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"If and only if" statements

It was explained in the last chapter that, when an "All As are Bs" statement is true, its converse, "All Bs are As", may or may not be true. If both "All As are Bs" and "All Bs are As" are true, then the As are exactly the same things as the Bs. Such a situation is usually expressed by an "if and only if" statement:

"Something is an A if and only if it is a B"

You will recall, from Chapter 2, that "Something is an A only if it is a B" is equivalent to "All As are Bs", and that "Something is an A if it is a B" is equivalent to "All Bs are As". Therefore, the "if and only if" statement, "Something is an A if and only if it is a B" is equivalent to,

"All As are Bs and all Bs are As"

"If and only if" is sometimes abbreviated to "iff" and symbolised by $\iff$. In accordance with the use of "necessary" and "sufficient" introduced in Chapter 2, the above "iff" statement may also be expressed as:

"Being an A is a necessary and sufficient condition for being a B"

Some famous "if and only if" statements in mathematics are:

- A triangle has three equal sides if and only if it has three equal angles.
- A polyhedron is regular if and only if it is one of the five Platonic solids (the regular tetrahedron, cube, octahedron, dodecahedron and icosahedron). A polyhedron is said to be regular if all its faces are the same, all its edges are the same and all its vertices are the same, that is, at each vertex the same number of edges meet and the angles between them are all the same.
- A number is divisible by 9 if and only if the sum of its digits is divisible by 9.
- A real number is rational if and only if its decimal expansion is terminating or (eventually) repeating.

"If and only if" statements are very highly prized in mathematics. If an, "All As are Bs" statement is found, the mathematician asks, "What conditions are there on a B which will be just enough to make sure it is an A?" If such a condition C is found, the result will be an "if and only if" statement:

"Something is an A if and only if it is a C and a B"
This will completely express the connection between the concepts A and B. An important example is Euler’s formula for polyhedra:

\[ V - E + F = 2 \]

where \( V \) is the number of vertices of a polyhedron, \( E \) the number of its edges and \( F \) the number of faces. This formula is true for most simple polyhedra. It can be checked for the Platonic solids by counting, and, in the eighteenth century, Euler proved that it was true for many others. But finding exactly which polyhedra it was true for, that is, finding a condition \( C \) such that,

“A polyhedron is a \( C \) if and only if \( V - E + F = 2 \)”


Since an “if and only if” statement really makes two assertions, its proof must contain two parts. The proof of “Something is an A if and only if it is a B” will look like this:

Let \( x \) be an A,

therefore, \( x \) is a B.

Let \( y \) be B,

therefore, \( y \) is an A.

So, something is an A if and only if it is a B.

**Example 1**

Prove that a whole number is even if and only if its square is even.

**Finding a proof**

The proof should look like this:

Let \( x \) be even.

Therefore, \( x^2 \) is even.

Let \( y \) be a number such that \( y^2 \) is even.

Therefore, \( y \) is even.

Therefore, \( y^2 \) is even.

So a number is even if and only if its square is even.

The first half of this proof was an exercise in the last chapter. In the second half of the proof, we begin with,

Let \( y^2 \) be even,

and then write this in symbols,

\[ y^2 = 2K \]

for some whole number \( K \).

We then look for a reason why \( y \) should be even. As 2 divides the right-hand side, it also divides the left-hand side. So 2 divides one (or both) of the factors on the left. These are both \( y \), so 2 divides \( y \).

**Proof**

The final proof (including both parts) is:

Let \( x \) be even,

so that,

\[ x = 2K \]

for some whole number \( K \).

Then,

\[ x^2 = (2K)^2 \]
\[ = 4K^2 \]
\[ = 2(2K^2) \]

which is even.

Therefore, if \( x \) is even then \( x^2 \) is even.

Conversely, let \( y \) be a whole number such that \( y^2 \) is even.

So,

\[ y^2 = 2K \]

for some whole number \( K \).

As 2 divides \( 2K \), 2 divides \( y^2 \).

So 2 divides either \( y \) or \( y \), that is, 2 divides \( y \).

Therefore, if \( y^2 \) is even then \( y \) is even.

Therefore, a whole number is even if and only if its square is even.
Example 2

Prove that a whole number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Finding a proof

The main problem is to find a way to symbolise the number so that the sum of its digits can also be written in symbols. It will be necessary to have a symbol for each digit. If the number written in the usual decimal form is,

\[ a_n a_{n-1} \ldots a_2 a_1 a_0 \]

(i.e. \(a_0\) in the units column, \(a_1\) in the tens column, and so on), then the number itself is,

\[ 10^n a_n + \ldots + 100a_2 + 10a_1 + a_0 \]

and the sum of the digits is,

\[ a_n + \ldots + a_2 + a_1 + a_0 \]

The reason why one of these is divisible by 9 if and only if the other one is, is that the difference between the two numbers is,

\[ 99 \ldots 9a_n + \ldots + 99a_2 + 9a_1 \]

\[ \underbrace{99 \ldots 9}_{n \text{ nines}} \]

which is obviously divisible by 9. The proof could be set out as below.

