A dynamic, embedded Lagrangian model for ocean climate models. Part I: Theory and implementation

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\textbf{A B S T R A C T}

A framework for embedding a Lagrangian model within ocean climate models that employ horizontal Eulerian grids is presented. The embedded Lagrangian model can be used to explicitly represent processes that are at the subgrid scale to the Eulerian model. The framework is applied to open ocean deep convection and gravity driven downslope flows, both of which are subgrid-scale in the present generation of level coordinate ocean climate models. In order to apply the embedded Lagrangian framework to these processes, it is necessary to partition the mass and momentum of the model into an Eulerian system and a Lagrangian system. This partitioning allows the Lagrangian model to transport seawater using a more appropriate set of dynamics.

A number of schemes suitable for implementation in the embedded Lagrangian model are derived. Two dynamically passive schemes are derived that emulate existing parameterisations and two dynamically active schemes are also derived that evolve Lagrangian parcels of water (termed “blobs”) according to a set of physical equations. Some details of the implementation into the Geophysical Fluid Dynamics Modular Ocean Model are also given. Finally, results are presented that show that the dynamically passive schemes are able to emulate their Eulerian counterparts to within roundoff error in idealised test cases.

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\section{1. Introduction}

For many applications of computational fluid dynamics, the greatest limiting factor is computational resources. The limitation of computational resources and the very large scale of the domain in global ocean climate modelling means that a number of climatically important processes are unresolved in the present generation of global scale climate models. There has thus been significant effort in the ocean model development community to formulate subgrid-scale (SGS) parameterisations for all such processes (e.g. see the recent review by Griffies et al. (2010)).

All global scale ocean models that are used for realistic coupled climate experiments utilise a fixed Eulerian grid in the horizontal. There are some models under development that are fully Lagrangian (e.g. Haertel and Randall, 2002), and there are others that have an adaptive mesh (e.g. Piggott et al., 2007), however, none of these other formulations have yet reached a stage where they are able to be used for realistic, coupled global scale studies. Lagrangian coordinates in the vertical (for example, isopycnal coordinate models) are finding much utility in ocean circulation studies (Bleck and Boudra, 1986; Meghann et al., 2010; Dunne et al., 2012). However, the most common class of ocean model used in realistic global scale studies remains the “level” coordinate model (which we take here to mean geopotential, \( z \), pressure and \( p \) coordinates). Henceforth, when we say “Eulerian model” we are referring to an ocean model that solves the primitive equations and has a horizontal grid that is fixed in time.

The fundamental equations of numerical ocean models are typically applied throughout the computational domain, with SGS parameterisations augmenting those equations. For global scale climate models the fundamental equations are the hydrostatic primitive equations. One variation on this idea is the so-called super-parameterisation, first applied in the oceanographic context by Campin et al. (2011). The super-parameterisation is a two dimensional non-hydrostatic model that is embedded into a hydrostatic model. The admission of non-hydrostatic dynamics is fundamentally different to the (hydrostatic) primitive equations of the main model and is well suited for the study of regions of deep convection. Another proposal by Duan et al. (2010) has been to use spatio-temporal filtering of the fully non-hydrostatic Boussinesq equations to better represent vertical motions without the computational cost of using the fully non-hydrostatic equations.

In this paper we propose a rather different means by which to admit dynamics of unresolved or poorly represented physics in
ocean climate models. To overcome the problems associated with applying a single dynamic regime over the entire computational domain, the mass of the system is partitioned and certain parts of the domain are treated with a different set of dynamics to the bulk of the system. Specifically, we provide a framework in which a Lagrangian model is embedded within the main Eulerian model. The Lagrangian parcels (which we shall refer to as “blobs”) generally follow a different set of dynamics to that of the main model. The proposed framework most resembles the cloud-in-cell method (Christiansen, 1973; Mohammadian and Marshall, 2010), however it is not restricted to two dimensional vorticity as is the case in the traditional cloud-in-cell methods. The iceberg model of Martin and Adcroft (2010) is a precedent for our proposed framework within the global scale ocean modelling community. The present framework generalises the ideas behind the iceberg model to be fully three dimensional and fully interactive with the Eulerian model.

Open ocean deep convection and gravity driven downslope flows are our main focus for applying the proposed framework. Both processes are essentially sinking plumes where a relatively narrow water mass is intruding into a much larger water mass. The blobs can thus be considered a Lagrangian discretisation of a plume (or group of plumes). Additionally, blobs can be formed when a certain condition is satisfied, meaning that if the condition is not met, a blob is not formed. This is particularly useful in the context of ocean convection, which is sporadic in both space and time.

This paper presents details of the theory and implementation of the embedded Lagrangian model with application to modelling open ocean deep convection and gravity driven downslope flows. A companion paper, Bates et al. (2012), hereafter BGE-II details the results of tests carried out in model configurations employing idealised bathymetry. This paper consists of the following sections:

- Section 2 outlines the requirements for an Eulerian model to admit an embedded Lagrangian model. Furthermore, it discusses the nature of the Lagrangian parcels and derives tracer mass, seawater mass and momentum budgets for the combined Eulerian and Lagrangian system. The mechanical energy budget is derived in Appendix B.
- Section 3 gives details of the implementation in the Modular Ocean Model (MOM; Bates et al. (2012)), including algorithms for evolving blob properties in time. The aspects of the Eulerian model that must be altered in order to admit the embedded Lagrangian model are also discussed. This section also details some simplifications to the momentum budget of the combined Eulerian and Lagrangian systems.
- Section 4 derives specific equations for open ocean deep convection and gravity currents to illustrate an application of the framework to primitive equation level models. Some “dynamically passive” schemes are presented, the aim of which is to emulate selected parameterisations that are already in existence in purely Eulerian models. Results from these schemes are briefly presented. Other “dynamically active” schemes are also presented, where the equations are derived and discussed.
- Section 5 is a discussion and summary of the theory and background presented, along with prospects for future research and applications.
- Appendix A includes a list of common symbols used throughout this paper.
- Appendix B derives the mechanical energy budget for the combined Eulerian and Lagrangian systems.
- Appendix C derives the blob size required by the NCon-like scheme (see Section 4.3.1).
- Appendix D derives expressions for grid cell variables that are for the combined Eulerian and Lagrangian systems. These are, the total thickness of a grid cell, the total tracer concentration and the total density.

2. Formulation

The purpose of this section is to outline the framework required to embed a Lagrangian model within an Eulerian model when the application is to represent open ocean deep convection and gravity driven downslope flows. Specifically, we discuss implications for the budgets of tracer mass, seawater mass, momentum and mechanical energy.

Large scale ocean models such as MOM are hydrostatic, and there is no overflow parameterisation for hydrostatic models that conserves momentum in the non-hydrostatic sense. Indeed, most existing overflow parameterisations modify only the tracer equation, as the key aim of such parameterisations is to improve water-mass properties as well as transport and formation/convolution rates that are impacted directly by water-masses (e.g. Kösters et al., 2005; Wu et al., 2007; Danabasoglu et al., 2010; Briegleb et al., 2010). In contrast to other schemes, the method proposed in the present paper considers a three-dimensional momentum equation for the Lagrangian blobs, with the vertical momentum equation including acceleration terms absent under the hydrostatic approximation. These vertical accelerations are essential for the vertical movement of the blobs downslope or within a convectional plume. However, we are not introducing a fully non-hydrostatic momentum budget for the Lagrangian submodel, since we base the pressure gradients acting on the blobs on the hydrostatic pressure obtained from the Eulerian parent model. Furthermore, by introducing a vertical acceleration to the Lagrangian sub-model, the sum of the Eulerian plus Lagrangian models no longer conserves momentum for the full system. This property is a consequence of allowing for the sub-model to be governed by more complete dynamics than the parent model. This is a property shared by the super-parameterisation scheme of Campin et al. (2011). Nonetheless, we insist on the conservation of scalar properties for the combined model. Conservation of mass, tracer, and heat is a consequence of the partitioning between the E and L systems used in our formulation, and such conservation is an essential property of any parameterisation employed by long-term ocean simulations such as those considered by climate models.

The embedding of a Lagrangian model within an Eulerian model is distinct from the Arbitrary Lagrangian–Eulerian approach (Hirt et al., 1972), in which a mesh may be fixed in the Eulerian sense or move in the Lagrangian sense. In our approach, the Eulerian model remains a true Eulerian model in the sense that it has a fixed mesh and prognostically solves for velocity and tracer concentration at a fixed point in space. The Lagrangian model is a true Lagrangian model in that it solves prognostically for the position and seawater mass and tracer content of a water parcel. To embed the Lagrangian model there must be a rigorous accounting processes in place to ensure that the properties of the combined system remain conservative. We therefore spend a significant amount of this paper deriving budgets for the individual and combined systems.

In order to admit the embedded Lagrangian model the Eulerian model must have, as part of its formulation:

1. Source terms for seawater mass, tracer mass, and momentum, 2. A free surface.

As shall be seen, without source terms in the Eulerian model’s formulation, there is no way to transfer seawater mass and tracer mass between the Eulerian model and the Lagrangian model. Without a free surface, there is no way for the Lagrangian blobs to move freely throughout the model domain, and, for the Eulerian model to respond to that movement of seawater mass. The rigid lid approximation (Bryan, 1969) dictates that the volume of water in a given water column must be conserved, thus, to move a blob from
one water column to another with a rigid lid would require that some return exchange of volume must be prescribed. Prescribing such an exchange would require a priori assumptions about that return flow, thereby negating one of the major advantages of the employment of the Lagrangian framework.

As discussed in Section 5, there are other potential applications of the embedded Lagrangian framework. Not all of the requirements for the schemes presented here are requirements for other applications of embedded Lagrangian schemes. Conversely, other schemes may impose other requirements.

2.1. What is a blob?

In the Lagrangian computational fluid dynamics literature, the term “blob” refers to a desingularised point vortex (Chorin, 1978). It is from the vortex blob literature that the embedded Lagrangian blob model drew its inspiration, and so, has been named accordingly. Here, a blob is considered to have all of its properties concentrated at a point. However, the Eulerian model can still “feel” finite properties, such as the mass and volume of a blob.

MOM, being an Eulerian model, has a fixed horizontal area for each grid cell. Thus, the thickness and density weighted thickness are important variables in the model’s formulation. In the context of the technology required by MOM to calculate the thickness of a grid cell it is required that the Lagrangian system’s contribution to the total thickness be calculated. So, even though all of the properties of a blob are concentrated at a point, they are also assumed to fill a certain volume. The relation between volume and mass can be found using density, which in turn is found using an equation of state for seawater (IOC et al., 2010). There is nothing intrinsic in the formulation that requires a blob to have a particular shape in order to calculate its volume, or other properties. One of the advantages of considering all of a blob’s properties to reside at a point in space is that a blob may only reside entirely within one grid cell, and may not partially reside in two or more. Such an approach significantly reduces the complexity of the algorithm required to implement the embedded Lagrangian model.

A consequence of having all blob properties (e.g. mass, volume, tracer content, etc.) defined at a single point is that no statement need be made regarding the shape of a blob in order for it to contribute to the total system’s properties. However, to parameterise certain interactions between the Eulerian system (hereafter E system) and the Lagrangian System (hereafter L system) we may be motivated to prescribe an assumed shape in order to formulate the parameterisation (e.g. entrainment). In the case of an entrainment parameterisation, we want to be able to relate the volume of a blob to the surface area of the blob in order to diagnose the amount of entrained material from an entrainment velocity. The precise details are parameterisation specific and shall be discussed in Section 3 for our implementations. Fig. 1 shows a schematic diagram which illustrates how blobs interact with the Eulerian model. It shows blobs being formed, blobs being destroyed, blobs moving through the Eulerian model and exchanging properties, as well as blobs changing dynamic regimes.

In the remainder of this section, we derive the budgets for seawater mass, tracer mass, and momentum for the E, L, and combined E + L systems, with Appendix B also presenting a discussion of mechanical energy. We are most concerned with maintaining exact conservation of the scalar properties mass and tracer content in our formulation, in particular when exchanges occur between the E and L sub-systems. In contrast, momentum conservation is not enforced. Momentum conservation is not enforced for two reasons. First, the L system generally satisfies a different momentum budget from the hydrostatic primitive equations of the E system. For example, the L system in the convection and overflow applications has a non-zero vertical acceleration. A conservative exchange of momentum between the E and L system is therefore precluded. Second, there are ambiguities associated with the exchange of horizontal momentum between the E and L system that require a choice to be made regarding the precise value of the exchange. At this stage of our development, we have made choices based on perceived simplicity of the algorithm. These choices are detailed in Section 3.3, with further research required to fully examine their fidelity.

2.2. Seawater mass and tracer mass

We insist that mass be conserved both locally and globally in a non-Boussinesq formulation. We shall only consider the non-Boussinesq case, with the volume conserving Boussinesq case regained by setting the density to a reference density, \( \rho_0 \), except when coupled to the acceleration due to gravity, \( g \) (Spiegel and Veronis, 1960). A corollary of seawater mass conservation is that tracer content must also be conserved locally and globally, and that there exists a compatibility condition between seawater mass conservation and tracer conservation (Griffies et al., 2001). We present here some rudimentary issues in order to carefully formulate the conservation laws in the presence of the embedded Lagrangian model. The partitioning of tracer mass and seawater mass may be written as

![Fig. 1](image-url)
(mc)_i = (mc)_k + (mc)_L,  
(1a)

m_T = m_e + m_L,  
(1b)

where m is seawater mass and C is tracer concentration. A subscript T indicates the combined system, or total system, E refers to the Eulerian model, and L refers to the Lagrangian model.

