \tag{C.4}
\]

Subtracting (C.4) from (C.3) and dividing through by \((\nabla_{\gamma} S \Theta_{z} - \nabla_{\gamma} \Theta S_{z})\), the \( D_{z} \) and \( w \) terms are eliminated and we find

\[
(\nabla \Theta / h) \cdot \mathbf{n} = K / \lambda^{\perp} + D / \lambda^{\gamma} + \nabla_{\gamma} K \cdot \mathbf{n} - \Theta_{t}|_{\gamma} / \nabla_{\gamma} \Theta \tag{C.5}
\]

where

\[
1 / \lambda^{\perp} = \frac{\Theta_{z} \nabla_{\gamma} \cdot (h \nabla_{\gamma} S) - S_{z} \nabla_{\gamma} \cdot (h \nabla_{\gamma} \Theta)}{h (\Theta_{z} |\nabla_{\gamma} S| - S_{z} |\nabla_{\gamma} \Theta|)}
\]

\[
1 / \lambda^{\gamma} = \frac{S_{zz} \Theta_{z} - \Theta_{zz} S_{z}}{\Theta_{z} |\nabla_{\gamma} S| - S_{z} |\nabla_{\gamma} \Theta|}
\]

Above we have noted that \( \alpha / \beta = |\nabla_{\gamma} S| / |\nabla_{\gamma} \Theta| = (S_{t} / \Theta_{t})|_{\gamma} \), simplifying the unsteady term in (C.5).

The stability ratio is \( R_{\rho} = \alpha \Theta_{z} / \beta S_{z} \) so that \( R_{\rho} \equiv |\nabla_{\gamma} S| / |\nabla_{\gamma} \Theta| S_{z} \). Hence neither \( 1 / \lambda^{\perp} \) nor \( 1 / \lambda^{\gamma} \) contain singularities if the water column is stably stratified (i.e. if \( R_{\rho} \neq 1 \Rightarrow \Theta_{z} |\nabla_{\gamma} S| - S_{z} |\nabla_{\gamma} \Theta| \neq 0 \)).

We may remove a thickness gradient term from \( 1 / \lambda^{\perp} \) such that

\[
1 / \lambda^{\perp} = \frac{\Theta_{z} \nabla^{2}_{\gamma} S - S_{z} \nabla^{2}_{\gamma} \Theta}{\Theta_{z} |\nabla_{\gamma} S| - S_{z} |\nabla_{\gamma} \Theta|} + \nabla_{\gamma} \log(h) \cdot \mathbf{n} \tag{C.6}
\]

Note here that \( \nabla_{\gamma} \log(h) \cdot \mathbf{n} \) does not represent the bolus transport. In the case of a linear equation of state \( \nabla_{\gamma} \alpha \equiv \nabla_{\gamma} \beta \equiv 0 \), hence \( 1 / \lambda^{\perp} \) simplifies to

\[
1 / \lambda^{\perp}_{\text{linear}} = \frac{\nabla^{2}_{\gamma} \Theta (\Theta_{z} (\alpha / \beta) - S_{z})}{|\nabla_{\gamma} \Theta| (\Theta_{z} (\alpha / \beta) - S_{z})} + \nabla_{\gamma} \log(h) \cdot \mathbf{n}
\]

\[
= \frac{\nabla^{2}_{\gamma} \Theta}{|\nabla_{\gamma} \Theta|} + \nabla_{\gamma} \log(h) \cdot \mathbf{n}
\]

\[
= \frac{\nabla^{2}_{\gamma} S}{|\nabla_{\gamma} S|} + \nabla_{\gamma} \log(h) \cdot \mathbf{n} \tag{C.7}
\]

156
Given the equality
\[ A_3 \frac{d^2 B}{dA^2} |_{x,y} = A_2 B_{zz} - B_2 A_{zz}, \quad (C.8) \]
where \(|_{x,y}\) signifies constant latitude and longitude, \(1/\lambda^7\) may be written
\[ 1/\lambda^7 = \frac{\Theta^2}{|\nabla \gamma S|} \frac{R_\rho}{(R_\rho - 1)} \frac{d^2 S}{d\Theta^2} |_{x,y}. \quad (C.9) \]

The thickness-weighted mean velocity, \(\overline{v h}/h\), may be represented as
\[ \overline{v h}/h = \nabla \gamma, \log(h). \quad (C.10) \]

Above, an additional mixing term may be added, \(K_{PV}(\beta/f) j\), which relates the bolus velocity, \(v^*\), to gradients of potential vorticity, \(PV\), rather than thickness, \(h\) (McDougall, 1991a).

