Stability in the water column
Gravity acts on vertical density gradients in the ocean to either stabilise or destabilise the water column:

Three examples of ocean stability are

- **Stable Stratification**: $\frac{de}{dz} > 0$
- **Neutral Stratification**: $\frac{de}{dz} = 0$
- **Unstable Stratification**: $\frac{de}{dz} < 0$
What is the vertical density gradient across the top 200m at (i) the equator, and (ii) 60S? (iii) Estimate the meridional (north-south) gradient at 60S.

Archimedes Principle:
The upward force acting on a body within a fluid, is equal to the weight of the fluid displaced by the body

Wood $\rho=500\text{kg/m}^3$

Weight = 5000N

Weight = $mg = \rho Vg$

$\rho=1000\text{kg/m}^3$

And a weight of $1000 \times 10 = 10000\text{N}$

Wood block will accelerate upwards!
What would happen with a concrete block $\rho=2400\text{kg/m}^3$
Archimedes Principle:
The upward force acting on a body within a fluid, is equal to the weight of the fluid displaced by the body.

PROOF

\[ F_1 = P_1A = D \rho gh^2 \]
\[ P_1 = D \rho g \]
\[ A = h^2 \]
\[ F_2 = P_2A = (D+h) \rho gh^2 \]
\[ P_2 = (D+h) \rho g \]

Pressure, \( P = \frac{h \rho g}{A} \)
\[ m = \rho V \]
\[ A = h^2 \]
\[ V = h^3 \]

Net Force on box, \( F = F_1 - F_2 \)
\[ F = D \rho gh^2 - (D+h) \rho gh^2 \]
\[ = -h \rho pg = -V \rho g \]
\[ = -mg \]
\[ = -\text{weight of water displaced} \]

Weight of displaced water
\[ = mg = \rho_1 V g \]
Imagine a balloon filled with seawater at rest (equilibrium depth)

If the balloon moves into a fluid of different density...

Buoyancy forces will try to return the balloon to its equilibrium depth

Imagine a balloon filled with seawater at rest (equilibrium depth) in two different density gradients

When set in motion the balloon will oscillate about its equilibrium depth with a frequency proportional to the density gradient

The larger the density gradient, the higher the frequency (Brunt-Väisälä or buoyancy frequency) and the shorter the period

Large density gradient therefore gives rise to stable conditions - with little mixing

Less energy required to mix

More energy required to mix
Where does the buoyancy frequency come from?

\[ N = \sqrt{\frac{g}{\rho}} \left( \frac{d\rho}{dz} \right) \]

\[ \rho_0 \text{ Density} \]

\[ \rho(z) = \rho_0 + kz \]

Net Force, \( F = \rho_1 V_g - \rho_2 V_g \)

\[ = V_g (\rho_0 + kh - (\rho_0 + k(h + z'))) \]

\[ = -V_g k z' \]

\[ \rho_2 > \rho_1 \]

Conservations of linear momentum: \( F = ma \)

\[ a = \frac{d^2z'}{dt^2} \]

\[ d^2z' = -\frac{g \rho}{\rho_1} \frac{d\rho}{dz} \]

Frequency \( f = \frac{N}{2\pi} \)

Period \( T = 2\pi/ N \)

Simple harmonic oscillator:

\[ \frac{d^2z}{dt^2} + N^2 z = 0 \]

Solving the equation:

\[ m^2 + N^2 = 0 \]

\[ m^2 = -N^2 \]

Two cases:

- \( N^2 < 0 \)
  \[ m_{1,2} = \pm \sqrt{-N^2} \]
  \[ z(t) = Ae^{\sqrt{-N^2}t} + Be^{-\sqrt{-N^2}t} \]

- \( N^2 > 0 \)
  \[ m_{1,2} \text{ are complex solutions} \]
  \[ z(t) = A \cos(Nt) + B \sin(Nt) \]
Where does the buoyancy frequency come from?

$$N = \sqrt{\frac{g}{\rho}} \frac{d\rho}{dz}$$

**Solution 1: run away solution (if \( N^2 \) is negative, if \( \frac{d\rho}{dz} < 0 \))**

$$z(t) = Ae^{\frac{g}{\rho} \frac{d\rho}{dz} t} + Be^{-\frac{g}{\rho} \frac{d\rho}{dz} t}$$