We define the tracer concentration as

\[ C_n = \frac{m_n}{V}, \]

where m is the mass of the seawater parcel being considered and \( m_n \) is the mass of the nth tracer constituent, each with units of kg. The density of the constituent is defined as

\[ \rho_n = \frac{m_n}{V}, \]

where V is the volume of the seawater parcel being considered, with units of m\(^3\) and \( \rho_n \) is the density of the constituent in kg m\(^{-3}\). Seawater density, \( \rho \) is thus defined as the sum of N constituents

\[ \rho = \sum_{n=1}^{N} \rho_n. \]

Some thermodynamic tracers may be defined in a similar fashion to material tracer concentration as in Eq. (2). In particular, the use of conservative temperature (McDougall, 2003) leads to the relation

\[ \Theta = \frac{\mathcal{H}}{m C_p^S}, \]

where \( \mathcal{H} \) is the potential enthalpy (J) of the seawater parcel being considered and \( C_p^S \) is the heat capacity, with units of J K\(^{-1}\) kg\(^{-1}\).

The continuous tracer conservation and seawater mass conservation equations are written as,

\[ \partial_t (\rho C) + \nabla \cdot (\rho \mathbf{V} C) = \rho S^{(C)} - \nabla \cdot (\rho \mathbf{F}), \]

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{V}) = \rho S^{(M)}, \]

where t is time, \( S^{(C)} \) is the tracer source (with units of tracer concentration per time) and \( S^{(M)} \) is the mass source (s\(^{-1}\)), \( \mathbf{V} = (u, v, w) \) is the three dimensional velocity (m s\(^{-1}\)) and \( \mathbf{F} \) is the subgrid scale tracer concentration flux (with units tracer concentration-metres per second). The compatibility condition between seawater mass conservation and tracer mass conservation means that we may obtain the mass Eq. (6b) from the tracer Eq. (6a) by setting the tracer concentration uniformly to a non-zero constant, noting that the net tracer concentration flux is zero under such circumstance, \( F(C = \text{const}) = 0 \). Any partitioning of the total system into submodels must respect this compatibility condition.

Application of Green’s Theorem (see, for instance, Section 3.31 of Aris (1962)) over a finite volume (e.g. a grid cell) allows us to compute a budget for that volume based on fluxes across the surface of that volume

\[ \partial_t \int_{V} \Psi dV = \int_{V} \nabla \Psi dV - \int_{S} \mathbf{n} \cdot \mathbf{J} \Psi dS, \]

where \( \Psi \) is an arbitrary conserved scalar (e.g. \( \rho C \)), \( \mathbf{J} \) is the flux due to advection plus subgrid scale processes, \( \mathbf{n} \) is the outward pointing normal vector to the surface being considered and S is the surface of the volume, V, being considered.

If we consider the volume, V, to be a grid cell and \( \Psi = \rho C \), then the finite volume conservation law (7) is given by

\[ \partial_t \int_{V} (\rho C) dV = \int_{V} \rho S_{(C)} dV - \int_{S} \mathbf{n} \cdot \mathbf{J} \Psi dS, \]

where \( \mathbf{J} \) is the subgrid scale flux.
the E system to the blob and (L2E) indicates the transfer of mass from the blob to the E system (noting that by convention \( \langle \text{dm/dt} \rangle_{\text{L2E}} \leq 0 \)).

New blobs or blobs that are destroyed also represent a transfer of tracer between the E system and the L system. New blobs represent a tracer sink for the E system and a tracer source for the L system of \((mc)_{\text{new}}\), while destroyed blobs represent a tracer source for E system and a tracer sink for the L system of \((mc)_{\text{dstry}}\).

Combining all of the above gives tracer source terms for the E and the L systems

\[
A \int_{E} \rho \mathcal{S}_{E}^{(C)} \, dz = - \sum_{q=1}^{Q_{\text{new}}} \frac{(mc)^{q}_{\text{new}}}{\Delta \tau} + \sum_{q=1}^{Q_{\text{dstry}}} \frac{(mc)^{q}_{\text{dstry}}}{\Delta \tau} - \sum_{q=1}^{Q_{\text{new}}} \left[ c_{q} \left( \frac{dm}{dt} \right)_{\text{blob}}^{q} + c_{q}^{D} \left( \frac{dm}{dt} \right)_{\text{blob}}^{q} \right] + A \int_{E} \rho \mathcal{S}_{E}^{(C)} \, dz \quad (11a)
\]

\[
A \int_{L} \rho \mathcal{S}_{L}^{(C)} \, dz = \sum_{q=1}^{Q_{\text{new}}} \frac{(mc)^{q}_{\text{new}}}{\Delta \tau} - \sum_{q=1}^{Q_{\text{dstry}}} \frac{(mc)^{q}_{\text{dstry}}}{\Delta \tau} + \sum_{q=1}^{Q_{\text{new}}} \left[c_{q} \left( \frac{dm}{dt} \right)_{\text{blob}}^{q} + c_{q}^{D} \left( \frac{dm}{dt} \right)_{\text{blob}}^{q} + m_{q}^{D} \left( \frac{dc}{dt} \right)^{q} \right] , \quad (11b)
\]

where we sum over all blobs, \( Q \), all new blobs, \( Q_{\text{new}} \), and all destroyed blobs, \( Q_{\text{dstry}} \), in the volume under consideration. \( \mathcal{S}_{E}^{(C)} \) is the E system sources that are not associated with the L system. Note that for brevity we have written

\[
A \equiv \int_{\Omega} dA , \quad (12)
\]

since the horizontal area of a grid cell is constant in time. Note that the sum of the exchange terms between the E and L sources vanishes in Eqs. (11), as it should since we are merely moving mass between the two submodels. The only source terms left are E system sources not involved in the exchange with the L system and non-material changes in tracer content for the L system,

\[
\int_{V_{L}} \rho \mathcal{S}_{L}^{(C)} \, dV = \int_{V_{L}} \rho \mathcal{S}_{L}^{(C)} \, dV + \sum_{q=1}^{Q_{\text{new}}} m_{q}^{D} \left( \frac{dc}{dt} \right)^{q} . \quad (13)
\]

Additionally, the compatibility condition is satisfied by Eqs. (11), which can be confirmed by setting \( C = \text{const} \), reducing the tracer source equations to seawater mass equations.

The second term on the right hand side of Eq. (7) is the flux across the boundaries of the cell. For the Lagrangian system, this flux is simply the tracer content of all blobs entering the cell, less the tracer content of all blobs exiting the cell in a given time interval. We may thus write the evolution of tracer content of the L system in a grid cell as

\[
\partial_{t} \left( \int_{V_{L}} \rho (C) \, dV \right) = \sum_{q=1}^{Q_{\text{new}}} \frac{(mc)^{q}_{\text{new}}}{\Delta \tau} - \sum_{q=1}^{Q_{\text{dstry}}} \frac{(mc)^{q}_{\text{dstry}}}{\Delta \tau} + \sum_{q=1}^{Q_{\text{new}}} \left[ c_{q} \left( \frac{dm}{dt} \right)_{\text{blob}}^{q} + c_{q}^{D} \left( \frac{dm}{dt} \right)_{\text{blob}}^{q} + m_{q}^{D} \left( \frac{dc}{dt} \right)^{q} \right] , \quad (14)
\]

where \( Q_{\text{new}} \) and \( Q_{\text{dstry}} \) are the number of blobs entering and leaving a grid cell in the time interval \( \Delta \tau \). The first four terms on the right hand side thus represent, respectively, the transfer of tracer from the E system to the L system via the creation of new blobs, the transfer tracer from the L system to the E system via the destruction of blobs, the amount of tracer brought into a grid cell by blobs entering the grid cell and the amount of tracer taken out of a grid cell by blobs leaving the grid cell. The final three terms in the square brackets represent, respectively, the amount of tracer transferred from the E system to the L system by blob entrainment, the amount of tracer transferred from the L system to the E system by blob detrainment and the final term is the change of a tracer due to non-material processes.

Eq. (8), the rate of change of tracer content of a cell, will apply to the E system of a partitioned system, except there will be additional source and sink terms arising from the exchange of properties with the L system, as described by Eq. (11a). Adding the E system contribution of the rate of change of tracer content of a grid cell to the L system contribution, Eq. (14), gives the rate of change of tracer content of the combined system

\[
\partial_{t} \left( \int_{V_{L}} \rho (C) \, dV \right) = - \int_{S_{L}} \mathbf{n} \cdot \mathbf{J}^{(C)}_{E} dS + \int_{V_{L}} \mathcal{S}_{E}^{(C)} \, dV + \sum_{q=1}^{Q_{\text{new}}} \frac{(mc)^{q}}{\Delta \tau} , \quad (15)
\]

where, \( J^{(C)}_{E} \) is the tracer flux arising from advection and SGS physics represented by the E system. Eq. (15) satisfies the compatibility condition.

2.3. Momentum

The equations for the linear momentum per unit volume of a continuous rotating fluid, based on Newton’s second and third laws is given by

\[
\frac{\rho}{\text{d}t} \mathbf{v} + \rho(2\Omega \times \mathbf{z}) \wedge \mathbf{v} = \mathbf{S}^{(V)} - \nabla p - \rho g \mathbf{z} + \nabla \cdot \tau , \quad (16)
\]

where \( \mathbf{S}^{(V)} \) is a momentum source, \( \tau \) is the stress tensor (a second order symmetric tensor), \( p \) is pressure, \( 2\Omega = (0, f', f) \) is twice the angular velocity of the earth and \( M \) is the advection metric frequency. Here, \( f = 2\Omega \sin \phi \) and \( f' = 2\Omega \cos \phi \), where \( \phi \) is latitude. For a derivation and discussion of the linear momentum budget, the reader is referred to Chapter 4 of Griffies (2004).

Integrating the momentum Eq. (16) over a grid cell and invoking Green’s Theorem, the momentum budget of a grid cell may be stated as

\[
\partial_{t} \left( \int_{V_{L}} \rho \mathbf{v} \, dV \right) = \int_{S_{L}} \left( \mathbf{S}^{(V)} - \rho(2\Omega \times \mathbf{z}) \wedge \mathbf{v} \right) dS + \int_{V_{L}} \nabla \cdot \tau - \nabla p \, dV - \int_{S_{L}} \rho \mathbf{g} \wedge (2\Omega \times \mathbf{z}) \wedge \mathbf{v} dV , \quad (17)
\]

where \( \mathbf{v} \) is the velocity relative to the bounding surface, \( S_{L} \), of the volume being considered. We write the sum of the momentum for the E plus L systems as

\[
\partial_{t} \left( \int_{V_{L}} \rho \mathbf{v} \, dV \right) = \partial_{t} \left( \int_{V_{E}} \rho \mathbf{v} \, dV \right) + \partial_{t} \left( \int_{V_{L}} \rho (\mathbf{v}) \, dV \right) , \quad (18)
\]

where the E system only has a horizontal component since it is assumed to be hydrostatic. Eq. (18) shows how the addition of the embedded Lagrangian system creates a component of momentum in the vertical which would not be there if there were no L system. As is shown below, the addition of the L system allows the representation of physics that is not generally admitted by the present generation of global scale hydrostatic ocean climate models. The inclusion of vertical momentum terms does not create an inconsistency with the hydrostatic assumption of the E system, aside from the fact that vertical momentum cannot be transferred between the E and L systems (see later in this section). The inclusion of the vertical momentum into the total system is a step towards more complete representation of the dynamics of the system under consideration.
The linear momentum of a blob can be expressed as

$$\frac{d}{dt}(m\mathbf{x}) = m\frac{d\mathbf{x}}{dt} + \mathbf{x} \frac{dm}{dt}, \quad (19)$$

where we have used Newton’s notation $\mathbf{x} = \frac{d\mathbf{x}}{dt}$ to help emphasise the particle-like nature of a blob. The first term on the right hand side represents the change in momentum due to the change in velocity, while the second term represents the change in momentum due to the change in mass of a blob.

Multiplication of the momentum per unit volume, Eq. (16), by the volume of a blob gives the change in momentum of a blob due to the change in velocity

$$m\frac{d\mathbf{x}}{dt} = -m(2\mathbf{\Omega} + \mathbf{Mz}) \cdot \mathbf{x} - V\nabla p - V\rho g \mathbf{z} + V\nabla \cdot \mathbf{\tau}. \quad (20)$$

The change in momentum of a blob due to the change in mass can result from mass that is transferred from the E system to the L system and from the L system to the E system

$$\mathbf{x} \frac{dm}{dt} = \mathbf{u}_e \left( \frac{dm}{dt} \right)_{(L2E)} + \mathbf{x} \left( \frac{dm}{dt} \right)_{(E2L)}), \quad (21)$$

again noting that $\left( \frac{dm}{dt} \right)_{(L2E)} \leq 0$.

