lines $Ax = By$ and $Bx = Cy$. A construction for the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$
was related to the parametric form $x = a(t^2)/(1 + t^2)$, $y = 2bt/(1 + t^2)$. The
series for $\log_e N$ was exhibited as the result of certain rearrangements of the
series $1 - 1/2 + 1/3 - 1/4 + \cdots$. Unusual weighted sums were considered as
approximations for a definite integral and contrasted with the usual "rules." Finally, $\pi$ was computed by the use of the inverse sine.

7. Professor Pawley described extensions of Newton’s method for approxi-
mating real roots in which the desired root is approximated by $x$ intercepts of
curves of higher order of contact than the tangent. He derived an upper bound
to the error involved in these approximations. In particular, he simplified the
well known parabolic approximation by expanding an $x$ intercept of the parab-
ola into a convergent alternating series. An upper limit to the error involved
in this approximation was derived and illustrated by an example.

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**EQUATIONS IN QUATERNIONS**

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1. **Introduction.** We prove the existence of a quaternion root of the equation

$$a(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_m = 0, \quad a_m \neq 0,$$

with coefficients from the algebra of real quaternions. The writer had proved this result when $m$ is odd, but the proof was rendered obsolete when Nathan Jacobson pointed out that the result (without restriction on $m$) can be obtained as a simple consequence of some work of Ore [1]. This is given in detail in §2.

In §3 we give a method for obtaining the roots of (1), which is not very practical in the sense that it involves the simultaneous solving of two real equations of degree $2m-1$. The method used is a generalization of Sylvester’s treatment [2] of the quadratic equation corresponding to (1). Sylvester’s conclusion that a quadratic equation has six roots is incorrect because he neglects to show that they exist, and also overlooks the possibility of an infinite number of roots; a complete analysis is given in §4 (Theorem 2). The number of roots of (1) is discussed in §5 (Theorem 3), necessary and sufficient conditions being given for an infinite number of roots.

The proof given here of the existence of a root of (1) is stated for the general case where the coefficients of the equation are quaternions over any real-closed field $R$ (i.e., no sum of squares in $R$ is equal to $-1$, and no algebraic extension of $R$ has this property).

Reinhold Baer, on hearing of this existence proof, proved the converse, so that we have the following strong result:

**Theorem 1.** Let $D$ be a non-commutative division algebra with centrum $C$. Then every equation (1) with coefficients from $D$ has a solution in $D$ if and only if $C$ is a real-closed field, and $D$ is the algebra of real quaternions over $C$. 
The necessity of these conditions is shown in §6; the proof gives a slightly stronger result than stated in the theorem above, since only those equations with coefficients from \( C \) are used. The writer is indebted to Jacobson and Baer for permission to give their proofs here.

Note that (1) is a special equation. The most general quadratic term, for example, would have the form \( bxxcd \), involving three coefficients. However, the results are valid for equations similar to (1) having all coefficients to the right of the powers of the unknown.

2. **The existence of a root.** Let the coefficients of (1) be quaternions over a real-closed field \( R \). By replacing the quaternions \( a_r \) by their conjugates \( \overline{a_r} \), we obtain a polynomial \( \overline{a}(x) \). We multiply this on the right by \( a(x) \), and allow \( x \) to be commutative with the coefficients. Thus we obtain a polynomial \( a(x) \) with coefficients in \( R \), which is, by the fundamental theorem of algebra, factorable into linear factors in \( R(i, x) \), and hence a fortiori in \( R(i, j, x) \). Theorem 1 on p. 494 of Ore's paper [1] states that any other factorization of \( a(x) \) in \( R(i, j, x) \) must also have linear factors. Now \( a(x) \) can be factored into irreducible factors, and these are factors of \( a(x) \). Hence by Ore's theorem they are linear. Taking \( x-c \) as the right linear factor, we can write

\[
a(x) = (x^{m-1} + b_1x^{m-2} + \cdots + b_{m-1})(x - c).
\]

That \( x = c \) is a root of \( a(x) = 0 \) is not apparent from this equation, since we have assumed that \( x \) commutes with the coefficients. However, upon rewriting the above equation in a form analogous to (1),

\[
a(x) = x^m + (b_1 - c)x^{m-1} + (b_2 - b_1c)x^{m-2} + \cdots + (b_{m-1} - b_{m-2}c)x - b_{m-1}c,
\]

we verify immediately that \( c \) is a root.