- \( \rho_2 > \rho_1 \)
- Positively buoyant
- Net Force, \( F = \rho_1 Vg - \rho_2 Vg \)
  \( = Vg((\rho_1 + kh) - (\rho_0 + k(h + z'))) \)
  \( = Vg(kz) \)
  \( = Vgkz \)
  \( a = -\frac{1}{\rho}(gkz) \)

**Solution 2: simple harmonic oscillator (if \( N^2 \) is positive, if \( \frac{d\rho}{dz} > 0 \))**

$$z(t) = A \cos\left(\sqrt{\frac{g}{\rho}} \frac{d\rho}{dz} t\right) + B \sin\left(\sqrt{\frac{g}{\rho}} \frac{d\rho}{dz} t\right)$$

- \( \rho_2 > \rho_1 \)
- Positively buoyant
- Net Force, \( F = \rho_1 Vg - \rho_2 Vg \)
  \( = Vg((\rho_1 + kh) - (\rho_0 + k(h + z'))) \)
  \( = Vg(kz) \)
  \( = Vgkz \)
  \( a = -\frac{1}{\rho}(gkz) \)

\( F = ma \)
\( = \rho_1 Va \)
\( a = \frac{d^2z}{dt^2} = \frac{g}{\rho} \frac{d\rho}{dz} \)
The influence of stability is usually expressed by the buoyancy frequency which is also referred to as the Brunt-Väisälä Frequency:

\[ N = \sqrt{\frac{g}{\rho} \frac{d\rho}{dz}} \]

- The buoyancy frequency (N) quantifies the importance of stability
- Larger frequency (N) the more stable the water column


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**Buoyancy frequency**

- The buoyancy frequency is the frequency at which a vertically displaced parcel will oscillate within a stable environment.
- Parcels that are vertically perturbed are accelerated back to their initial position by a restoring buoyancy force.
- In an unstable environment, vertically displaced fluid parcels do not oscillate, but instead continue to move in the direction of their initial displacement.
Brunt-Väisälä Frequency

Sir David Brunt


Vilho Väisälä


Brunt-Väisälä frequency (N)

Fig. 2.8 A profile of buoyancy frequency N, temperature T, and density ρ in the North Atlantic Ocean.
Brunt-Väisälä Oscillation or Buoyancy Oscillation

• The frequency with which a parcel of seawater will oscillate when displaced a small vertical distance from its equilibrium position

\[ T = \frac{2\pi}{N} \]

If the stratification is strong, then \( N \) is large and \( T \) is small

⇒ Fast oscillations in a strongly stratified ocean.

If the stratification is weak (deep ocean) then \( N \) is small and \( T \) is large

• These oscillations are called internal waves – they occur along density gradients

Internal Waves

Strait of Gibraltar

Somalia
An internal wave travelling up the Derwent River estuary in Hobart, Tasmania. The effect of the wave is visible by streaks of smooth water produced by the convergences above the wave troughs. The wave seen here is of the shortest possible wavelength (compare the distance between the streaks with the yacht) and has a period of 10 - 20 minutes.

Taken from Matthias Tomczak’s webpage:


Brunt Väisälä frequency

\[ N = \sqrt{\frac{g \, dp}{\rho \, dz}} \]
Brunt Väisälä frequency

\[ N = \sqrt{\frac{g}{\rho}} \frac{dp}{dz} \]

Units: \( s^{-1} \)

\[ T = \frac{2\pi}{N} \]

Acts as a barrier for mixing of nutrients up into photic zone.

Classify these ocean scenarios:
Are they stable, unstable or neutral?

Increasing Property

Depth (m)
What happens if 
\[ \frac{d\rho}{dz} < 0 \]
- The fluid parcel will continue to move in the direction pushed - results in **convective overturning**

Convective overturning
- Leads to the formation of winter mixed layers
- E.g. night time cooling during winter, cold water overlies warm water... UNSTABLE

SHALLOW THERMOCLINE FROM SOLAR HEATING ELIMINATED DUE TO SURFACE COOLING / OVERTURN AND DEEPER MIXING
Mixing

• Water can be mixed through convective overturning (previous examples…)

• Water can also be mixed through turbulent diffusion (e.g. wind driven mixing)

How are properties transferred in water?

• **Molecular diffusivity** / **Molecular viscosity** – momentum is transferred to lower layers as the result of the interaction of molecules of water – Laminar flow.
How are properties transferred in water?