There are other ways in which momentum can be transferred between the E and L systems. Two mechanisms relate to other transfers of mass between the E and L systems, namely via the creation and destruction of blobs. The other is via interfacial drag. Interfacial drag shall be discussed later in this section. Here, we write how the source terms of momentum are affected for the E and L systems

$$A \int_E \mathbf{S}^w_{\mathbf{E}} \, dz = -\sum_{q=1}^{Q_{\text{new}}} \frac{m^q}{\Delta t}(\mathbf{x}^q, \mathbf{y}^q)^9 + \sum_{q=1}^{Q_{\text{new}}} \frac{m^q}{\Delta t}(\mathbf{x}^q, \mathbf{y}^q)^9$$

$$- \sum_{q=1}^{Q_{\text{new}}} \left[ \mathbf{u}_e \left( \frac{dm}{dt} \right)_{(L2E)}^q + (\mathbf{x}^q, \mathbf{y}^q)^9 \right] \left( \frac{dm}{dt} \right)_{(L2E)}^q + A \int_E \mathbf{S}^w_{\mathbf{E}} \, dz. \quad (22a)$$

$$A \int_L \mathbf{S}^w_{\mathbf{L}} \, dz = \sum_{q=1}^{Q_{\text{new}}} \frac{(\mathbf{m}^q)^9}{\Delta t} - \sum_{q=1}^{Q_{\text{new}}} \frac{(\mathbf{m}^q)^9}{\Delta t} + \sum_{q=1}^{Q_{\text{new}}} \left[ \mathbf{u}_e \left( \frac{dm}{dt} \right)_{(L2E)}^q + (\mathbf{x}^q, \mathbf{y}^q)^9 \right] \left( \frac{dm}{dt} \right)_{(L2E)}^q, \quad (22b)$$

where $\mathbf{S}^w_E$ is a momentum source in the horizontal only and $\mathbf{S}^w_L$ is a non-blob related momentum sources for the E system. It is important to note that there is an imbalance in these equations since the E system does not have a vertical momentum component. Thus, the E system can only lose or gain momentum in the horizontal, while the L system can lose or gain momentum in the horizontal and the vertical.

We now examine each of the terms in the momentum budget (17) in turn. The first term, being the source term, has already been discussed. The second term represents the transport of momentum across a cell boundary. There may be the advection across the cell face, which may arise from a disasurface velocity of the E system or from blobs crossing the surface

$$- \int_{\Delta t} \int_{V_L} \mathbf{n} \cdot \mathbf{v}^{(\rho)}_{\mathbf{E}} \rho_d \, dS \, dt = - \int_{\Delta t} \int_{V_L} \mathbf{n} \cdot \mathbf{v}^{(\rho)}_{\mathbf{E}} (\rho_d) \, dS \, dt$$

$$+ \sum_{q=1}^{Q_{\text{new}}} \frac{(\mathbf{m}^q)^9}{\Delta t} - \sum_{q=1}^{Q_{\text{new}}} \frac{(\mathbf{m}^q)^9}{\Delta t}. \quad (23)$$

Here, it is important to note that the advection of momentum is only for horizontal momentum in the E system, as the system is assumed to be in hydrostatic balance.

The third term of Eq. (17) represents the modification due to stresses and pressure. The change of momentum due to stress and pressure may be computed by evaluating the E system pressure and stress at the surface of the volume, $V$. In addition to the stress and pressure on the surface, $S$, there may be surface stress and pressure at the interface between a blob and the E system. We recall, however, that all of a blob’s properties are concentrated at a point in space. Therefore, the “surface stress” between the E and L systems must be parameterised if it is to exist. Here, we shall denote the net force on a blob as a result of the parameterised surface stress as $\mathbf{G}_{(E2L)}$. The net force on the E system as a result of the parameterised surface stress of a blob is written as $\mathbf{G}_{(E2L)}$. If the interaction is perfectly elastic, then, these forces will be equal and opposite, $\mathbf{G}_{(E2L)} = -\mathbf{G}_{(L2E)}$. However, an interfacial drag parameterisation may require that the interaction is not perfectly elastic. Thus, we do not assume these two forces cancel

$$\int_{V_L} \mathbf{n} \cdot \mathbf{\tau} \cdot dS = \int_{V_L} \mathbf{n} \cdot \mathbf{\tau} \cdot dS + \sum_{q=1}^{Q_{\text{new}}} \mathbf{G}_{(E2L)} + \sum_{q=1}^{Q_{\text{new}}} \mathbf{G}_{(L2E)} - \sum_{q=1}^{Q_{\text{new}}} \mathbf{V}^q (\mathbf{x}_{\text{d}q}, \mathbf{y}_{\text{d}q}) \rho_d \mathbf{p}$$

$$+ \sum_{q=1}^{Q_{\text{new}}} \frac{(\mathbf{m}^q)^9}{\Delta t} - \sum_{q=1}^{Q_{\text{new}}} \frac{(\mathbf{m}^q)^9}{\Delta t} + \sum_{q=1}^{Q_{\text{new}}} \left( \frac{dm}{dt} \right)^0_{(L2E)}. \quad (27)$$

The linear momentum budget (17) of the gravitational force, the Coriolis force and the advection metric frequency. Eulerian models generally do not include the horizontal component of planetary rotation, while it may be admitted by the L system

$$- \int_{V_L} \mathbf{\rho} \mathbf{g} \mathbf{z} + (2\mathbf{\Omega} + \mathbf{Mz}) \wedge \mathbf{v} \, dV$$

$$= -A \int_{V_L} \mathbf{\rho} \mathbf{g} \mathbf{z} \, dV - A \int_{V_L} \mathbf{\rho} (f + \mathbf{Mz}) \mathbf{u}_{\text{e}} \, dV + \sum_{q=1}^{Q_{\text{new}}} \mathbf{m}^q \mathbf{g} \mathbf{z}$$

$$- \sum_{q=1}^{Q_{\text{new}}} \mathbf{m}^q (2\mathbf{\Omega} + \mathbf{Mz}) \wedge \mathbf{x}^q. \quad (25)$$

For brevity, the hydrostatic E system momentum budget is written as

$$\partial_t \int_{V_L} (\mathbf{v}^{(\rho)}_{\mathbf{E}}) \mathbf{v}^{(\rho)}_{\mathbf{E}} \, dV = = \partial_t \mathbf{M}_E = \sum_{q=1}^{Q_{\text{new}}} \left( \frac{(\mathbf{m}^q)^9}{\Delta t} - \sum_{q=1}^{Q_{\text{new}}} \frac{(\mathbf{m}^q)^9}{\Delta t} \right)$$

$$\sum_{q=1}^{Q_{\text{new}}} \mathbf{G}_{(E2L)} + \mathbf{G}_{(L2E)} - \mathbf{V}^q (\mathbf{x}_{\text{d}q}, \mathbf{y}_{\text{d}q}) \mathbf{p}$$

$$+ \sum_{q=1}^{Q_{\text{new}}} \left( \frac{dm}{dt} \right)^0_{(L2E)} \mathbf{m}^q \mathbf{g} \mathbf{z}$$

$$+ \sum_{q=1}^{Q_{\text{new}}} \frac{(\mathbf{m}^q)^9}{\Delta t} - \sum_{q=1}^{Q_{\text{new}}} \frac{(\mathbf{m}^q)^9}{\Delta t} + \sum_{q=1}^{Q_{\text{new}}} \left( \frac{dm}{dt} \right)^0_{(L2E)}. \quad (27)$$
To reach Eq. (27) we have used the hydrostatic balance for the pressure
\[ \nabla p(x, y, z) = (\hat{x} \partial_x, \hat{y} \partial_y, \hat{z} \partial_z) p - \rho g \hat{z}. \]  
(28)

Eq. (27) shows the additional terms that arise from partitioning the model into an E system and an L system.

3. Implementation

Embedding a dynamically active Lagrangian model into an ocean climate model requires fundamental changes to a number of aspects of the Eulerian model. Lagrangian models that are dynamically passive (i.e. do not move mass, or move mass instantaneously) require far fewer modifications. As such, the following mostly pertains to the changes required to admit dynamically active schemes.

Considerable care has been taken in the implementation of the Lagrangian model to ensure that basic design principles are adhered to, despite the partitioning of the domain's mass into two separate but intricately linked systems. Namely, the code conserves tracer mass and seawater mass to the same accuracy as when only running a purely Eulerian model. The code is also bitwise reproducible across restarts and has an option to be bitwise reproducible regardless of computer processor domain decomposition.

3.1. Implementation of Grid Cell Partitioning

The movement of blobs from one grid cell to another requires alteration to the convergence of blobs, Eq. (15). Blob convergence has implications for the calculation of a number of model prognostic variables. The evolution of sea surface height, \( \eta \), which is a prognostic variable for depth based vertical coordinates is written as
\[ \partial_t \eta = -\nabla \cdot \mathbf{U} + Q_{\text{sw}}/\rho_{\text{sw}}, \]  
(29)

where \( \mathbf{U} \) is the depth integrated velocity (m s\(^{-1}\)), \( Q_{\text{sw}} \) is the mass flux per unit area (kg s\(^{-1}\) m\(^{-2}\)) crossing the surface and \( \rho_{\text{sw}} \) is the density of the water crossing the boundary. To include blobs in the water column, we write the semi-discrete equation for the evolution of sea surface height as
\[ \partial_t \eta^i = -\nabla \cdot \mathbf{U}^i + Q_{\text{sw}}^i/\rho_{\text{sw}} + \frac{1}{A^i \Delta t} \left( \sum_{q=1}^{q_{\text{in}}} V^q + \sum_{q=1}^{q_{\text{out}}} V^q \right), \]  
(30)

where it should be noted that \( q_{\text{in}} \) is the number of blobs entering a water column and \( q_{\text{out}} \) is the number of blobs leaving the water column. Here, the labels \((i,j)\) are the horizontal grid cell indices. For pressure based vertical coordinates the prognostic variable is the bottom pressure. In analogy to sea surface height, the semi-discrete expression for the time evolution of bottom pressure is given by
\[ \partial_t (p_b - p_h)^i = -g \nabla \cdot (\mathbf{U}^i) + g Q_{\text{sw}}^i + \frac{g}{A^i \Delta t} \left( \sum_{q=1}^{q_{\text{in}}} m^q + \sum_{q=1}^{q_{\text{out}}} m^q \right), \]  
(31)

where \( p_b \) is the bottom pressure, \( p_h \) is the applied pressure at the surface and \( \mathbf{U}^i \) is depth integrated, density weighted velocity (kg m\(^{-1}\) s\(^{-1}\)).

Blob convergence also has implications for advection of properties by the E system. This convergence manifests itself as a vertical velocity in the seawater mass budget of an interior grid cell
\[ (\rho W_{E}^{i})^k = \partial_t (\rho dz)^i + \nabla \cdot (\rho dz \mathbf{u})^i + (\rho W_{E}^{i})^{k-1} + \frac{1}{A^i \Delta t} \left( \sum_{q=1}^{q_{\text{in}}} m^q + \sum_{q=1}^{q_{\text{out}}} m^q \right). \]  
(32)

where \( k \) is the vertical grid cell index and
\[ dz_{E} = A_{E}^{-1} \int_{V_{E}} dV \]  
(33a)

is the total thickness of a grid cell, and
\[ dz_{L} = A_{L}^{-1} \int_{V_{L}} dV \]  
(33b)

is the contribution from the E system to the thickness of a grid cell.

The first term on the right hand side of Eq. (32) is the rate of change of the total density weighted thickness of the grid cell (which is found from knowledge about the evolution of the vertical coordinate), the second term is the mass convergence of the E system, the third term is the mass flux across the upper face of the cell and the final term is the blob convergence. It can be seen from Eq. (32) that the blob convergence term directly affects the diagnostic vertical velocity. The mass of a grid cell is controlled by the vertical spacing of coordinate surfaces, \( ds \). The total mass of a grid cell is given by
\[ \int_{V} \rho dV = A \int_{\partial V} \rho \frac{\partial z}{\partial s} ds, \]  
(34)

which is the sum of the mass of the combined E and L systems. However, with the point-like nature of the Lagrangian model, it is possible for the L system to have more mass in a grid cell than what is allowable, given the coordinate spacing. We refer to this as the grid cell mass constraint. In the instance where the mass contained within the L system of a grid cell is greater than the mass in the E system
\[ \sum_{q_{\text{in}}} m^q > A \int_{\partial V} \rho \frac{\partial z}{\partial s} ds, \]  
(35)

we must return mass to the E system in order to maintain conservation and numerical stability. In the implementation in MOM, a scaling parameter (between 0 and 1) is introduced such that if the mass of the L system in a cell exceeds some proportion of the total allowable mass in that grid cell (usually 0.8), then blobs are destroyed until the mass of the L system is sufficiently small. This operation is required to ensure that mass is conserved locally (and globally in a non-Boussinesq model), as well as to ensure the numerical stability of the Eulerian system since MOM is not designed to have vanishingly small or even negative mass grid cells. A more detailed discussion on the details of the implementation in MOM is given in Section 6.5 of Bates (2012).

In order to calculate the pressure gradient force, it is necessary to alter how pressure is calculated, taking into account the presence of blobs. Making use of the combined density of the E and L systems the pressure is given by
\[ p = g \int_{z}^{q} \rho_{E} dz', \]  
(36)

where an expression for the total density is derived in D.3 and given by Eq. (D.11).

MOM is an Arakawa (1966) B-grid. Since velocity grid points and tracer grid points are not coincident on a B-grid, there are strategies required to allow mass to be partitioned in a consistent manner on both the velocity grid and the tracer grid. Details of the strategy employed by MOM are given in Chapter 6 of Bates (2012).