The cross-contour velocity \(\nabla \cdot n\) may now be represented
\[ \nabla \cdot n = \frac{K}{\lambda^h} + D/\lambda^7 + K_{PV}/\lambda^h + \nabla_\gamma K \cdot n - \Theta |_{\gamma}/\nabla_\gamma \Theta \quad (C.11) \]
where \(1/\lambda^h = \nabla_\gamma \log(h) \cdot n\).
C.2 Derivation of the density equation

Multiplying (C.1) by $|\nabla_\gamma \Theta|$ and (C.2) by $|\nabla_\gamma S|$ and expanding the along-isopycnal mixing terms in both equations we have

\[
S_t \nabla_\gamma \Theta + (\nabla h/h) \cdot n |\nabla_\gamma S| |\nabla_\gamma \Theta| + w^7 S_z |\nabla_\gamma \Theta| = h^{-1} \nabla_\gamma (hK) \cdot n |\nabla_\gamma S| |\nabla_\gamma \Theta| + K \nabla^2_\gamma S |\nabla_\gamma \Theta| + (DS_z)_z |\nabla_\gamma \Theta|,
\]

(C.12)

\[
\Theta_t \nabla_\gamma S + (\nabla h/h) \cdot n |\nabla_\gamma \Theta| |\nabla_\gamma S| + w^7 \nabla_\gamma \Theta |\nabla_\gamma S| = h^{-1} \nabla_\gamma (hK) \cdot n |\nabla_\gamma S| |\nabla_\gamma \Theta| + K \nabla^2_\gamma S |\nabla_\gamma \Theta| + (D\Theta_z)_z |\nabla_\gamma S|,
\]

(C.13)

Subtracting (C.12) from (C.13) the $\nabla_\gamma (hK)$, $(\nabla h/h) \cdot n$, $S_t$ and $\Theta_t$ terms are eliminated (recalling that $|\nabla_\gamma S|/|\nabla_\gamma \Theta| = S_t/\Theta_t$) and we find

\[
w^7 = K/\eta^\perp + D/\eta^\gamma + D_z \tag{C.14}
\]

where

\[
1/\eta^\perp = \frac{\nabla^2_\gamma S |\nabla_\gamma S| - \nabla^2_\gamma S |\nabla_\gamma \Theta|}{\Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta|} \tag{C.15}
\]

\[
1/\eta^\gamma = \frac{\Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta|}{\Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta|}. \tag{C.16}
\]

Again, neither $1/\eta^\perp$ nor $1/\eta^\gamma$ are singular if the water column is stably stratified.

If a linear equation of state is assumed such that $\rho_z \propto -\alpha \Theta_z + \beta S_z$, for $\alpha$ and $\beta$ constant, and again $\alpha/\beta = |\nabla_\gamma S|/|\nabla_\gamma \Theta|$, the vertical mixing term $1/\eta^\gamma$ becomes

\[
1/\eta^\gamma_{\text{Linear}} = \frac{\alpha \Theta_{zz} - \beta S_{zz}}{\alpha \Theta_z - \beta S_z} = \rho_{zz}/\rho_z. \tag{C.17}
\]

Also, as $\nabla_\gamma \alpha \equiv \nabla_\gamma \beta \equiv 0$, the along-isopycnal mixing term $1/\eta^\perp$ becomes

\[
1/\eta^\perp_{\text{Linear}} = \frac{\nabla^2_\gamma \Theta |\nabla_\gamma S| - (\alpha/\beta) \nabla^2_\gamma \Theta |\nabla_\gamma \Theta|}{\Theta_z |\nabla_\gamma S| - S_z |\nabla_\gamma \Theta|} = 0. \tag{C.18}
\]
Hence for a linear equation of state, the vertical velocity through isopycnals simplifies to $w^* = D \rho_{zz}/\rho_z + D_z$. 
C.3 Derivation of an isopycnal advective-diffusive balance equation for a conservative tracer

We will now derive an equation which relates advection down the isopycnal gradient of a conservative tracer, $C$, to mixing. Consider the steady state conservation equation for $C$ in an incompressible ocean, distant from sources and sinks

$$C_t|_\gamma + \left( \nabla \kappa / h \right) \cdot \nabla_\gamma C + w^\cdot C_z = h^{-1} \nabla_\gamma \cdot \left( h K \nabla_\gamma C \right) + (DC_z)_z. \quad (C.19)$$