- **Eddy Diffusivity / Eddy viscosity** – (Turbulent Diffusion) momentum is transferred to lower layers as a result of the interaction of water parcels (turbulence) – Turbulent flow
- Eddy viscosity is usually more important.

Molecular diffusion is much slower than turbulent diffusion

**Molecular Diffusion**

**Turbulent Diffusion**

\[ \text{clear fluid} + \text{red dye} = \text{pink water} \]

Just like shaking a container will speed up the mixing rate
Salt transport and molecular diffusion

\[ \Gamma = \rho s \vec{u} \]

\( \vec{u} \) = velocity of the fluid (in m/s)

\( \rho = \) density of fluid (in kg/m\(^3\))

\( \Gamma \) = rate of transport of salt across the unit area normal to the direction of \( \vec{u} \)

\( \Gamma \) is the flux of salinity per unit area due to fluid motion (in kg/(m\(^2\)s)), it is also called the advective flux.

There is another means of transporting salt: molecular diffusion

The diffusive flux is proportional to the gradient (and of opposite sign)

Fick’s Law of diffusion

\[ \Gamma = -\rho K_D \nabla s \]

\[ \Gamma_x = -\rho K_D \frac{\partial s}{\partial x} \]

\[ \Gamma_y = -\rho K_D \frac{\partial s}{\partial y} \]

\[ \Gamma_z = -\rho K_D \frac{\partial s}{\partial z} \]

Our total flux per unit area becomes:

\[ \Gamma = \rho s \vec{u} - \rho K_D \nabla s \]

\( K_D \) is the molecular diffusivity of salt in water (~10\(^{-9}\) m\(^2\)/s)
But do we really know $\bar{u}$?

We cannot calculate/measure changes in $\bar{u}$ on very short timescales and very small spatial scales…

$$\bar{\Gamma} = \rho s \bar{u} = ?$$

BUT we try to take their effects on the main circulation into account!

In analogy with molecular diffusion (which is based on thermodynamics), we define an “eddy viscosity” (eddy diffusion, which is based on nothing) to take into account the mixing due to turbulence that we cannot resolve.

Flux of salinity per unit area due to turbulence:

$$\bar{\Gamma} = -\rho K_E \bar{\nabla} s$$
The full flux of salinity per unit area is therefore:

\[ \tilde{\Gamma} = \rho s \bar{u} - \rho K_E \nabla \bar{S} - \rho K_D \nabla \bar{S} \]

Flux due to large scale fluid motion (the mean flow)
Flux due to molecular diffusion
Flux due to "eddy diffusion" (= non-resolved fluid motion)

\[ K_E \gg K_D \]

In oceanography we usually do not take molecular diffusion into account.

\[ \tilde{\Gamma} \approx \rho s \bar{u} - \rho K_E \nabla \bar{S} \]

This transport equation can be written for every scalar in the fluid (oxygen, nutrients, temperature, CFCs, ...).

**Diffusion** - the flow from higher to lower concentrations

If box 2 contains water containing 1kg of salt per cubic meter and box 1 contains fresh water. If \( L = 1 \text{m} \) and \( K_D = 10^{-9} \text{m}^2 \text{s}^{-1} \) what will the initial flux be?

\[ \Gamma_x = -K \frac{dC}{dx} \]

What will the initial flux be if we consider eddy diffusion instead of molecular diffusion (\( K_E = 10^{-1} \text{m}^2 \text{s}^{-1} \))?
Flows from high concentrations to low concentrations

\[ \Gamma = -\rho K_E \nabla S \]

\[ \Gamma_x = -\rho K_{E,h} \frac{\partial S}{\partial x} \]
\[ \Gamma_y = -\rho K_{E,h} \frac{\partial S}{\partial y} \]
\[ \Gamma_z = -\rho K_{E,v} \frac{\partial S}{\partial z} \]

Concentration gradient (could be salinity, temperature, oxygen, …)

Diffusion effects will therefore be strongest in regions with high concentration gradients

This is very important in estuaries where there is a SALT gradient.

Molecular versus Turbulent (Eddy) Diffusivity

Eddy (turbulent) diffusivities are usually many orders of magnitude larger than their molecular counterparts.