3.2. Computational methods for blobs

The natural numerical representation of the blobs in Fortran code in doubly linked lists\(^1\) (Newell and Shaw, 1957; Ellis et al., 1994), as

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\(^1\) Linked lists rely on pointers, which are not available in Fortran prior to the Fortran 90 standard.
linked lists are much more convenient than arrays for handling an arbitrary and potentially highly variable number of nodes. Special sorting algorithms are required to ensure bitwise reproducibility across restarts and domain decomposition when running parallel code. Details of the sorting algorithms employed in MOM are given in Section 7.2 of Bates (2012).

We have seen in the previous section that it is important to know when a blob has moved from one grid cell to another. For the horizontal, we use an efficient technique from computational geometry (Cormen et al., 2001) that tests whether a vector joining the grid cell centre with the blob’s final position intersects any of the horizontal grid cell walls. If so a blob is deemed to have crossed into a neighbouring cell. The code has been tested on regular longitude-latitude grids (both with solid boundaries and periodic boundaries), as well as the Murray (1996) triporal grid and has been found to be robust.

Blob trajectories are calculated by subcycling them relative to the E system. The trajectories of the blobs are integrated using adaptive step Runge–Kutta–Fehlberg methods (Atkinson, 1989). Two methods of varying order are available (Bogacki and Shampine, 1989; Cash and Karp, 1990). The desired accuracy is set and the methods adjust the step size of individual blobs to achieve that accuracy. All blob time steps periodically coincide with the E system’s time step in order to properly calculate the combined system’s properties.

3.3. The Implementation of momentum partitioning

There are some differences between the momentum theory outlined in Eq. (27) of Section 2.3 and that which is implemented in MOM. Referring to the linear momentum of a blob, Eq. (19), we ignore the component of the change in momentum that results from the change in mass

$$\frac{d}{dt}(mx) = m \frac{dx}{dt}.$$  (37)

This approximation makes the equations of motion for the blobs much easier to solve. As a trade off, however, the momentum budget for the combined E and L systems is no longer closed. The linear momentum for a blob thus effectively takes the form of Eq. (20). However, there are still terms in the E and L momentum source, Eq. (22), that arise from the transfer of mass between the E and L systems. Yet, those terms do not appear in the momentum equation for a blob. The implications of this choice are now detailed.

Similar to the transfer of mass between the E and L systems due to entrainment and detrainment, new blobs are assumed to act as a sink to the E system with the momentum per unit mass of the E system, but do not necessarily have that momentum per unit mass when being added to the L system. Blobs that are destroyed are also assumed to be a source for the E system at the momentum per unit mass of the E system, and a sink for the L system at the momentum per unit mass of the blob. The momentum sources (22) thus become

$$A \int_E S_{E}^{\text{u}} d\mathbf{z} = - \sum_{q=1}^{Q_{\text{new}}} \frac{m^E_q}{\Delta t} \mathbf{u}^E_q + \sum_{q=1}^{Q_{\text{detry}}} \frac{m^E_q}{\Delta t} \mathbf{u}^E_q$$

$$+ \sum_{q=1}^{Q_{\text{new}}} \frac{m^L_q}{\Delta t} \mathbf{u}^L_q - \sum_{q=1}^{Q_{\text{detry}}} \frac{m^L_q}{\Delta t} \mathbf{u}^L_q + A \int_E S_{E}^{\text{u}} d\mathbf{z},$$  (38a)

$$A \int_L S_{L}^{\text{v}} d\mathbf{z} = \sum_{q=1}^{Q_{\text{new}}} \frac{m^L_q}{\Delta t} + \sum_{q=1}^{Q_{\text{detry}}} \frac{m^L_q}{\Delta t}$$

$$+ \sum_{q=1}^{Q_{\text{new}}} \frac{m^E_q}{\Delta t} \mathbf{u}^E_q - \sum_{q=1}^{Q_{\text{detry}}} \frac{m^E_q}{\Delta t} \mathbf{u}^L_q + A \int_E S_{E}^{\text{v}} d\mathbf{z}.$$  (38b)

The assumption (37) simplifies the implementation of the blob schemes, and reduces the computational expense and memory footprint. The implication of the approximation (37) can be seen in Eqs. (38), where the blob assumes that mass that is being entrained and detrained is done so with a momentum per unit mass of the blob, while the E system assumes that mass being entrained and detrained to the blob at the momentum per unit mass of the E system. This simplification was made to the momentum budget and not the seawater mass or tracer budgets as we consider a closed momentum budget to be subservient to having a closed tracer mass and seawater mass budget.

Finally, it is also assumed that the reaction force of a blob travelling through the E system is not felt by the E system, i.e. it is assumed that $\mathbf{G}_{E2L} = 0$.  (39)

The momentum budget, as implemented in MOM, is thus written as

$$\partial_t \int_V (\rho \mathbf{v}) dV = \partial_t M_E + \frac{1}{\Delta t} \left( \sum_{q=1}^{Q_{\text{new}}} (mx^E_q)^{\text{new}} - \sum_{q=1}^{Q_{\text{new}}} (mx^L_q)^{\text{new}} \right)$$

$$- \sum_{q=1}^{Q_{\text{new}}} m^E_q (2\Omega + M\mathbf{z}) \times \mathbf{x}^q$$

$$+ \sum_{q=1}^{Q_{\text{new}}} \mathbf{G}_{E2L} \times (\hat{x} \partial \mathbf{y}_q, \hat{y} \partial \mathbf{y}_p) + \sum_{q=1}^{Q_{\text{new}}} \mathbf{z} \mathbf{G}^\mathbf{v} (\rho_E - \rho_L^E)$$

$$+ \sum_{q=1}^{Q_{\text{new}}} \frac{m^E_q (\mathbf{x}^q_{\text{new}} - \mathbf{u}_E)}{\Delta t}$$

$$+ \sum_{q=1}^{Q_{\text{new}}} \frac{m^E_q (\mathbf{x}^q_{\text{dstry}} - \mathbf{u}_E)}{\Delta t} + \frac{\partial}{\partial t} \left[ \int_L (mx^L_q)^{\text{new}} + \int_E (mx^E_q)^{\text{new}} \right].$$  (40)

When compared to Eq. (27) it can be seen that there are additional non-conservative terms in the combined momentum budget that arise due to the difference in horizontal velocity between the E and L systems.

4. Application to convection and gravity currents

In this section we discuss how the embedded Lagrangian model is applied to two dynamic processes, being open ocean deep convection and gravity driven downslope flows. We then develop four Lagrangian schemes in total; two for each process. Two schemes are dynamically active and two schemes are dynamically passive. Dynamically passive schemes do not directly affect the momentum budget of the model, while dynamically active schemes do. The dynamically passive schemes are designed to emulate existing Eulerian schemes. The emulation of existing schemes serves two main purposes. First, as a developmental step for the implementation of the embedded Lagrangian framework. Second as a conceptual tool to show how the framework works without the additional complexity of active dynamics. The dynamically active schemes introduce more realistic physics, and are a more complete representation of the processes under examination.

4.1. Open ocean deep convection

Open ocean deep convection is a process that is fundamentally non-hydrostatic. Hence, the hydrostatic primitive equations do not explicitly represent such processes (Marshall et al., 1997). Convective adjustment schemes capture the integral effects of convection (Klinger et al., 1996) and some more sophisticated schemes can
parameterise penetrative convection by taking into account non-local effects (Large et al., 1994). On the other hand, Hughes et al. (2009) indicate that caution needs to be exercised when drawing conclusions about the meridional overturning circulation when using convective adjustment schemes.

Deep convection in the ocean is an intrinsically non-hydrostatic process, with relatively large vertical velocities. Unlike the atmospheric analogue, the time scale of oceanic deep convection means that rotation becomes an important factor (Marshall and Schott, 1999). The non-trivial vertical velocities, coupled with the time-scale of deep convective processes, also means that the traditional practice of ignoring the horizontal component of rotation becomes questionable (Wirth and Barnier, 2008). The effect of the horizontal component of the Earth’s rotation is to induce “tilted convection” that causes convecting plumes to deflect (Denbo and Skillingstad, 1996; Sheremet, 2004; Wirth and Barnier, 2006).

4.2. Gravity driven downslope flows

The other dynamic process we consider concerns gravity currents, which are poorly represented in coarse resolution level models (Winton et al., 1998; Legg et al., 2006). Dense water that sinks near boundaries usually accumulates in a basin (sometimes a marginal sea, such as the Weddell or the Mediterranean). The dense water is formed by brine rejection, evaporation or cooling. The accumulated water then overflows over sills or through channels into the deep ocean, acting as a bottom boundary layer (Killworth, 1977). As the waters descend along the continental slope, they are subject to geostrophy, bottom drag (Smith, 1975; Killworth, 1977; Price and Baringer, 1994; Özgökmen and Fischer, 2008) and topographic steering (Muench et al., 2009; Ilicak et al., 2011). These overflow waters play a large part in forming the water-masses of the world’s deep and bottom waters (Killworth, 1983), such as North Atlantic Deep Water (Dickson and Brown, 1994) and Antarctic Bottom Water (Gordon et al., 2001). In recent years, there has been considerable effort put into improving our understanding of overflow processes (for instance, Legg et al. (2009)).

4.3. Dynamically passive schemes

Dynamically passive schemes do not obey momentum equations, but their blobs are formed, moved and destroyed instantaneously. These schemes were developed to show that the principles of seawater mass and tracer mass conservation can be maintained with an ocean model that is partitioned in E and L systems.

4.3.1. A blob-NCon scheme

The first scheme is based on the NCon convective adjustment scheme of Cox (1984) in which vertical pairs of grid cells are tested for gravitational instability and homogenised if an instability exists. We achieve the same end by forming a pair of blobs, swapping their positions and then destroying them instantaneously. Blobs are formed with just the right tracer content to ensure that the tracer concentration of the pair of grid cells is homogeneous (to round off error) after the blobs have been swapped. The process is illustrated in Fig. 3. The mass of the two blobs is identical and is given by

\[ m = \frac{m^b m^k}{m^b + m^k} \]

for which a derivation is provided in Appendix C.

We test the blob-NCon scheme against the standard NCon scheme of Cox (1984) in the experimental setup of Jones and Marshall (1993), but largely using the same parameters as Campin et al. (2011). The domain is a flat bottom, doubly periodic configuration of size \( 40 \times 40 \) km and \( 2000 \) m deep. There are \( 20 \) grid points in each direction, giving a resolution of \( 2 \text{ km} \times 2 \text{ km} \times 100 \text{ m} \). A linear equation of state, dependent on temperature only, is used. The domain is initially stratified with a buoyancy frequency of \( N = 3 \times 10^{-4} \text{ s}^{-1} \). An 800 W m\(^{-2}\) cooling disc, with radius of 10 km is applied over the centre of the domain. Rotation is included as an f-plane with \( f_0 = 10^{-4} \text{ s}^{-1} \). We use a constant Laplacian viscosity of \( 1.0 \text{ m}^2 \text{ s}^{-1} \) in the vertical and \( 50 \text{ m}^2 \text{ s}^{-1} \) in the horizontal. Tracer may only be advected, or convectively adjusted (i.e. there is no explicit horizontal or vertical diffusion). Both cases use \( z^* \) vertical coordinates with a free surface.

Two experiments are run, one using the original NCon scheme and the other using the blob-NCon scheme. The experiments are named NCon and blob-NCon, respectively. The NCon and blob-NCon experiments test the water column for instabilities seven times at each time step (i.e. NCon = 7). Fig. 4 shows the average temperature with depth and time of NCon. There is very good agreement between the solutions of NCon and blob-NCon experiments with differences in the order of machine error. Such a small difference is to be expected as the schemes are effectively equivalent in so much as both schemes homogenise tracer between two spatially unstable adjacent grid cells and they have the same search algorithm to find the unstable pairs of cells.

4.3.2. A blob-overflow scheme

The second dynamically passive scheme is based on the overflow scheme of Campin and Goosse (1999), in which shelf waters
are transported to their neutrally buoyant level in an adjoining deep ocean water column if it is found that the off-shelf density is greater than the on shelf density, as illustrated in Fig. 5. For the blob analogue to the Campin and Goosse (1999) scheme, the mass of each blob is calculated as,

\[ m = \int_{\Delta t} \mathbf{u}_{\text{slope}} \rho dt, \tag{42} \]

where \( \mathbf{u}_{\text{slope}} \) is the calculated volume flux. To find locations where shelf waters are more dense than offshelf waters the model conducts the search illustrated in Fig. 6.

The parameterisation is active when the condition \( \rho_{\text{onshelf}} > \rho_{\text{offshelf}} \) is satisfied, where \( l \) is defined in Fig. 6. The bottom cell of each water column is searched at each time step for \( l = 1, 2, 3, 4 \). The volume flux is calculated as

\[
\mathbf{u}_{\text{slope}} = \begin{cases} 
\frac{g \delta}{\rho_0 H} (\partial_j H, \partial_i H) (\rho_{\text{onshelf}} - \rho_{\text{offshelf}}), \\
\frac{g \delta}{\rho_0 H} (\partial_j H, \partial_i H) (\rho_{\text{onshelf}} - \rho_{\text{offshelf}}), \\
\frac{g \delta}{\rho_0 H} (\partial_j H, \partial_i H) (\rho_{\text{onshelf}} - \rho_{\text{offshelf}}), \\
\frac{g \delta}{\rho_0 H} (\partial_j H, \partial_i H) (\rho_{\text{onshelf}} - \rho_{\text{offshelf}}), \\
\end{cases} \tag{43} \]

where \( \delta \) is a nondimensional number that is the fraction of the grid cell participating in the overflow event and has a value \( 0 \leq \delta \leq 1 \) and \( \mu \) is a prescribed frictional dissipation parameter with units of \( \text{s}^{-1} \).