Here, $C_t|_\gamma$ represents the slow ‘trend’ in $C$ and not the variability that is encompassed in the mixing coefficients $K$ and $D$. Substituting (4.17) and expanding the right hand side, (C.19) becomes

$$C_t|_\gamma + \left( \nabla \kappa / h \right) \cdot \nabla_\gamma C + KC_z / \eta^\perp + DC_z / \eta^\perp + D_z C_z = h^{-1} \nabla_\gamma \cdot \left( h K \nabla_\gamma C \right) + DC_{zz} + D_z C_z. \quad (C.20)$$

Clearly the last terms on the left and right hand sides of (C.20) cancel. Expanding the mean and bolus velocities, grouping the along isopycnal and diapycnal mixing terms and dividing through by the absolute tracer gradient, $|\nabla_\gamma C|$, (C.20) may thus be written

$$\mathbf{v} \cdot \mathbf{n}_C = D/\lambda_C^\perp + K/\lambda_C^\perp + K_{PV} / \lambda_C^K + \nabla_\gamma K \cdot \mathbf{n}_C - C_t|_\gamma / |\nabla_\gamma C|. \quad (C.21)$$

where

$$1/\lambda_C^\perp = \frac{\nabla_\gamma \cdot (h \nabla_\gamma C) }{h |\nabla_\gamma C|} - \frac{C_z}{|\nabla_\gamma C|} \frac{1}{\eta^\perp}, \quad (C.22)$$

$$1/\lambda_C^K = \frac{C_{zz}}{|\nabla_\gamma C|} - \frac{C_z}{|\nabla_\gamma C|} \frac{1}{\eta^K}, \quad (C.23)$$
and

\[ \frac{1}{\lambda^h_C} = \nabla \gamma \log(h) \cdot n_C \]  \hspace{1cm} (C.24)

Above, \( n_C \) is the unit vector in the direction of the along-isopycnal gradient of \( C \) (i.e. \( n_C = \nabla \gamma C / |\nabla \gamma C| \)).

Assuming a linear equation of state \( 1/\eta^h_{\text{Linear}} = 0 \) and \( 1/\eta^l_{\text{Linear}} = \rho_z/\rho \).

Hence

\[ \frac{1}{\lambda^h_{C-\text{Linear}}} = \frac{\nabla \gamma \cdot (h \nabla \gamma C)}{|\nabla \gamma C|} = \frac{\nabla^2 \gamma C}{|\nabla \gamma C|} + \nabla \gamma \log(h) \cdot n_C. \]  \hspace{1cm} (C.25)

and

\[ \frac{1}{\lambda^l_{C-\text{Linear}}} = \frac{C_{zz}}{|\nabla \gamma C|} - \frac{C_z \rho_{zz}}{|\nabla \gamma C| \rho_z} \]

\[ = \frac{1}{\rho_z |\nabla \gamma C|} (\rho_z C_{zz} - C_z \rho_{zz}). \]  \hspace{1cm} (C.26)

Given (C.8), the vertical mixing scale length, assuming a linear equation of state, may be written

\[ \frac{1}{\lambda^l_{C-\text{Linear}}} = \frac{\rho_z^2}{|\nabla \gamma C|} \frac{d^2 C}{d\rho^2}|_{x,y}. \]  \hspace{1cm} (C.27)

So the isopycnal advection-diffusive balance equation for the tracer \( C \) is analogous to the water mass equation [\((4.1)\) or \((4.6)\)]. As such, along-isopycnal mixing enters through the ratio of along-isopycnal curvature to along-isopycnal gradient, \( \nabla^2 \gamma C / |\nabla \gamma C| \), and vertical mixing largely enters through the curvature of the tracer with respect to density, \( d^2 C / d\rho^2 \) (note: if a linear equation of state is assumed, then \( d^2 \Theta / dS^2 \) can be written in terms of the curvature of either \( \Theta \) or \( S \) with respect to \( \rho \)).

As (C.21) is written in terms of a conservative tracer \( C \), it also holds for \( \Theta \) and \( S \). That is, substituting \( \Theta \) or \( S \) for \( C \) in (C.21), one yields the water mass equation in terms of the mean velocity \((4.6)\).
C.4 Reference level streamfunction at a fixed pressure

Here we formulate the difference in geostrophic streamfunction on a neutral density surfaces $\Delta \Psi^\gamma$, relative to a difference on a pressure surface. When solving for a streamfunction near topography or outcropping regions, it is cumbersome to use a density surface which is non-existent at some latitudes and longitudes. As discussed in McDougall and Klocker (Submitted), errors arise in defining a streamfunction on a density surface over basin to global scales. Although these errors can be reduced by considering appropriate specific volume anomaly and correction terms, it is likely to be best to reference to a constant pressure, at or near the mean sea level.