Example:
Molecular diffusivity of salt in water \( K_D \sim 10^{-9} \) m\(^2\)/s
Vertical diffusion in the ocean, \( K_{E,v} \sim 10^{-5} - 10^{-3} \) m\(^2\)/s
Horizontal diffusion in the ocean, \( K_{E,h} \sim 1 - 1000 \) m\(^2\)/s

- Deep ocean mixing occurs near rough topography
- Surface layer turbulence can be caused by wind mixing

More appropriate to consider mixing along and across isopycnals
The Gulf Stream

- Where will diffusive effects be strongest? … In regions with high concentration gradients

Richardson Number:

Predicts the onset of turbulence

\[ Ri = \frac{N^2}{\left(\frac{du}{dz}\right)^2} \]

Remember \( N^2 \) can be <0

Where \( N \) is the Brunt Väisälä frequency, and \( u \) the horizontal velocity.

If \( Ri < 0 \) (i.e. \( N^2 < 0 \)) the water column is unstable – turbulent convective mixing

If \( Ri > 0 \) (i.e. stable stratification) then velocity shear \( \frac{du}{dz} \) must be large to generate turbulence.

- If \( Ri > \frac{1}{4} \) no turbulence is found
- If \( Ri < \frac{1}{4} \) turbulence may be found.
Determine if turbulence will occur in the following examples:

What is $du/dz$?

\[ Ri = \frac{N^2}{(du/dz)^2} \]

### Upper Ocean

1. Flux: Amount of heat, salt or volume that flows through a unit area per unit time (e.g. in $\text{J m}^{-2} \text{s}^{-1}$ for heat), $\text{kg m}^{-2} \text{s}^{-1}$ for salt or $\text{m}^3 \text{m}^{-2} \text{s}^{-1} = \text{ms}^{-1}$ for volume).

2. Volume transport: The volume of moving water measured between two points of reference and expressed in cubic meters per second or in Sverdrup (Sv).

\[ 1 \text{ Sv} = 1 \text{ million cubic meters per second} = 10^6 \text{ m}^3/\text{s} \]

**Harald Sverdrup**
The amount of salt in a parcel of water moving through the ocean remains relatively constant (unless the parcel is in contact with the surface, then river discharge, evaporation, precipitation, sea ice, will change it!)

\[ \Rightarrow \text{Salinity is conservative.} \]

The flux of salt into the cube includes an advective contribution (\( \rho \)) and a diffusive contribution (-K\( \frac{\partial s}{\partial x} \))

Diffusive flux (turbulent and molecular) of salt has a negative sign because it is directed in the opposite direction to the salinity gradient

The conversation equation for salt per unit volume is given by

\[
\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s = \nabla \cdot (K \nabla s) - \frac{\partial s}{\partial x} + \frac{\partial s}{\partial y} + \frac{\partial s}{\partial z} + \frac{\partial s}{\partial z} \frac{\partial t}{\partial x} - \frac{\partial s}{\partial y} \frac{\partial t}{\partial y} - \frac{\partial s}{\partial z} \frac{\partial t}{\partial z} = 0,
\]

But don't panic...

In most instances the advective flux will dominate, so the diffusive part can be ignored. We are also usually interested in 1D flow (e.g. flow through a channel or across a section)

Volume transport of an ocean current with velocity \( u \) through an area \( A \) is given by:

Volume transport = \( u \cdot A = u H L \)
Transport of Volume, Heat & Salt

If we know the heat (or salt) per m$^3$ we can convert volume flux into heat (or salt) flux.

Heat per m$^3 = \rho c_p T$

Salt per m$^3 = \rho S$

So, multiplying by $uA$

$Q = \rho c_p T u A$

$S_f = \rho S u A$

$\rho$ is density, $c_p \approx 4000$ J/Ckg, salinity is in absolute units, 35kg/1000kg

UNITS heat flux – Watts (1 Watt = 1 J/s). Salt flux - kg/s

Total Heat and Salt Fluxes

$V_T = u \times L \times H$ \hspace{1cm} m$^3$/s

$Q_T = \rho c_p u T \times L \times H$ \hspace{1cm} J/s

$S_f = \rho u S \times L \times H$ \hspace{1cm} Kg/s
Example

Calculate the total volume, heat and salt fluxes

a) Past Tasmania

b) Through the Drake Passage.

Assume the ocean currents persist over the upper 1000m only.

c) Why might the volume flux be different?