3. **A right-division algorithm.** The norm \( n \) of any quaternion \( x \) is defined as the product of \( x \) and its conjugate \( \overline{x} \); and the addition of \( x \) and \( \overline{x} \) gives \( t \), the trace of \( x \). It is well known that \( x \) satisfies the equation

\[
x^2 - tx + n = 0.
\]

We now divide \( a(x) \) on the right by the expression in this equation, and obtain the algorithm

\[
a(x) = q(x^2 - tx + n) + f(a_r, t, n)x + g(a_r, t, n),
\]

the remainder being comprised of the last two terms, polynomials in \( t, n \) and the coefficients of \( a(x) \). The nature of the quotient \( q \) does not interest us. Note that the remainder vanishes for any root of equation (1), and conversely.

When \( f \neq 0 \), the vanishing of this remainder can be expressed in the form

\[
x = -\frac{1}{f} \cdot g = -\frac{1}{ff} (\overline{f}g),
\]

where \( \overline{f} \), the conjugate of \( f \), is obtained by replacing each \( a_r \) in \( f \) by its conjugate
\( \bar{a}_r \). Since the conjugate of a product equals the product of the conjugates in reverse order, we have

\[ \bar{x} = -\frac{1}{f \overline{f}} (\bar{g}f). \]

By multiplication and addition of the last two equations we get the norm and trace of \( x \); thus

\[ n = \frac{1}{f \overline{f}} (\overline{g}g), \quad t = -\frac{1}{f \overline{f}} (\overline{f}g + \bar{g}f), \]

since \( f \overline{f} \) and \( \bar{g}g \) have real coefficients and are commutative with the other polynomials. These equations may be written in the forms

\[ N(t, n) = n \overline{f}f - \overline{g}g = 0, \quad T(t, n) = t \overline{f}f + \overline{f}g + \bar{g}f = 0, \]

where \( N(t, n) \) and \( T(t, n) \) are polynomials in \( t \) and \( n \) with real coefficients.

First we note that any root \( x_0 \) of (1) has a trace \( t_0 \) and a norm \( n_0 \) which satisfy equations (6). This is apparent except when \( f(a_r, t_0, n_0) = 0 \), in which case equation (4) is meaningless. But in this case equation (3) implies that \( g(a_r, t_0, n_0) \) vanishes, and equations (6) are satisfied.

Conversely, any simultaneous real solution \((t_0, n_0)\) of (6) gives one or more roots of (1). First suppose that \( f(a_r, t_0, n_0) \neq 0 \). The values \( t_0 \) and \( n_0 \) can be substituted in (4) to give a quaternion \( x_0 \), and since these quantities satisfy (2), equation (3) indicates that \( x_0 \) is a root of \( a(x) = 0 \). It is important to note that in this case one solution of equations (6) gives exactly one solution of (1).

On the other hand, if \( f(a_r, t_0, n_0) = 0 \), then the first equation (6) gives

\[ \bar{g}(a_r, t_0, n_0)g(a_r, t_0, n_0) = 0. \]

But the product of a quaternion and its conjugate is zero only if the quaternion is zero, and hence \( g(a_r, t_0, n_0) = 0 \). Returning to (3), we see that any solution of

\[ x^2 - t_0x + n_0 = 0 \]

is also a solution of (1). The above analysis enables us to inquire into the number of roots of (1), but first we must know the number of solutions in quaternions of equation (7).

4. Quadratic equations. Consider the equation

\[ x^2 + bx + c = 0, \quad c \neq 0, \]

\( b \) and \( c \) being real quaternions. We assume that \( t(b) \), the trace of \( b \), is zero, for otherwise the substitution \( x = y - \frac{1}{4}t(b) \) gives a quadratic equation with the required property. For example, the substitution \( x = y + \frac{1}{4}t_0 \) in equation (7) gives

\[ y^2 - d = 0, \quad d = \frac{1}{4}t_0^2 - n_0. \]
We shall need the treatment of this equation to complete the discussion of (8). Suppose that \( y = y_0 + y_1 i + y_2 j + y_3 i j \), each \( y \) with a subscript being real. We substitute in (9) and separate the result with respect to the linearly independent units 1, \( i \), \( j \), and \( i j \), to obtain
\[
y_0^2 - y_1^2 - y_2^2 - y_3^2 = d, \quad y_0 y_1 = y_0 y_2 = y_0 y_3 = 0.
\]
If \( d \geq 0 \), then \( y_1 = y_2 = y_3 = 0 \), and the roots of (9) are \( \pm \sqrt{d} \). If \( d < 0 \), then \( y_0 = 0 \), and we obtain an infinitude of quaternion solutions of (9), corresponding to the real solutions of \( y_1^2 + y_2^2 + y_3^2 = -d \). Henceforth we take \( b \) and \( c \) to be not both real.