Choosing the reference level to be a pressure surface, (4.11) simplifies to

$$
\Delta \Psi^{p_0} - \Delta \Psi^\gamma = - \left[ (p - \tilde{p}) \delta - \int_{p_0}^{p} \delta dp' \right]_{(x_2,y_2)}^{(x_1,y_1)} \quad (C.28)
$$

For each equation of type (C.28) the choice of $\Theta_{const}$ and $S_{const}$ and hence the nature of $\delta$ is arbitrary. For a specific tracer-contour of temperature $\Theta_{contour}$ and salinity $S_{contour}$ respectively, a specific volume anomaly variable $\delta_{contour}$ may be defined as

$$
\delta_{contour} = \frac{1}{\rho(\Theta, S, P)} - \frac{1}{\rho(\Theta_{contour}, S_{contour}, P)}. \quad (C.29)
$$

Along such a contour $\delta_{contour} = 0$ and hence (C.28) simplifies further to

$$
\Delta \Psi^{p_0} - \Delta \Psi^\gamma = \left[ \int_{p_0}^{p} \delta_{contour} dp' \right]_{(x_2,y_2)}^{(x_1,y_1)}. \quad (C.30)
$$

Equation (C.30) offers a possible increase in accuracy but requires the redefinition of a specific volume anomaly variable $\delta_{contour}$ for each individual tracer-contour.
C.5 Defining tracer-contours

Here we describe how tracer-contours have been defined from fields of the Hallberg Isopycnal Model (HIM). The North Pacific region (210°E - 230°E and 10°N - 30°N) is used as an example.

We start with 3 dimensional fields of $\theta$, $S$, $p$, $h$ and the length scales $\lambda_\perp$, $\lambda_\gamma$ and $\lambda_h$. These variables are defined between the latitudes and longitudes of the region and for all definable isopycnals. The data are spaced by approximately one degree of latitude and longitude. We do not consider isopycnals which interact with the ocean floor or those completely above 500m depth. The North Pacific region has 19 longitudinal points by 21 latitudinal points by 15 isopycnals ($N = 15$).

From each point on the boundary, on each isopycnal a temperature contour is followed until it exits the domain. Doing this for the North Pacific region results in 78 contours on each isopycnal with a total of 1170 contours (i.e. $78 \times N$). Contours are removed if they are less than two degrees long, have an average pressure less than 500 db and/or have a minimum temperature gradient less than $10^8$ K m$^{-1}$ along them.

In the inversion, contour ends are linearly interpolated onto a set of points around the domain. A streamfunction is solved for at these points on a reference level. If no contour starts or ends close to a given point on the boundary, on any level in the vertical, no solution can be found for the streamfunction difference at that point. An example of this is the north-west corner of the North Pacific region. As the temperature gradient is to the south-west on each isopycnal, no contour starts or ends near the north-west corner. The corner point is thus isolated from the inversion.

In order to avoid this problem we solve for the streamfunction on a set of points around the domain such that each interacts with at least 10 contours (10 is an arbitrary choice). In practice this is done by moving around the boundary points removing each point which is associated with less than 10 contours. The contours
that the original point was associated with are then assigned to the remaining points and the procedure is repeated. For the North Pacific Region 56 boundary grid points are defined \((L = 56)\). The streamfunction is solved for on these points.

The data are at 1° resolution. Thus tracer-contours should be spaced by about 1°. As the contours are defined at every point on the boundary of the domain, the spacing of tracer-contours is about 0.5°. Hence, we expect the effective number of contours to be half those defined. That is, half the number of boundary grid points (28 for the North Pacific region).
Appendix D

Appendix: Chapter 5

D.1 Uncertainties and Sensitivity Analysis

Mixing parameters and diapycnal fluxes determined using inverse methods are known to be highly sensitive to the model details and the data sources used (Tziperman, 1988; Lux et al., 2001; St.Laurent et al., 2001). It is pertinent to test whether the results presented here, for the vertical and along-isopycnal mixing coefficients, are robust in terms of their sensitivity to changes to key assumptions and parameterizations, equation weighting and data sources. In order to test the robustness of our results, we conduct a series of sensitivity tests, in each case assessing the quantitative and qualitative changes made to the vertical and along-isopycnal mixing rates inferred. We show that the choice of equation weighting, parameterizations and data source has a largely negligible effect, the later being the most influential.