We test the blob version of the Campin and Goosse (1999) scheme against the original scheme using the Dynamics of Overflows, Mixing and Entrainment (DOME) experimental setup. The DOME setup is a domain with a shallow embayment at the north, and uniformly sloping topography to a flat bottom. A linear equation of state, dependent on temperature only, is used. A region in the northern part of the embayment is held at a (constant) cold temperature and is tagged with a passive dye tracer of value one, while the dye tracer is initially set to zero everywhere else. The domain is initially stratified with \( N = 2.3 \times 10^{-3} \text{ s}^{-1} \) and the buoyancy anomaly of the inflow (relative to the initial conditions) is \( \Delta b_0 = 0.019 \text{ m s}^{-2} \). The domain is on an \( f \)-plane, with a value of \( F_0 = 10^{-4} \text{ s}^{-1} \). We use a constant horizontal viscosity of \( 500 \text{ m}^2 \text{ s}^{-1} \) and a constant vertical viscosity of \( 0.5 \text{ m}^2 \text{ s}^{-1} \). There is no explicit tracer diffusion, except that a vertical diffusivity of \( 50.0 \text{ m}^2 \text{ s}^{-1} \) is prescribed in vertically unstable regions.

Fig. 7 shows the value of the passive tracer in the bottom cells and the centre of mass of the plume at day 360 for a run that uses the Campin and Goosse (1999) overflow parameterisation. The horizontal position of the centre of mass of the plume is also shown. These results are commensurate with the coarse resolution results of Legg et al. (2006). Similar to the deep convection test case, the difference between the run using the original Campin and Goosse (1999) scheme and the blob version of that scheme is of the order of machine error.

4.3.3. A modified blob-overflow scheme

Since the formulation of the overflow scheme of Campin and Goosse (1999), the use of free surface models in ocean climate models has become common. The use of the rigid lid approximation dictates that the volume of water in a water column must remain constant. A free surface obviates this restriction, meaning that mass can be moved from one water column to another without prescribing a compensating return flow. Utilising this property of free surface models, we alter the Campin and Goosse (1999) scheme such that the blob originating on the shelf is transported off the shelf, and none of the other blobs are formed. Such a scenario is shown diagrammatically in Fig. 8.

As can be seen in Fig. 5, there should be no barotropic response in the traditional Campin and Goosse (1999) scheme as there is no

Fig. 5. Illustration of the overflow scheme, adapted from Campin and Goosse (1999). Here, shaded boxes are dry cells, while unshaded boxes are wet ocean cells. When the density on a shelf is greater than the density at the same level off the shelf in a deep ocean column, blobs of equal mass are formed. The deep ocean column, \( l \), can be one of four surrounding grid cells from the shallow cell, as shown in Fig. 6. These blobs are moved around between the shelf and the neutral density level in the deep ocean, as indicated by the density, \( \rho \), relations on the right hand side.

Fig. 6. The search pattern for on-shelf off-shelf instabilities. The four neighbouring water columns, \( l \), are checked to see whether an overflow event should occur. The process of an overflow event is shown in Fig. 5.

Fig. 7. The value of the passive tracer in the bottom grid cells at day 360 for a DOME run using the Campin and Goosse (1999) overflow parameterisation. The black line represents the centre of mass of the plume in longitude-latitude space. A grid cell is defined as being part of the plume if its dye tracer concentration is greater or equal to 0.01.
net movement of seawater mass, only a baroclinic response from the redistribution of density thickness classes. In contrast, there is a barotropic response in the modified Campin and Goosse (1999) scheme, as the \((i,j)\) water column has lost thickness \(\delta \eta < 0\) and the \(l\) water column has gained thickness \(\delta \eta > 0\), where \(\eta\) is the sea surface height anomaly. We also expect a baroclinic response, as the thickness of water that is approximately \(\rho_{l}^{i}\) in the \(l\) water column tends to increase, while the thickness in the \((i,j)\) water column tends to decrease. The original Campin and Goosse (1999) scheme does capture a baroclinic response, however it is different to the response of the modified scheme since, rather than making one particular density class thicker in the deep ocean than making one particular density class thicker in the deep ocean column (Fig. 5).

The modified Campin and Goosse (1999) scheme has been implemented in MOM in both the Lagrangian blob framework as well as in a traditional Eulerian framework. BGE-II investigate the properties of the modified Campin and Goosse (1999) parameterisation in an Eulerian framework and find that this simple modification produces surprisingly different results to the original scheme.

4.4. Dynamically active schemes

We develop two dynamic schemes, one for open ocean convection and the other for down slope flows. Firstly, we state a general momentum equation for a water parcel

\[
\frac{d\mathbf{v}}{dt} - (2\Omega + \ell_{z}) \times \mathbf{v} = -g \frac{\rho_{l}}{\rho} \mathbf{z} - \nabla p + \rho \mathbf{G}, \tag{45}
\]

where \(\mathbf{G}\) is \((G^{x}, G^{y}, G^{z})\) represents body forces.

Using Newton’s notation and recalling the approximation made for an individual blob in Eq. (37) we can obtain a momentum evolution equation for an individual blob by multiplying Eq. (45) by the volume of the blob

\[
\frac{dm_{x}}{dt} = -(2\Omega + \ell_{z}) \times m_{x} - g \frac{\rho_{l}}{\rho} \mathbf{z} - \mathbf{V} \nabla p + m \mathbf{G}, \tag{46}
\]

where we have used \(m = \rho V\).

Note that it is not assumed that a blob is in hydrostatic balance. Since we consider the blobs to be point particles, we interpolate the \(E\) system’s horizontal pressure gradient to the blob’s position using the inverse distance weighting method (Shepard, 1968). For the vertical pressure gradient, the \(E\) system is assumed to be in hydrostatic balance,

\[
\frac{\partial p}{\partial t} = -g \frac{\rho_{l}}{\rho} \eta, \tag{47}
\]

thus giving the blobs a reduced gravity contribution to its vertical acceleration

\[
-g \frac{\rho_{l}}{\rho} \mathbf{z} = -g(\rho - \rho_{l}) \mathbf{z} - (\mathbf{x} \partial_{x} + \mathbf{y} \partial_{y}) p. \tag{48}
\]

The reduced gravity term makes the embedded Lagrangian model pseudo-nonhydrostatic.\(^2\) Thus, the evolution equation for a blob’s position can be written as

\[
\frac{d\mathbf{x}}{dt} = -(2\Omega + \ell_{z}) \times \mathbf{x} - g \frac{(\rho - \rho_{l})}{\rho} \mathbf{z} - \mathbf{V} \nabla p + \mathbf{G}. \tag{49}
\]

4.4.1. A free dynamically active scheme

We now use the general formulation (49) to find a specific formulation to allow us to represent open ocean deep convection. It is assumed that the only body force acting on a blob is a Rayleigh drag with a coefficient of drag, \(\alpha (s^{-1})\). This assumption yields the following equations of motion for the blobs

\[
\begin{align}
\tilde{x} &= f y + f^{*} \tilde{z} - \rho^{-1} \partial_{x} p - \alpha (x - x_{0}), \tag{50a} \\
\tilde{y} &= -f x - \rho^{-1} \partial_{y} p - \alpha (y - y_{0}), \tag{50b} \\
\tilde{z} &= -f^{*} x - g \rho^{-1} (\rho - \rho_{l}) - \alpha (z - z_{0}). \tag{50c}
\end{align}
\]

A key driver for the vertical acceleration, \(\tilde{z}\), in Eq. (50c) is the pseudo-nonhydrostatic term which is proportional to the density difference between the blob and the surrounding \(E\) system. Also the horizontal component of the Earth’s rotation provides an important piece to admit slantwise convection (Wirth and Barnier, 2006; Wirth and Barnier, 2008).

4.4.2. A bottom dynamically active scheme

The ideas for the dynamically active bottom blobs are largely taken from streamtube models (Smith, 1975; Killworth, 1977; Price and Barnier, 1994), which is illustrated in Fig. 9. In some regards, we can consider this dynamically active implementation as a Lagrangian discretisation of a streamtube model. One big difference, however, is the coordinate system. Streamtube models use the inverse distance weighting method (Shepard, 1968). For the vertical pressure gradient, the \(E\) system is assumed to be in hydrostatic balance,

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\frac{\partial p}{\partial t} = -g \frac{\rho_{l}}{\rho} \eta, \tag{47}
\]

thus giving the blobs a reduced gravity contribution to its vertical acceleration

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\tilde{z} &= -f^{*} x - g \rho^{-1} (\rho - \rho_{l}) - \alpha (z - z_{0}). \tag{50c}
\end{align}
\]

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---

\(\Omega\): angular velocity of the Earth

\(\ell\): horizontal component of the Earth’s rotation

\(\rho\): density

\(\mathbf{G}\): body forces

\(\mathbf{V}\): horizontal velocity

\(\alpha\): coefficient of drag

\(f\): Coriolis parameter

\(f^{*}\): modified Coriolis parameter

\(g\): gravitational acceleration

\(\partial_{x}\) and \(\partial_{y}\): partial derivatives

\(\mathbf{x}\) and \(\mathbf{y}\): position vectors

\(\mathbf{z}\): vertical position

\(p\): pressure

\(\rho_{l}\): density of the streamtube

\(\rho_{s}\): density of the ambient fluid

\(\mathbf{u}\): along slope velocity

\(H\): depth of the ocean

\(u_{0}\) and \(w_{0}\): velocities at the surface and bottom, respectively

\(\circ\): neutral buoyancy level

---

\(^2\) The term quasi-nonhydrostatic may be a better description, however, the term quasi-nonhydrostatic is commonly used to refer to the relaxation of the hydrostatic balance by including, in a consistent manner, the horizontal component of the Earth’s rotation (see, for instance Marshall et al. (1997)).
two dimensional along slope coordinates, while we use three dimensions.

Starting with the general blob momentum Eq. (49) and looking to streamtube models for inspiration, we make the following assumptions:

1. the horizontal component of the Earth’s rotation is unimportant, \( f = 0 \),
2. the vertical pressure gradient is given by the hydrostatic balance of the E system, as described by Eq. (47),
3. there are no lateral pressure gradient forces acting on the blob, \((\mathbf{u}\hat{x} + \mathbf{v}\hat{y})_p = 0\),
4. there is no flow across solid boundaries,
5. the topographic slope is small for the Eulerian system,
6. the bottom drag and interfacial drag may be parameterised, based on Price and Baringer (1994).

The body force term encompasses bottom drag, interfacial drag and a normal reaction force

\[
\rho \mathbf{G} = -\frac{\tau_{(\text{bot})}}{h_b} - \frac{\rho}{h_b} \left( \xi - \mathbf{u}_b \right) + \rho \mathbf{G}_{(\text{normal})},
\]

where \( \tau_{(\text{bot})} = \left( \tau_{(\text{bot})}^x, \tau_{(\text{bot})}^y, \tau_{(\text{bot})}^z \right) \) is the parameterised bottom stress \((N \, m^{-2})\). The height of the blob, \( h_b \), is analogous to the height of the streamtube, \( h_s \), in Fig. 9. The normal reaction force is required to enforce the solid bottom boundary condition. It acts orthogonally to the boundary and is proportional to the buoyancy force

\[
\rho \mathbf{G}_{(\text{normal})} = -\frac{\rho}{h_b} \left( \xi - \mathbf{u}_b \right) \mathbf{n},
\]

where \( \mathbf{n} = -\frac{\nabla H}{\sqrt{\nabla H \cdot \nabla H}} \) is the outward pointing normal unit vector. The minus sign in Eq. (52) arises because the normal force is inward pointing. The above assumptions give the equations of motion

\[
\ddot{x} = fy + \partial_x H \left( \frac{\rho (\rho - \rho_b)}{\rho (\rho + \rho_b)} \right) - \frac{\tau_{(\text{bot})}^x}{h_b} - \frac{\rho}{h_b} \left( \xi - \mathbf{u}_b \right),
\]

\[
\ddot{y} = -fx + \partial_y H \left( \frac{\rho (\rho - \rho_b)}{\rho (\rho + \rho_b)} \right) - \frac{\tau_{(\text{bot})}^y}{h_b} - \frac{\rho}{h_b} \left( \xi - \mathbf{u}_b \right),
\]

\[
\ddot{z} = - \frac{\rho (\rho - \rho_b)}{\rho (\rho + \rho_b)} \frac{1}{\sqrt{h_b}} - \frac{\tau_{(\text{bot})}^z}{h_b} - \frac{\rho}{h_b} \frac{\rho z}{h_b},
\]

Using the parameterisation of Price and Baringer (1994), the bottom stress and entrainment velocity are given by

\[
\tau_{(\text{bot})} = \rho C_D \xi \mathbf{x},
\]

\[
\mathcal{E} = \begin{cases} 
\frac{\rho (\rho - \rho_b)}{\rho (\rho + \rho_b)} \frac{\xi}{h_b} & \text{if } Ri \leq 0.8 \\
0 & \text{if } Ri > 0.8
\end{cases}
\]

\[
\text{Ri} = \frac{\rho (\rho - \rho_b) h_b}{\rho (\rho - \rho_b) \xi h_b},
\]

where \( \text{Ri} \) is the Richardson number. The form for the entrainment velocity is due to the experiments of Ellison and Turner (1959) and subsequent analysis (Turner, 1986). In the Price and Baringer (1994) streamtube model, the streamtube height, \( h_b \), is related to the spreading of the gravity current. Here, the height scale is constant and is specified a priori.