Applying the division algorithm of §3 to (8), we obtain the following values for the functions \( f \) and \( g \):
\[
\begin{align*}
f &= b + t, \\
g &= c - n.
\end{align*}
\]
Hence equations (6) become
\[
\begin{align*}
nt^2 + nb^2 - cc + n(c + \bar{c}) - n^2 &= 0, \\
& \text{(11)} \\
t^3 + tbb - 2nt + t(c + \bar{c}) + bc + \bar{c}b &= 0, \\
& \text{(12)}
\end{align*}
\]
since \( b + \bar{b} \) vanishes. Following the theory of §3, we see that if a real solution \( t = t_0, \ n = n_0 \) of these equations satisfies \( f = 0 \) and \( g = 0 \), then we have \( b = -t_0 \) and \( c = n_0 \). But \( t_0 \) is real, and \( b \) has zero trace, so that both are zero; also, \( c \) must be real. Hence equation (8) reduces to one of type (9), contrary to hypothesis. Consequently the solutions of (11) and (12) do not satisfy \( f = 0 \), and this, by §3, implies that each of these real solutions gives exactly one solution of (8); the solution is given by the substitution of the functions (10) in equation (4).

We introduce the notation
\[
\begin{align*}
B &= bb \bar{b} + c + \bar{c}, \\
C &= cc \bar{c}, \\
D &= \bar{b}c + \bar{c}b,
\end{align*}
\]
noting that \( B, C, \) and \( D \) are real. First we consider solutions of (11) and (12) with \( t = 0 \), so that \( D = 0 \), by (12). Then the possible values of \( n \) are given by (11), which reduces to
\[
n^2 - Bn + C = 0. 
\]
We want real roots; any real root will be positive because of the manner in which equations (11) and (12) were set up. Thus we obtain 0, 1, or 2 roots of (8) according as \( B^2 - 4C \) is negative, zero, or positive.

Finally, we search for solutions of (11) and (12) with \( t \neq 0 \). We solve (12) for \( n \); thus
\[
n = (t^3 + tB + D)/2t,
\]
and we substitute this value in (11) to obtain
\[
\begin{align*}
t^6 + 2Bt^4 + (B^2 - 4C)t^2 - D^2 &= 0.
\end{align*}
\]
Each distinct real root of this equation gives us, by use of (15), a root of (8). In order to find the number of real roots of (16), we prove the following:

**Lemma 1.** If $B$ is negative, so is $B^2 - 4C$.

**Proof.** Since $bb$ is not negative, $c + \bar{c}$ must be negative by the hypothesis. We can write

$$B^2 - 4C = bbB + bb(c + \bar{c}) + (c - \bar{c})^2.$$ 

If $c$ has the form $c_0 + c_1i + c_2j + c_3k$, then the last term on the right side of this equation equals $-4(c_1^2 + c_2^2 + c_3^2)$. Hence the three terms on the right are real, and none of them is positive. They cannot all be zero, for that would imply that $b = 0$ and $c = \bar{c}$, contrary to hypothesis, and this proves the lemma.

We consider (16) as a cubic in $t^2$, and look for positive roots. If $D \neq 0$, the number of positive roots is one by Descartes’ rule of signs and Lemma 1. Thus equation (16), considered as a sextic, has two real roots when $D \neq 0$.

If $D = 0$, we divide the obvious zero roots out of (16), and have

$$(17) \quad t^4 + 2Bt^2 + B^2 - 4C = 0.$$ 

Considering this as a quadratic equation, we see that the discriminant is not negative, so that the roots are real. If $B^2 - 4C$ is positive, $B$ is positive by Lemma 1, and the quadratic (17) has negative roots. Hence the quartic (17) has no real root. Similarly, if $B^2 - 4C$ is zero, we find that the quartic (17) has no real roots other than zeros. In both these cases, all roots of (8) are obtained from (14). Finally, if $B^2 - 4C$ is negative, the quartic (17) has exactly two real roots, giving two solutions of (8). Note that in this case no solutions of (8) result from (14).

We summarize these results in the following:

**Theorem 2.** Consider the quaternion equation (8), the trace of $b$ being zero. If $b$ and $c$ are real (so that $b = 0$), the equation has an infinite number of roots or just two roots according as $c$ is positive or negative. Otherwise, the equation has one or two roots according as the quantities defined in (13) satisfy the relations $D = B^2 - 4C = 0$ or not.