In this study, the bolus velocity \( v^* \) is represented as a down gradient diffusion of the thickness of isopycnal layers \( h \). The rate of diffusion of thickness is controlled by the mixing coefficient \( K_{PV} \) such that \( v^*h = -K_{PV}\nabla\gamma(h) \). In the analysis presented in section 5.4, we assume the coefficient for thickness is the same as that for temperature and salinity (i.e. \( K_{PV} = K \)). These choices are arbitrary and one could equally mix potential vorticity, \( f/h \), instead of \( h \), and/or
allow $K$ and $K_{PV}$ to vary independently. To test the influence of this parameter in
the eastern North Atlantic we remove the thickness term from the inverse method.
The effect on the mixing coefficients and their standard error is negligible (dotted
line; Fig.D.1a and Fig.D.1b). This suggests that the choice of parameterization is
inconsequential in this region and the method would be unable to diagnose $K_{PV}$
independently here. However, the bolus velocity is likely to be of leading order
importance in the Southern Ocean and in regions of steeply sloping isopycnals
(Gent et al., 1995; McDougall and McIntosh, 2001).

Here a statistically steady balance is assumed such that the trend in $\Theta$ and $S$
on isopycnal surfaces, is negligible when compared to the mean advection, along-
isopycnal mixing and diapycnal mixing terms (i.e. we have assumed $\Theta_t|_{\gamma} = 0$). In
the climatology of DW09, trends have been quantified and are discussed in DW09.
We test the importance of the trend term by retaining the $\Theta_t|_{\gamma}/\nabla_{\gamma}\Theta$ term in
equation (5.1). This term effectively represents the gradual movement of temper-
ature contours on isopycnals. The effect on the inverse calculation in the eastern
North Atlantic is negligible (dashed line; Figs. D.1a and D.1b) suggesting a steady
state balance is a valid assumption in the region considered. This assumption may
break down in regions where strong changes are detected on isopycnals and/or
along-isopycnal gradients are particularly low.

In ZMS09 and here, the relative weighting of the contour equations (5.3) versus
the large-scale conservation, or box equations (5.4) is chosen by minimizing the
condition number of the matrix, $A$, in the system $Ax = b$. It is conventional, in
box inverse modeling studies, to weight equations by an a priori error estimate. As
there is significant uncertainty in the estimation of prior error, the choice is still
largely arbitrary. To test the sensitivity of our mixing estimates, to the weighting of
contour versus box equations, we simply multiply and divide by two, the weighting
given by the minimum condition number criterion. This change in the weights gives
a negligible impact on the along-isopycnal mixing coefficient profile in the upper
2000m and the vertical mixing profile in the upper 1000m (Figs. D.1c and D.1d).
Figure D.1: Vertical mixing coefficient, $D$, (left) and along-isopycnal mixing coefficient, $K$, (right) as determined using the tracer-contour inverse method. Black lines represent the estimated value while the area between the grey lines represent the standard error. a and b: for the standard settings (solid line), no bolus velocity (dotted line) and including an unsteady term (dashed line). c and d: for the standard settings (solid line), increasing the weight on the contour equations (dashed line) and increasing the weight on the box equations (dotted line). e and f: as in c and d but using the climatology of GK.
There is a non negligible effect on the vertical mixing coefficient, $D$, in the 1000 m - 2000 m depth range, although they show an increase with depth for all choices of weights. There is a small amount of sensitivity to the choice of reference level, partly because this choice effects the magnitude of $b$, which in turn effects the relative weights of each equation.

The scale lengths: $\lambda_\gamma$, $\lambda_\perp$ and scale heights: $\eta_\gamma$, $\eta_\perp$ are functions of the curvatures and gradients of temperature and salinity determined from the climatology. It is likely that the way in which a climatology is constructed and the amount and quality of data which goes into it, will have a bearing on the length scales and hence the resultant mixing coefficients determined by the tracer-contour inverse method. To test the sensitivity of our results to these differences, we redo the inversion using the climatology of (Gouretski and Koltermann, 2004, hereafter GK). The climatology of GK is constructed using data from the WOCE and pre-WOCE periods and hence contains no data from Argo floats. Both GK and DW09 are averaged on neutral density surfaces, although using different approaches.