The small topographic slope assumption has implications for the interfacial drag terms in the momentum Eq. (53). The bottom boundary conditions dictate that there is no normal flow, \(- \mathbf{v} \cdot \mathbf{n} = 0\). The small slope assumption says that

\[
- \mathbf{v} \cdot \mathbf{n} \approx \mathbf{w}_b.
\]

This assumption implies that the E system’s vertical velocity is approximately zero near the bottom. Therefore, while it is necessary for the horizontal Rayleigh drag terms to be taken as the difference between the E system velocity and the blob velocity, it is acceptable for the vertical Rayleigh drag term to only use the blob’s vertical velocity.

It is important to note that the form of bottom drag and entrainment velocity specified by Eq. (54) are not intrinsic to the embedded Lagrangian model and other forms may be admitted. Indeed, it has been shown by Chang et al. (2005) that there are several drawbacks to the formulation used here, with an alternative formulation suggested by Xu et al. (2006).

4.4.3. Free dynamically active blob creation

The creation criteria for the formation of blobs must be specified. The chosen condition for the convective scheme is that a blob is formed when the square of the buoyancy frequency is less than a reference value

\[
N^2 < N^2_0,
\]

where \( N_0 \) is a reference buoyancy frequency. The reference value is not necessarily set to zero as there may be some arguments for having a value less than zero. Alternatively, the value of \( N_0^2 \) could be stochastic, in order to take account of the stochastic nature of deep convection. In the implementation in MOM, \( N_0^2 \) is a constant parameter that is specified a priori. We calculate the square of the buoyancy frequency as

\[
N^2 = -\frac{g}{\rho} \left( \frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial S} \frac{\partial S}{\partial z} \right),
\]

where \( S \) is the salinity and \( \Theta \) is conservative temperature. We use the combined system value for vertical distance and tracer concentration, calculated as,

\[
\frac{dz_{ik}}{\tau} = \int dz + A \sum_{q=1}^{Q_{ik}} \sum_{m=1}^{M_{ik}} \frac{\rho_{ik} dV + \sum_{q=1}^{Q_{ik}} \sum_{m=1}^{M_{ik}} \rho_{ik} dV}{\sum_{q=1}^{Q_{ik}} m dV},
\]

where \( dz_{ik} \) is the total thickness of a grid cell. The combined total thickness Eq. (58) is derived in Appendix D.1 and the combined tracer concentration Eq. (59) is derived in Appendix D.2.

The square of the buoyancy frequency of the combined system is used when testing the water column for instability. We use the value for the combined system because that is the measure of the stability of the water column under consideration. In addition we require the E system to be unstable, otherwise the pseudo-nonhydrostatic term would be positive and the initial velocity of a blob would be upward rather than downward.

4.4.4. Bottom dynamically active blob creation

The formation criterion and initial position of a bottom blob is schematically illustrated in Fig. 10. A blob is formed when the difference in density of a shallow ocean grid box \((i, j, k)\) and the bottom grid cell in the deep column, is greater than some threshold

\[
\Delta \rho < \rho_{ik} - \rho_{k+n},
\]

where \( \Delta \rho \) is the threshold density difference for a blob to be formed, and the bottom grid cell of the deep ocean water column \((i, k+n)\). For instance in Fig. 10 we have \( n = 3 \). \( \Delta \rho \) is a parameter that is specified a priori.

4.4.5. Transfer between dynamic regimes

The dynamically active blobs have the capacity to transfer between regimes. For instance, a free blob may become a bottom blob after interacting with topography. On the other hand, when a bottom blob becomes less dense than its surroundings

\[
\rho^3 < \rho_b
\]
it becomes a free blob. There is an option for blobs to be destroyed instead of transferring from one scheme to the other, that is, a free blob that interacts with topography is destroyed and a bottom blob that satisfies the separation condition \( \text{(61)} \) is destroyed.

### 4.4.6. Dynamic blob destruction

We also require a mechanism by which to destroy blobs. One choice might be to destroy them once they reach their neutrally buoyant level. However, this choice precludes the possibility of admitting any overshoot, such as occurs with penetrative convection. Instead we choose to have a background detrainment velocity that has a small value when the difference in density between the blob and its surroundings is large, and a large value when the difference is small. A detrainment velocity, \( \mathcal{D} \), that satisfies these conditions is

\[
\mathcal{D} = -\frac{\Gamma}{|\rho - \rho_E|},
\]

where \( \Gamma \) is a parameter (kg m\(^{-2}\) s\(^{-1}\)), specified \textit{a priori}. There are two ways that we limit the amount of material detrained by a blob in a given time step. One is to place an upper bound on the detrainment velocity, \( \mathcal{D} \leq \mathcal{D}_{\text{max}} \). The other limitation is to ensure that the change in mass of a blob over a time step is not greater than the mass of the blob itself, \( dm \leq m \).

The rate of change of tracer content must also satisfy the compatibility condition with the rate of change of seawater mass. The rate of change for tracer content is given by the entrainment (from the \( \text{E} \) system to the \( \text{L} \) system) minus the detrainment (from the \( \text{L} \) system to the \( \text{E} \) system). The rate of entrainment is given by the tracer content per unit volume of the \( \text{E} \) system, multiplied by the entrainment velocity and the area over which the entrainment is taking place. Conversely, the rate of detrainment is given by the tracer content per unit volume of the \( \text{L} \) system, multiplied by the detrainment velocity and the area over which the detrainment is taking place.

\[
\left( \frac{\partial \rho}{\partial t} \right)^q = (\rho E \mathcal{C}_E) - (\rho L \mathcal{D}) \mathcal{A}_{\text{interface}},
\]

with the compatible expression for rate of change of seawater mass being

\[
\left( \frac{\partial m}{\partial t} \right)^q = \int (\rho E \mathcal{C}_E - \rho L \mathcal{D}) \mathcal{A}_{\text{interface}}.
\]

In the present scheme, we assume a spherical shape for the interface area, with only half of the area participating in entrainment and detrainment for the bottom dynamically active scheme. We note that in the free dynamic scheme, the entrainment velocity is zero, \( \mathcal{E} = 0 \).

There are several other mechanisms by which blobs can be destroyed:

- If the mass of blobs within a grid cell exceeds a proportion (usually taken as 80%) of the total mass of the grid cell, blobs are then destroyed to ensure the stability of the \( \text{E} \) system and to ensure the conservation of seawater mass. We refer to this condition as the grid cell mass constraint.
- If a blob penetrates the free surface, its properties are immediately returned to the \( \text{E} \) system of the surface grid cell where it penetrated.
- If a blob leaves a water column that has some wet grid cells and enters a completely dry water column (i.e. a land column), then the blob is destroyed and all of its properties are returned to the \( \text{E} \) system in the grid cell that the blob was in immediately prior to being grounded.

### 5. Discussion and conclusion

A framework has been introduced for embedding a Lagrangian model within ocean climate models that have fixed Eulerian grids in the horizontal. The framework is general enough to be applied to a variety of physical processes. We also presented the methods and techniques required to implement the embedded Lagrangian framework to represent open ocean deep convection and gravity driven downslope flows. The embedded Lagrangian model has been implemented in the Modular Ocean Model (Griffies, 2012), which is an open source and publicly available ocean climate model. Details of simulations in two idealised test cases are the subject of a companion paper, (Bates et al. (2012) BGE-II).

Two implementations were discussed for each of the two applications. The most simple implementations emulated existing Eulerian parameterisations, such as the NCon-like scheme, based on the NCon scheme of Cox (1984) for convection and the blob-overflow scheme for overflows, based on the scheme of Campin and Goosse (1999). These schemes were dubbed “dynamically passive” schemes, as the blobs exist only instantaneously and they do not obey any momentum equations. These dynamically passive schemes were tested here and were shown to successfully emulate their Eulerian analogues. The dynamically passive schemes, while not providing any immediate advantages over their Eulerian analogues, were an important stepping stone in the development process towards the “dynamically active” blobs. We also derived one dynamically active scheme for open ocean deep convection and one dynamically active scheme for gravity driven downslope flows. The dynamically active bottom blobs, which model gravity driven downslope flows are thoroughly tested in BGE-II. The dynamically active free blobs, for open ocean deep convection, will be the subject of future testing and are not addressed in the companion paper.

The computational cost for the blobs is highly dependent on their number, which is itself a function of the surface forcing. In general, we have found that the blobs add to the overall model cost by an amount consistent with subgrid scales schemes used for the planetary boundary layer and mesoscale eddy parameterisations. Nonetheless, we note that our emphasis at this stage of implementation has been physical and numerical integrity, with optimization secondary to our focus. That is, the code has been written.
with the famous words of Knuth (1974) in mind: “premature optimization is the root of all evil.”.

There are some other limitations in the present implementation. Namely, the embedded Lagrangian model has yet to be extended for use with non-material tracers (e.g. age and biogeochemical tracers), though we believe there are no fundamental limitations for doing so. There is also an approximation made in the momentum exchange between the blobs and the Eulerian system which results in some non-conservation of momentum. We hypothesise that the error introduced by this approximation is negligible, however, this hypothesis remains untested and cannot be tested without additional significant development effort.

It has been pointed out by Chang et al. (2009) that bottom topography, including small scale features, have a very large influence on the pathway and entrainment of overflowing waters. As presently implemented the dynamically active bottom blob scheme uses the topography of the Eulerian model. This is not an intrinsic requirement of the scheme and it is in principle possible to supply the Lagrangian model with a finer resolution topography. The ability for the Lagrangian model to accept a different topography to that of the Eulerian model is the next planned significant development activity.

There are many opportunities for other applications of the embedded Lagrangian framework to be applied to other physical processes and phenomena, including

**Argo blobs** similar to passive Lagrangian particles (Böning and Cox, 1988) that are sometimes used as diagnostics in ocean climate models, Argo blobs could be given a varying density that allows them to sink and rise in a manner analogous to an Argo float. It could record properties of the Eulerian model thereby modelling the profiling nature of an Argo float on-line. Argo blobs could be used as another tool for model evaluation by comparing trajectories and measurements with historical floats. Kamenkovich et al. (2011) used Lagrangian drifters to simulate an array of Argo floats, with the floats instantaneously moving between 1500 m and the surface. Due to the instantaneous nature of the change of depth, the floats only took vertical profiles. One advantage of an Argo blob is that the float would park at a particular density surface, rather than a particular depth. Using the Argo blobs would also give non-vertical profiles as it would take a finite amount of time for the float to travel between the float’s neutral density and the surface of the ocean.

**Point vorticity blobs** one extension of the work of Mohammadian and Marshall (2010) could be to include point vortices to explicitly represent SGS eddies and calculate the effect on tracers and other fields.

**Wave packet blobs** Lagrangian parcels could act as wave packets, transporting information from a source region, following a certain set of physics, and giving information about mixing back to the Eulerian model according to a set of rules. A set of rules regarding the formation of internal gravity waves, their propagation and dissipation would need to be formulated. There exist some ray tracing methods (e.g. Rainville and Pinkel (2006)) for internal waves in the ocean which may be suitable for implementation in an embedded Lagrangian model.

The introduction of an embedded Lagrangian model opens many new opportunities in ocean climate model development by exploiting the advantages of both Eulerian and Lagrangian methods within a single model. The technique has already been implemented into a publicly available ocean climate model, which can be found at http://www.gfdl.noaa.gov/fms. Along with the opportunities offered by this new technique comes much work that is required to properly understand the system and the parameter space that results. The first results from the embedded Lagrangian model in the context of overflows are described in BGE-II, however, there is ongoing research to fully develop and exploit the approach across other physical regimes.

### Acknowledgements

Sincere thanks go to Alistair Adcroft for a number of enlightening and very helpful discussions. Thanks also go to the staff at GFDL for hosting MLB several times during the study, and also to Trevor McDougall for hosting MLB and SMG at CSIRO Hobart during the first half of 2011. We appreciate the input of Sonya Legg and Mehmet Llicak, whose helpful comments improved the manuscript. We would also like to acknowledge the constructive comments from two anonymous reviewers, whose input improved the manuscript. This work was supported by an award under the Merit Allocation Scheme on the NCI National Facility at the ANU. This study was supported by the Australian Research Council. MLB is grateful for support from the ARC Centre of Excellence for Mathematics and Statistics of Complex Systems and the ARC Network for Earth System Science.