5. **The number of roots of (1).** We suppose first that no real solution of equations (6) satisfies $f = g = 0$, so that there is a one-to-one correspondence between the roots of (1) and the real solutions of (6). We now need some information about the nature of the functions $f$ and $g$ of equation (3).

**Lemma 2.** The functions $f$ and $g$ of equation (3) are of degree $m - 1$ in $n$ and $t$; moreover, $f$ has only one term of this degree, namely $t^{m-1}$. Also, every term of $g$ is divisible by $n$, with the exception of $a_m$.

**Proof.** The proof is by induction on $m$. Equations (10) indicate the truth of the lemma in case $m = 2$. We now obtain recurrence relations for $f$ and $g$. The polynomial of degree $m+1$ analogous to $a(x)$ can be written in the form
$a(x) \cdot x + a_{m+1}$, and corresponding to the algorithm (3) we have
\[
a(x) \cdot x + a_{m+1} = qx(x^2 - tx + n) + fx^2 + gx + a_{m+1} = (qx + f)(x^2 - tx + n) + ftx - fn + gx + a_{m+1}.
\]
Calling the remainder in the last expression above $Fx + G$, we have the relations $F = ft + g$ and $G = -fn + a_{m+1}$. The induction is completed by noting that if the functions $f$ and $g$ have the properties stated in the lemma, so do $F$ and $G$, $m$ being replaced by $m+1$.

It is a consequence of the above lemma that equations (6) are of degree $2m-1$ in $n$ and $t$. Now it is known [3] that the curves represented by (6) cannot have more than $(2m-1)^2$ intersections provided that the polynomials $N$ and $T$ are relatively prime; and this is the case when neither of the two resultants of $N$ and $T$ vanishes identically. We now show that this is the case.

By Lemma 2, equations (6) can be written in the forms
\[
N(t, n) = c_0 + c_1 t + \cdots + c_{2m-2} t^{2m-2} + c_{2m-1} t^{2m-1},
\]
\[
T(t, n) = d_0 + d_1 t + \cdots + d_{2m-2} t^{2m-2} + t^{2m-1},
\]
the coefficients $c_r$ and $d_r$ being polynomials in $n$. Then the resultant obtained by eliminating $t$ is
\[
\begin{vmatrix}
  c_0 & c_1 & \cdots & c_{2m-2} & 0 & \cdots & 0 \\
  0 & c_0 & c_1 & \cdots & c_{2m-3} & c_{2m-2} & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & c_0 & c_1 & \cdots & c_{2m-2} \\
  d_0 & d_1 & \cdots & d_{2m-2} & 1 & 0 & \cdots & 0 \\
  0 & d_0 & \cdots & \cdots & d_{2m-2} & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & d_0 & d_1 & \cdots & d_{2m-2} & 1
\end{vmatrix}
\]

This is a polynomial in $n$; in order to show that it does not vanish identically, we prove that it has a non-zero constant term. When we set $n = 0$, Lemma 2 and equation (6) show that each $c_r = 0$ for $r = 1, 2, \cdots, 2m-2$, whereas $c_0$ assumes the value $-a_m a_m$; that is, all elements of the determinant above the principal diagonal vanish. Hence the value of the constant term of this resultant is $(-a_m a_m)^{2m-1}$, which is not zero.

Having shown that the polynomials $f$ and $g$ have no common factor involving $t$, we now eliminate the possibility that they have a polynomial in $n$ alone as a common factor. We could show that this is not possible by proving that the resultant which eliminates $n$ does not vanish identically. But it is easier to proceed directly. If a polynomial in $n$ divides $T$, it must divide the coefficient of the highest power of $t$. But this coefficient is unity.
Having shown that equation (1) cannot have more than \((2m-1)^2\) roots when no real solution of (6) satisfies \(f=g=0\), we turn to the case where these equations are satisfied by certain solutions of (6). Let there be \(s\) such real solutions. Then we have \(s\) equations of type (7), each having either two roots or an infinite number of roots (by Theorem 2). If the number of roots is finite, that is, if these equations have two roots each, we divide them out of equation (1). Thus we obtain an equation of degree \(m-2s\), which has no factors of the form (7), and hence has at most \((2m-4s-1)^2\) quaternion roots. Adding \(2s\) to account for the roots of the quadratic equations, we note that

\[2s + (2m - 4s - 1)^2 < (2m - 1)^2,\]

when \(m \geq 2s\) and \(s > 0\). Hence we have shown that if the number of roots of (1) is finite, it cannot exceed \((2m-1)^2\). Theorem 2, and in particular equation (9), can be used now to give the following result:

**Theorem 3.** Equation (1) has an infinite number of quaternion roots if and only if \(a(x)\) is divisible by an expression of type (7), with the real values \(t_0\) and \(n_0\) satisfying the inequality \(t_0^2 < 4n_0\). If the number is finite, it cannot exceed \((2m-1)^2\).