We have mapped the GK climatology onto neutral-density surfaces ($\gamma^N$, Jackett and McDougall, 1997). We interpolate onto the same grid as DW09 so that a clear comparison between the two can be made between the two. The velocity and mixing coefficients generated by the tracer-contour inverse method are generally more noisy when using the GK climatology. When using GK, the results are more sensitive to the proximity of the analysis to the mixed layer. For this reason we avoid using data below average depths of 250m. The vertical mixing coefficient, inferred using GK, is weak, $O(10^{-5} \text{ m}^2 \text{ s}^{-1})$, through all of the water column increasing somewhat in the 1000 m - 2000 m depth range (Fig.D.1e). The along-isopycnal mixing coefficient, $K$, inferred using GK, reduces from highs of $O(1000 \text{ m}^2 \text{ s}^{-1})$ in the upper 500 m of the water column, to $O(100 \text{ m}^2 \text{ s}^{-1})$ below 1000 m (Fig.D.1f). These results are very similar to those derived using DW09 and earlier estimates summarized in Figs. 5.3 and 5.4.
D.2 Mixing lengths, heights and thermal wind

The following scale lengths, dependent on the mean hydrography, are derived
in ZMS09 from linear combinations of the Θ and S conservation equations in
isopycnal coordinates.

\[
\begin{align*}
1/\lambda_\perp &= \frac{\Theta_z \nabla \cdot (h \nabla \gamma S) - S_z \nabla \cdot (h \nabla \gamma \Theta)}{h(\Theta_z |\nabla \gamma S| - S_z |\nabla \gamma \Theta|)} \\
1/\lambda_\gamma &= \frac{S_{zz} \Theta_z - \Theta_{zz} S_z}{\Theta_z |\nabla \gamma S| - S_z |\nabla \gamma \Theta|} \\
1/\lambda_h &= \nabla_\gamma \log(h) \cdot n \\
1/\eta_\perp &= \frac{\nabla^2 \Theta |\nabla \gamma S| - \nabla^2 S |\nabla \gamma \Theta|}{|\nabla \gamma S| \Theta_z - |\nabla \gamma \Theta| S_z} \\
1/\eta_\gamma &= \frac{\Theta_{zz} |\nabla \gamma S| - S_{zz} |\nabla \gamma \Theta|}{|\nabla \gamma S| \Theta_z - |\nabla \gamma \Theta| S_z}.
\end{align*}
\]

The following equivalent form of the thermal wind equation, in finite difference
form and in terms of a reference level streamfunction, is used in this study

\[
\int_i^{i+1} (\mathbf{v}^p_0 - \mathbf{v}) \cdot \mathbf{m} \, dx = \\
fi+1/2 [\Psi^p_0 - \Psi^\gamma]_{i+1}^{i+1} = -fi+1/2 \left[ p - pi+1/2 \right] \delta(p) - \int_{p_0}^{p} \delta(p') dp' \right]_{i+1}^{i+1}.
\]

(D.1)

In (D.1) \(i\) and \(i + 1\) are closely spaced ‘casts’ moving clockwise (outwards to the
left) along a bounding ‘section’ of the domain (Fig. 5.2). Values at the midpoint
between cast \(i\) and \(i + 1\) on the isopycnal \(\gamma\) are given the subscript \(i + 1/2\), for
example the Coriolis frequency \(fi+1/2\) and pressure \(pi+1/2\). The specific volume
anomaly is \(\delta(p)\), with reference values 0°C and 35 respectively. In (5.3), \(\delta_{\text{contour}}(p)\)
is used, where the reference values are the conservative temperature and salinity
of each contour on each isopycnal. Some improved accuracy in (D.1) may be
possible with the use of a more locally referenced form of \(\delta\) and additional terms
((McDougall and Klocker, Submitted)), although the difference is negligible in this
study as isopycnals are not steeply sloped.
For $L$ casts, the lateral advection terms in (5.4) are then

$$\oint (v^0 - v) \cdot m h C dx = - \sum_{i=1}^{L} f_{i+1/2} h_{i+1/2} C_{i+1/2} \left[ \left\{ p - p_{i+1/2} \right\} \delta(p) - \int_{p_0}^{p} \delta(p') dp' \right]_{i}^{i+1} \quad (D.2)$$

$$\oint (v^0) \cdot m h C dx = \sum_{i=1}^{L} f_{i+1/2} h_{i+1/2} C_{i+1/2} \left[ \Psi p_0 \right]_{i}^{i+1} \quad (D.3)$$

where $N + 1 = 1$. 

170
Bibliography


180