### Appendix A. List of common symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>horizontal area (m²)</td>
</tr>
<tr>
<td>( \mathcal{A} )</td>
<td>the horizontal area of a grid cell (m²)</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Rayleigh drag coefficient (s⁻¹)</td>
</tr>
<tr>
<td>( C )</td>
<td>tracer concentration</td>
</tr>
<tr>
<td>( C_p^0 )</td>
<td>heat capacity (J K⁻¹ kg⁻¹)</td>
</tr>
<tr>
<td>( D )</td>
<td>detrainment velocity (m s⁻¹)</td>
</tr>
<tr>
<td>( \Delta t )</td>
<td>Eulerian model time step (s)</td>
</tr>
<tr>
<td>( \delta )</td>
<td>fraction of a grid cell participating in an overflow event</td>
</tr>
<tr>
<td>( E2L )</td>
<td>indicates that a property is being transferred from the E to the L system when subscripted</td>
</tr>
<tr>
<td>( E )</td>
<td>indicates a variable for the Eulerian system when subscripted</td>
</tr>
<tr>
<td>( E )</td>
<td>entrainment velocity (m s⁻¹)</td>
</tr>
<tr>
<td>( \eta )</td>
<td>sea surface height deviation from the geoid (m)</td>
</tr>
<tr>
<td>( F )</td>
<td>subgridscale tracer concentration flux (tracer concentration-metres per second)</td>
</tr>
<tr>
<td>( f )</td>
<td>the Coriolis parameter (s⁻¹)</td>
</tr>
<tr>
<td>( f^* )</td>
<td>the horizontal Coriolis parameter (s⁻¹)</td>
</tr>
<tr>
<td>( G )</td>
<td>acceleration due to the body force (m s⁻²)</td>
</tr>
<tr>
<td>( g )</td>
<td>acceleration due to gravity (m s⁻²)</td>
</tr>
<tr>
<td>( H )</td>
<td>the depth of topography (m)</td>
</tr>
<tr>
<td>( h )</td>
<td>the radius of a blob (m)</td>
</tr>
<tr>
<td>( h_e )</td>
<td>potential enthalpy (J)</td>
</tr>
<tr>
<td>( J )</td>
<td>subgridscale scalar flux</td>
</tr>
<tr>
<td>( K )</td>
<td>kinetic energy (J)</td>
</tr>
<tr>
<td>( K )</td>
<td>kinetic energy per unit mass (J kg⁻¹)</td>
</tr>
<tr>
<td>( L )</td>
<td>indicates a variable for the Lagrangian system when subscripted</td>
</tr>
<tr>
<td>( L2E )</td>
<td>indicates that a property is being transferred from the L to the E system when subscripted</td>
</tr>
<tr>
<td>( l )</td>
<td>grid cell search index</td>
</tr>
<tr>
<td>( M )</td>
<td>the advection metric frequency (s⁻¹)</td>
</tr>
<tr>
<td>( m )</td>
<td>mass (kg)</td>
</tr>
<tr>
<td>( \mu )</td>
<td>frictional dissipation parameter (s⁻¹)</td>
</tr>
<tr>
<td>( N )</td>
<td>buoyancy frequency (s⁻¹)</td>
</tr>
<tr>
<td>( n )</td>
<td>outward pointing normal vector</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>the rotation vector of the Earth (s⁻¹)</td>
</tr>
</tbody>
</table>
List of common symbols (continued)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>potential energy (J)</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure (N m$^{-2}$)</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>gravitational potential (m$^2$s$^{-2}$)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>latitude (°N)</td>
</tr>
<tr>
<td>$\psi$</td>
<td>an arbitrary scalar</td>
</tr>
<tr>
<td>$Q$</td>
<td>the number of blobs in a given volume</td>
</tr>
<tr>
<td>$Q_{\text{new}}$</td>
<td>the number of new blobs in a given volume</td>
</tr>
<tr>
<td>$Q_{\text{new}}$</td>
<td>the number of new blobs that are destroyed in a given volume</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density (kg m$^{-3}$)</td>
</tr>
<tr>
<td>$S$</td>
<td>the surface of a volume</td>
</tr>
<tr>
<td>$s$</td>
<td>the vertical coordinate of the Eulerian model</td>
</tr>
<tr>
<td>$S$</td>
<td>a source</td>
</tr>
<tr>
<td>$T$</td>
<td>indicates a variable for the combined E and L systems when subscripted</td>
</tr>
<tr>
<td>$\tau$</td>
<td>the stress tensor (N m$^{-2}$)</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>conservative temperature (C).</td>
</tr>
<tr>
<td>$u$</td>
<td>zonal velocity (m s$^{-1}$)</td>
</tr>
<tr>
<td>$u$</td>
<td>horizontal velocity $u = (u, v)$</td>
</tr>
<tr>
<td>$V$</td>
<td>volume (m$^3$)</td>
</tr>
<tr>
<td>$v$</td>
<td>meridional velocity (m s$^{-1}$)</td>
</tr>
<tr>
<td>$v$</td>
<td>three dimensional velocity $v = (u, v, w)$</td>
</tr>
<tr>
<td>$v^{(s)}$</td>
<td>the three dimensional velocity relative to a surface $v^{(s)} = (u, v, w^{(s)})$</td>
</tr>
<tr>
<td>$w$</td>
<td>vertical velocity (m s$^{-1}$)</td>
</tr>
<tr>
<td>$w^{(h)}$</td>
<td>vertical disasurface velocity (m s$^{-1}$)</td>
</tr>
<tr>
<td>$x$</td>
<td>position of a blob $x = (x, y, z)$</td>
</tr>
<tr>
<td>$x$</td>
<td>velocity of a blob $\dot{x} = (x, y, z)$</td>
</tr>
<tr>
<td>$\dot{x}$</td>
<td>acceleration of a blob $\dot{x} = (\dot{x}, \dot{y}, \dot{z})$</td>
</tr>
<tr>
<td>$x$</td>
<td>zonal unit vector</td>
</tr>
<tr>
<td>$y$</td>
<td>meridional unit vector</td>
</tr>
<tr>
<td>$z$</td>
<td>vertical unit vector</td>
</tr>
</tbody>
</table>

Appendix B. Mechanical energy

The energetics of a turbulent stratified fluid may be partitioned into kinetic energy, potential energy and internal energy, with the potential energy able to be partitioned into background potential energy and available potential energy (Winters et al., 1995). Ocean models, and particularly Boussinesq models, are missing certain energetic conversions (Hughes et al., 2009; Tailleux, 2010), however, analysis of ocean model energetics remains an important tool. Here, we shall ignore internal energy, as global scale ocean models do not capture the conversion from turbulent kinetic energy to internal energy, nor do they capture the conversion of internal energy to background potential energy (Hughes et al., 2009).

The total mechanical energy, $E$, of the system is given by

$$E = P + K + \int_V \rho S_{\text{E}}(\mathbf{E}) \, dV, \quad \text{(B.1)}$$

where $S_{\text{E}}(\mathbf{E})$ is an energy source (e.g. wind stress). Here, the potential energy, $P$, and kinetic energy, $K$, of a finite volume is given by

$$P = \int_V \rho g \, dV, \quad \text{(B.2)}$$

$$K = \frac{1}{2} \int_V \mathbf{v} \cdot \mathbf{v} \, dV. \quad \text{(B.3)}$$

We partition ocean model energetics into contributions from the $E$ system and the $L$ system

$$P = gA \int_E (\rho g z) \, dz + g \sum_{q=1}^{\infty} (mz)^q, \quad \text{(B.4)}$$

$$K = \frac{A}{2} \int_{\mathbf{E}} \mathbf{v}_E \cdot \rho g \, dV + \frac{1}{2} \sum_{q=1}^{\infty} m^q \mathbf{x}_q \cdot \mathbf{x}_q. \quad \text{(B.5)}$$

We now examine the kinetic energy and gravitational potential energy in turn.

B.1. Kinetic energy

The Eulerian model is assumed to be in hydrostatic balance. The relevant formulation for the kinetic energy per unit mass of a hydrostatic fluid is given by

$$K^{(h)} = u^2/2. \quad \text{(B.6)}$$

The justification for ignoring the vertical component of velocity arises because the Hamiltonian of a hydrostatic fluid has kinetic energy only associated with horizontal motions (Bokhove, 2000). The Lagrangian model, on the other hand, is not assumed to be hydrostatic, hence the vertical component of velocity cannot be ignored, giving the kinetic energy per unit mass

$$K = \mathbf{x}^2/2. \quad \text{(B.7)}$$

Due to the different forms of the kinetic energy equations, it is more convenient to study the two systems individually and combine them at the end (rather than starting with the combined system and separating them, as was done for other components of the system, such as seawater mass). Starting with the $E$ system’s kinetic energy, we state the hydrostatic momentum equation (see Section 5.3.1 of Griffies (2004))

$$\partial_t (\rho \mathbf{u}) + (\mathbf{v} \cdot \nabla) (\rho \mathbf{u}) + (f + \nabla) \mathbf{z} \cdot (\rho \mathbf{u}) = - \nabla p + \rho F^{(u)} + S^{(u)}, \quad \text{(B.8)}$$

where $F^{(u)}$ is the horizontal frictional force. Taking the dot product of the horizontal velocity with the momentum Eq. (B.8), we obtain an evolution equation for the hydrostatic kinetic energy per unit volume

$$\partial_t (\rho K^{(h)}) + (\mathbf{v} \cdot \nabla) (\rho K^{(h)}) + K^{(h)} \cdot \nabla \cdot (\rho \mathbf{v}) = \mathbf{u} \cdot (-\nabla p + \rho F^{(u)}) + S^{(K)}, \quad \text{(B.9)}$$

where $S^{(K)}$ is a kinetic energy source per unit volume (kg m$^{-1}$ s$^{-2}$). Here, the third term on the left hand side is due to the compressibility of seawater, and, in a Boussinesq system $\nabla \cdot (\rho \mathbf{v}) = 0$.

We make use of the identity (from Eq. (14.3) of Griffies (2012))

$$\int_V \mathbf{u} \cdot \nabla p = \int_S p (\hat{n} \cdot \mathbf{v}) \, dS + \int_V \rho \frac{d\Phi}{dt} \, dV, \quad \text{(B.10)}$$

where $\Phi$ is the gravitational potential, defined as

$$\Phi = g z. \quad \text{(B.11)}$$

Integrating Eq. (B.9) and using identity (B.10) gives an expression for the kinetic energy budget of a grid cell for the $E$ system

$$\partial_t \left( \int_{\mathbf{E}} K^{(h)} \, dV \right) = - \int_{\mathbf{E}} (\mathbf{v} \cdot \hat{n}) \rho K^{(h)} \, dS - \int_{\mathbf{E}} K^{(h)} \mathbf{z} \cdot (\rho \mathbf{v}) \, dV - \int_{\mathbf{E}} p (\hat{n} \cdot \mathbf{v}) \, dS + \int_{\mathbf{E}} \rho \hat{x} \cdot \mathbf{F}^{(u)} \, dV + \int_{\mathbf{E}} S^{(K)} \, dV. \quad \text{(B.12)}$$

where Green’s Theorem has been used to reach this result. We write the $E$ system’s kinetic energy budget (excluding the exchange with the $L$ system) as
\[
\partial_t K_E = - \int_{V_L} (\mathbf{v}_E \cdot \hat{n}) \rho_L K^{(b)}_E \, dS - \int_{V_L} K^{(b)}_E \nabla \cdot (\rho \mathbf{w}) \, dV \\
- \int_{V_L} \rho_L \nabla \cdot \mathbf{w} \, dS - \int_{V_L} \rho_L \frac{d}{dt} \mathbf{w} \, dV + \int_{V_L} \rho_L \mathbf{F}_E^{(W)} \, dV \\
+ \int_{V_L} S^{(K)}_E \, dV. \tag{13.13}
\]

where \( S^{(K)}_E \) is kinetic energy sources not associated with the L system.

Using the chain rule, we write the rate of change of a blob's kinetic energy as
\[
\frac{d}{dt} (mK) = m \frac{dK}{dt} + K \frac{dm}{dt}. \tag{14.14}
\]
The second term on the right hand side is the change in kinetic energy due to the change in mass of a blob. This term may be written as
\[
K \frac{dm}{dt} = u_1 \left( \frac{dm}{dt} \right)_{(L2)} + u_2 \left( \frac{dm}{dt} \right)_{(L2)}. \tag{15.15.1}
\]

In addition to the transfer of mass between the E and L systems that can occur as a result of entrainment and detrainment, there is also kinetic energy transferred between the two systems when a blob is created and when a blob is destroyed. Using logic analogous to that used to derive Eq. (22a) it can be seen that the E system source term is given by
\[
\int_{V_L} S^{(K)}_E \, dV = \int_{V_L} S^{(K)}_E \, dV - \sum_{q=1}^{Q} \left[ (x^2 + y^2) \left( \frac{dm}{dt} \right)_{(L2)} + u_1 \left( \frac{dm}{dt} \right)_{(L2)} \right]^q \\
- \sum_{q=1}^{Q_{\text{new}}} m (x^2 + y^2) \left( \frac{dm}{dt} \right)_{(L2)} + \sum_{q=1}^{Q_{\text{out}}} m (x^2 + y^2) \left( \frac{dm}{dt} \right)_{(L2)}. \tag{16.1.a.a}
\]

where we note that only the horizontal component of velocity is involved in the transfer. In contrast, the L system is pseudo-non-hydrostatic (see 2 of Section 4.4), and thus the vertical component of velocity is included
\[
\int_{V_L} S^{(K)}_E \, dV = \sum_{q=1}^{Q_{\text{new}}} m (x^2 + y^2) \left( \frac{dm}{dt} \right)_{(L2)} + u_1 \left( \frac{dm}{dt} \right)_{(L2)} \right]^q \\
+ \sum_{q=1}^{Q_{\text{new}}} \frac{m (x^2 + y^2)^{1/2} \left( \frac{dm}{dt} \right)_{(L2)}}{\Delta \tau} + \sum_{q=1}^{Q_{\text{out}}} \frac{m (x^2 + y^2)^{1/2} \left( \frac{dm}{dt} \right)_{(L2)}}{\Delta \tau}. \tag{16.1.a.b}
\]

The change in kinetic energy for the L system component of a grid cell must be equal to the change in kinetic energy of the blobs that reside in that grid cell, less those that leave, plus those that enter, plus the source term
\[
\partial_t \left( \int_{V_L} \rho K_L \, dV \right) = \sum_{q=1}^{Q} \left[ \frac{m}{\Delta \tau} \right]^{q} + \frac{1}{\Delta \tau} \left( \sum_{q=1}^{Q_{\text{new}}} (mK)^q - \sum_{q=1}^{Q_{\text{out}}} (mK)^q \right) \\
+ \int_{V_L} S^{(K)}_L \, dV. \tag{17.1.1}
\]

We note that the second term in the blob rate of change of kinetic energy Eq. (14.14) is not explicitly written here because it is included as part of the last term on the right hand side (the source term) of Eq. (17.1).