6. **The necessity of the conditions of Theorem 1.** Denote by \(u\) the order of \(D\) over its centrum \(C\). Thus there exist \(u\) elements in \(D\) which are linearly independent over \(C\), but any \(u+1\) elements in \(D\) are linearly dependent over \(C\). Given any element \(x\) in \(D\), there exist therefore elements \(c_i\) in \(C\) such that

\[x^u + \sum_{i=0}^{u-1} c_ix^i = 0.\]

Since \(D\) is a division algebra, it follows now that the sub-field \(C(x)\) of \(D\) which is generated by adjoining the element \(x\) of \(D\) to \(C\), is a commutative field, finite over \(C\), and the irreducible equation in \(C\) which is satisfied by \(x\) has a degree not exceeding \(u\). Since every equation in \(C\) has a solution in \(D\), this implies that the degrees of irreducible equations in \(C\) do not exceed \(u\).

**Lemma 3.** Let \(A\) denote the essentially uniquely determined algebraically closed commutative field which contains \(C\) and is algebraic over \(C\). Then \(A\) is finite over \(C\).

**Proof.** Suppose first that \(C\) is of characteristic \(p \neq 0\). We prove that there is no element \(t\) in \(C\) such that the equation \(z^p-t=0\) has no solution in \(C\). For, if there were such an element, then none of the equations \(z^{p^i}-t=0\) would have a solution in \(C\). Since this last equation has the form \((z-t^i)^{p^i}=0\) in the field \(A\), it has one and only one solution in \(A\); since the \(p^{i-1}\)-th power of this solution is a solution of \(z^p-t=0\), it follows that each of these equations is irreducible in \(C\). But this is impossible, since some \(p^i\) is larger than \(u\). Hence every element in \(C\) is the \(p\)-th power of an element in \(C\). Consequently [4], \(A\) is separable over \(C\); and this result is also true when the characteristic of \(C\) is zero.

If \(B\) is some field between \(A\) and \(C\), and if \(B\) is finite over \(C\), then \(B\) is a simple extension of \(C\) since it is separable over \(C\). Thus the degree of \(B\) over \(C\) is equal to the degree of the irreducible equation in \(C\) whose solution generates \(B\) over \(C\). Since the degrees of irreducible equations in \(C\) do not exceed \(u\), it follows that the degrees of finite extensions of \(C\) do not exceed \(u\). Consequently,
there exists a field \( M \) between \( A \) and \( C \) which is finite of maximal degree over \( C \). If \( w \) is any element of \( A \), then \( M(w) \) is finite over \( C \). Since the degree of \( M \) over \( C \) is as large as possible, it follows that \( M \) and \( M(w) \) have the same degree over \( C \). Hence \( w \) is in \( M \), and \( M = A \), and the lemma is proved.

Now it follows from a theorem by Artin-Schreier [5] that either \( C \) is a real-closed field, or \( A \) equals \( C \). The latter is impossible since \( C \), the centrum of a non-commutative division algebra, cannot be algebraically closed. Hence \( C \) is a real-closed field; but the only non-commutative division algebra over \( C \) is the algebra of real quaternions [6], and this completes the proof.

References

3. Cf., Böcher's Introduction to Higher Algebra, Macmillan, 1927, problem 4 on p. 239, and the theorem on p. 202. The theorem on p. 202 is not precisely what we want here, since we need a proposition about polynomials in two variables. Two polynomials \( f(x, y) \), \( g(x, y) \) have two resultants, \( R_x \) obtained by eliminating \( x \), and \( R_y \) by eliminating \( y \). Böcher's proof shows that the vanishing of \( R_x \) identically is a necessary and sufficient condition for the two polynomials to have a common factor involving \( x \). But it is possible for \( f(x, y) \) and \( g(x, y) \) to have a common factor in \( y \) alone, with \( R_x \) not identically zero; in such a case \( R \) vanishes (e.g., consider the polynomials \( xy - 4x - 3y + 12 \), \( xy + y^2 - 4x - 4y \)).