Exercise 1

Prove the following properties of cyclotomic polynomials over a field for which the polynomials exist.

(a) \( Q_{mp}(x) = Q_m(x^p)/Q_m(x) \) for a prime \( p \) that does not divide \( m \in \mathbb{N} \)

(b) \( Q_{mp}(x) = Q_m(x^p) \) for all \( m \in \mathbb{N} \) divisible by the prime \( p \)

(c) \( Q_{mp^k}(x) = Q_{mp}(x^{p^k-1}) \) for a prime \( p \) and arbitrary \( m, k \in \mathbb{N} \)

(d) \( Q_{2n}(x) = Q_n(-x) \) for odd \( n \geq 3 \)

(e) \( Q_n(0) = 1 \) for \( n \geq 2 \)

(f) \( Q_n(x^{-1})x^{\phi(n)} = Q_n(x) \) for \( n \geq 2 \)

(g) \[
Q_n(1) = \begin{cases} 
0 & \text{if } n = 1 \\
p & \text{if } n \text{ is a power of the prime } p \\
1 & \text{if } n \text{ has at least two distinct prime factors}
\end{cases}
\]

(h) \[
Q_n(-1) = \begin{cases} 
0 & \text{if } n = 2 \\
-2 & \text{if } n = 1 \\
p & \text{if } n \text{ is 2 times a power of the prime } p \\
1 & \text{otherwise}
\end{cases}
\]

Exercise 2

(a) Determine the primitive 8th roots of unity in \( \mathbb{F}_9 \) and the primitive 9th roots of unity in \( \mathbb{F}_{19} \).

(b) Factor \( Q_5 \) and \( Q_{12} \) over \( \mathbb{F}_{11} \), and factor \( Q_4 \) over \( \mathbb{F}_5 \), \( \mathbb{F}_{13} \), and \( \mathbb{F}_{17} \).

Exercise 3

Let \( \zeta \) be an \( n \)th root of unity over a field \( K \). Then

\[
1 + \zeta + \zeta^2 + \ldots + \zeta^{n-1} = \begin{cases} 
0 & \text{if } \zeta \neq 1 \\
\frac{n}{\zeta} & \text{if } \zeta = 1.
\end{cases}
\]

Exercise 4

(a) Let \( K \) be an arbitrary field and \( n \geq 2 \). If the polynomial \( x^{n-1} + x^{n-2} + \ldots + x + 1 \) is irreducible over \( K \), then \( n \) is prime.

(b) Find the least prime \( p \) such that \( x^{22} + x^{21} + \ldots + x + 1 \) is irreducible over \( \mathbb{F}_p \).

(c) Find the ten least primes \( p \) such that \( x^{p-1} + x^{p-2} + \ldots + x + 1 \) is irreducible over \( \mathbb{F}_2 \).