The full rate of change of the kinetic energy is given by summing the E and L systems
\[
\partial_t \left( \int_{V_L} \rho K \, dV \right) = \partial_t \left( \int_{V_L} \rho K^{(b)}_E \, dV \right) + \partial_t \left( \int_{V_L} \rho K_L \, dV \right). \tag{18.1.1}
\]

Using Eqs. (13.13) and (17.1.1), and substituting the source Eq. (16.16) gives the rate of change for the combined system
\[
\partial_t \left( \int_{V_L} \rho K \, dV \right) = \partial_t K_E + \sum_{q=1}^{Q} \left[ \frac{m}{\Delta \tau} \right]^{q} + \frac{1}{\Delta \tau} \left( \sum_{q=1}^{Q_{\text{new}}} (mK)^q - \sum_{q=1}^{Q_{\text{out}}} (mK)^q \right) \\
+ \sum_{q=1}^{Q_{\text{new}}} \frac{m (x^2 + y^2)^{1/2} \left( \frac{dm}{dt} \right)_{(L2)}}{\Delta \tau} + \sum_{q=1}^{Q_{\text{out}}} \frac{m (x^2 + y^2)^{1/2} \left( \frac{dm}{dt} \right)_{(L2)}}{\Delta \tau}. \tag{19.1.1}
\]

It can thus be seen that there are significant alterations to the kinetic energy budget of the total system when compared to a purely E system. In particular, there are terms involving the vertical velocity of blobs.

In the implementation in MOM, as described in Section 3.2, there are some alterations that are made to the kinetic energy budget which yield
\[
\partial_t \left( \int_{V_L} \rho K \, dV \right) = \partial_t K_E + \sum_{q=1}^{Q} \left[ \frac{m}{\Delta \tau} \right]^{q} \\
+ \frac{1}{\Delta \tau} \left( \sum_{q=1}^{Q_{\text{new}}} (mK)^q - \sum_{q=1}^{Q_{\text{out}}} (mK)^q \right) \\
+ \sum_{q=1}^{Q_{\text{new}}} \frac{m (x^2 + y^2)^{1/2} \left( \frac{dm}{dt} \right)_{(L2)}}{\Delta \tau} + \sum_{q=1}^{Q_{\text{out}}} \frac{m (x^2 + y^2)^{1/2} \left( \frac{dm}{dt} \right)_{(L2)}}{\Delta \tau}. \tag{20.1.1}
\]

These alterations follow from those made to the momentum budget (40).

B.2. Gravitational potential energy

We now examine the evolution of gravitational potential energy. Multiplying the seawater continuity Eq. (6b) by the gravitational potential, \( \Phi = gz \), and integrating over the volume of a grid cell, we obtain
\[
\int_{V_L} \Phi (\partial_t \rho) \, dV + \int_{V_L} \Phi \nabla \cdot (\mathbf{v} \rho) \, dV = \int_{V_L} \Phi \rho S^{(M)} \, dV. \tag{21.1}
\]

Use of the chain rule, Leibnitz’s rule, and the application of Green’s Theorem yields
\[
\partial_t \left( \int_{V_L} \rho \Phi \, dV \right) = \int_{V_L} \rho (\partial_t \Phi) \, dV - \int_{S_{V_L}} \Phi \mathbf{n} \cdot (\nabla \rho) \, dS + \int_{V_L} S^{(\Phi)} \, dV \tag{22.1.1}
\]

for the potential energy budget of a grid cell. Here \( S^{(\Phi)} = \Phi \rho S^{(M)} \) is a gravitational potential energy source (kg m\(^{-1}\) s\(^{-3}\)).

The gravitational potential energy of an individual blob is given by \( m \Phi \). Thus, the rate of change of the potential energy of an individual blob is given by
\[
\frac{d}{dt} (m \Phi) = m \frac{d}{dt} \Phi + \Phi \frac{dm}{dt}. \tag{23.1}
\]

The change in gravitational potential due to a blob’s exchange with the E system is given by
\[
\Phi \frac{dm}{dt} = \Phi \left[ \left( \frac{dm}{dt} \right)_{(L2)} + \left( \frac{dm}{dt} \right)_{(E2)} \right]. \tag{24.1}
\]
The potential energy source term for the L system is given by the exchange of mass between the E and the L system, which comprises new blobs, destroyed blobs and mass exchanged by individual blobs

\[
\int_{V_L} S_i^{(L)} \, dV = - \sum_{q=1}^Q \phi_i \left[ \frac{d(m_i^q)}{dt} \right]_{(E-L)} + \left[ \frac{d(m_i^q)}{dt} \right]_{(L-E)} - \sum_{q=1}^{Q_{occ}} \frac{m_i^q}{\Delta t} + \sum_{q=1}^{Q_{unocc}} \frac{m_i^q}{\Delta t} + \int_{V_E} S_i^{(E)} \, dV
\]

where \( S_i^{(E)} \) is the gravitational potential energy source term not associated with the exchange between the E and L systems.

The L system’s change in potential energy is given by the source term, the convergence of blobs in/out of the grid cell, and the change of potential energy of blobs that already reside in the grid cell. We express this change mathematically as

\[
\frac{\partial}{\partial t} \int_{V_L} (\rho V)^i \, dV = \int_{V_L} S_i^{(L)} \, dV + \sum_{q=1}^Q \rho_i \frac{d(m_i^q)}{dt} + \int_{S_{E-L}} \Phi_i n \cdot (\nu S) \, dS + \sum_{q=1}^{Q_{occ}} \frac{m_i^q}{\Delta t} - \sum_{q=1}^{Q_{unocc}} \frac{m_i^q}{\Delta t}
\]

where we have used the identity \( m_i \, d\phi = \rho_i \, dV \).

Combining Eqs. (B.22)–(B.26) and revealing specific labels for the E and L system components, we obtain the semi-discrete equation for the evolution of the total system’s gravitational potential energy

\[
\frac{\partial}{\partial t} \int_{V_L} (\rho V)^i \, dV = \int_{V_L} S_i^{(L)} \, dV + \int_{V_L} \rho_i \frac{d(m_i^q)}{dt} \, dV - \int_{S_{E-L}} \Phi_i n \cdot (\nu S) \, dS + \sum_{q=1}^{Q_{occ}} \frac{m_i^q}{\Delta t} - \sum_{q=1}^{Q_{unocc}} \frac{m_i^q}{\Delta t}
\]

There are additional terms in Eq. (B.27) that are not present in a purely Eulerian system. These terms are the result of blob convergence, as well as the change in vertical position of blobs.

**Appendix C. Derivation of blob size for the NCON-like scheme**

Here we derive the mass of a blob required by the blob version of the Cox (1984) NCon scheme to homogenise the tracer concentration of two adjoining cells. First, assume mass conservation

\[
m_i^k + m_i^{k+1} = m_i^0 + m_i^{k+1},
\]

where superscripts \( k \) and \( k+1 \) are upper and lower adjacent grid cells and subscripts \( i \) and \( f \) indicate initial and final states with \( m = \rho V \) being the grid cell mass.

With reference to the schematic in Fig. 3, we can see that the final tracer content of each grid cell will be the initial content, less that transported out of the cell by the Lagrangian scheme, plus that which is transported into the cell by the Lagrangian scheme

\[
m_i^k c_i^k = m_i^0 c_i^0 - m_i^{k-1} c_i^{k-1} + m_i^k c_i^k + m_i^{k-1} c_i^{k-1},
\]

\[
m_i^{k+1} c_i^{k+1} = m_i^k c_i^k + m_i^{k+1} c_i^{k+1} - m_i^{k-1} c_i^{k-1}.
\]

The mass in each grid cell remains constant, so that the mass transferred between cells must be the same

\[
m_i^k = m_i^k, \quad m_i^{k+1} = m_i^{k+1} \quad \Rightarrow \quad m_i^0 = m_i^b.
\]

Dropping the subscripts for the mass and substituting \( m_i^b \) with \( m_i^b \) yields the final tracer concentration from Eqs. (C.2) and (C.3)

\[
c_i^k = c_i^k - c_i^0 \frac{m_i^0}{m_i^b} + c_i^{k+1} \frac{m_i^0}{m_i^b}.
\]

\[
c_i^{k+1} = c_i^{k+1} + c_i^k \frac{m_i^0}{m_i^b} - c_i^{k-1} \frac{m_i^0}{m_i^b},
\]

After the swapping occurs, we want the tracer concentrations to be homogenised between the two grid cells

\[
c_i^k \longrightarrow c_i^{k+1}.
\]

We set Eqs. (C.7) and (C.8) to be equal, eliminating \( c_i^k \) and \( c_i^{k+1} \). Now, we solve for \( c_i^b \)

\[
c_i^b = c_i^{k+1} - c_i^{k-1} + \frac{m_i^0}{m_i^{k+1}} (c_i^{k+1} - c_i^k) = (c_i^{k+1} - c_i^k)
\]

In the limit where \( m_i^b = m_i^{k+1} \), we see that the blobs are half the mass of the grid cells, \( m_i^b = \frac{1}{2} m_i^k \).

**Appendix D. Total thickness, total tracer and total density calculations**

The purpose of this appendix is to find the total thickness, total tracer and total density of the combined E and L systems.

**D.1. Thickness**

To find the volume of a grid cell in the combined system, we first write the volume of the combined system as the sum of the E system and L system

\[
\int_{V_L} dV = \int_{V_E} dV + \int_{V_L} dV.
\]

The volume of a grid cell is the volume of the E system and all of the blobs in that grid cell. Mass is a prognostic variable for the blobs, and density is found using an equation of state. Hence, we use \( V = m/\rho \) to find the volume of an individual blob, with the total L system volume in a cell given by the sum of blobs in that cell

\[
\int_{V_L} dV = \sum_{q=1}^Q m_i^q \rho_i.
\]

Combining these equations gives

\[
\int_{V_L} dV = \int_{V_E} dV + \sum_{q=1}^Q m_i^q \rho_i.
\]

To find the total thickness of a grid cell, and taking the horizontal area of the grid cell out as a common factor, we write the semi-discrete equation for the total thickness of a grid cell as

\[
dz^{z_1} = \int dz + A \sum_{q=1}^Q \frac{m_i^q}{\rho_i}
\]

**D.2. Concentration**

To find the total tracer concentration, we first write the tracer content of the combined system

\[
m_i^b = m_i^b.
\]

\[
m_i^{k+1} = m_i^{k+1} \quad \Rightarrow \quad m_i^0 = m_i^b.
\]

\[
\Rightarrow m_i^0 = m_i^b.
\]
\[ \int_{V_t} \rho \omega^2_{t} dV = \int_{V_t} \rho \omega^2_{t} dV + \sum_{q=1}^{m} (mC)^{q}. \]  \text{(D.5)}

In a grid cell, we say that the total density and tracer concentration is uniform, so we can take the concentration outside of the volume integral for the combined system:

\[ \int_{V_t} \rho \omega_C dV = \int_{V_t} \rho \omega_C dV + \sum_{q=1}^{m} (mC)^{q}. \]  \text{(D.6)}

In analogy to the combined system volume Eq. (D.2) we can write down the mass of the combined system as the sum of the E and L systems:

\[ \int_{V_t} \rho dV + \sum_{q=1}^{m} m^{q} = \int_{V_t} \rho_{e} dV + \sum_{q=1}^{m} (mC)^{q}. \]  \text{(D.7)}

Division by the total system mass yields the semi-discrete form for the total tracer concentration:

\[ \rho_t = \frac{m_{t}}{V_{t}}, \]  \text{(D.9)}

which we may then partition into the E and L systems:

\[ \rho_t = \frac{m_{t}}{V_{t}} + \frac{m_{L}}{V_{L}}. \]  \text{(D.10)}

In terms of model variables, we rewrite the total density as:

\[ \rho_{t} = \int_{V_t} \rho dV + \sum_{q=1}^{m} m^{q} \]  \text{(D.11)}

References


Cox, M.D. 1978. A Primitive Equation, 3-Dimensional Model of the Ocean. NOAA/Geophysical Fluid Dynamics Laboratory.