Proof

Let \(x\) be a number with digits \(a_n, a_{n-1}, \ldots, a_2, a_1, a_0\). If \(x\) is divisible by 9 then,

\[ 10^n a_n + \ldots + 100a_2 + 10a_1 + a_0 \]

is divisible by 9.

So,

\[ (10^n a_n + \ldots + 100a_2 + 10a_1 + a_0) - (99 \ldots 9a_n + \ldots + 99a_2 + 9a_1) \]

\[ \underbrace{99 \ldots 9}_{n \text{ nines}} \]

is divisible by 9,

that is,

\[ a_n + \ldots + a_2 + a_1 + a_0 \]

is divisible by 9,

that is, the sum of the digits is divisible by 9.

If the sum of the digits is divisible by 9,

\[ a_n + \ldots + a_2 + a_1 + a_0 \]

is divisible by 9,

is divisible by 9.

Note

In this case, where the steps in the second half are just the reverse of those in the first half, it is possible to combine the steps in a string of, "if and only if":

A number with digits \(a_n, \ldots, a_2, a_1, a_0\) is divisible by 9,

iff \(10^n a_n + \ldots + 100a_2 + 10a_1 + a_0\) is divisible by 9,

iff \((10^n a_n + \ldots + 100a_2 + 10a_1 + a_0) - (99 \ldots 9a_n + \ldots + 99a_2 + 9a_1)\) is divisible by 9

iff \(a_n + \ldots + a_2 + a_1 + a_0\) is divisible by 9,

iff the sum of the digits is divisible by 9.

However it is rare that this is possible, and it is usually better to keep separate the two parts of an "if and only if" proof.

Exercises

(Grading of exercises: * easy, ** moderate, *** difficult.)

1. Are the following two "iff" statements:

   "Something is an A if and only if it is a B"

   and,

   "Something is a B if and only if it is an A" equivalent?

2. Consider the statement,

   "Something is an A if and only if it is a B"

   (a) Write down the two assertions made by the above "iff" statement.

   (b) How many parts does the proof of the given "iff" statement contain?

   (c) Does the proof of the two "all" statements:

   "All As are Bs"

   and

   "All Bs are As"

   complete the proof of the given "iff" statement?

3. (a) Give one example of a true "iff" statement.

   (b) Give one example of a false "iff" statement.
**4.** Are the following "if" statements true or false?
   (a) "An even number is prime if and only if it is 2."
   (b) "An odd number is prime if and only if it is 3."

**5.** One of the following three statements is true. State which one it is.
   (a) A figure is a polygon if and only if it is a triangle.
   (b) A figure is a circle if and only if it is a polygon.
   (c) A figure is a polygon if it is a triangle.

**6.** Prove that a whole number is odd if and only if its square is odd.

**7.** Write the "if" statement:
   \( x \) is a non-zero real number if and only if \( x^2 \) is positive in:
   (a) "necessary and sufficient form.
   (b) "if \(...\) then \(...\) and conversely" form.

**8.** Rewrite the "if" statement:
   "Something is an A if and only if it is a B"
   in five different forms.

**9.** Prove that a number is divisible by 4 if and only if its last two digits form a number divisible by 4.

**10.** Prove that a triangle is isosceles if and only if two of its angles are equal. (An isosceles triangle is by definition a triangle with two equal sides.)

**11.** Let \( m \) and \( n \) be two integers. Show that \( m^3 - n^3 \) is even if and only if \( m - n \) is even.

**12.** Show that a triangle has 3 equal sides if and only if it has 3 equal angles.

**13.** Show that a number is divisible by 3 if and only if the sum of its digits is divisible by 3.

**14.** Prove that the lines,
   \[ Ax + By + E = 0 \]
   and,
   \[ Cx + Dy + F = 0 \]
   are parallel if and only if \( AD - BC = 0 \)

**15.** Prove that the simultaneous equations,
   \[ Ax + By + E = 0 \]
   and,
   \[ Cx + Dy + F = 0 \]
   have exactly one solution if and only if \( AD - BC \neq 0 \).

**16.** Prove that the lines,
   \[ Ax + By + E = 0 \]
   and,
   \[ Cx + Dy + F = 0 \]
   are perpendicular if and only if \( AC + BD = 0 \)
   (Do not assume without proof that the product of the gradients of two perpendicular lines is \(-1\))

**17.** Prove that three distinct points,
   \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\)
   are collinear if and only if,
   \((x_2y_3 - x_3y_2) \) \(= (x_1y_3 - x_3y_1) \) \(+ (x_1y_2 - x_2y_1)\) \(= 0\)

**18.** Prove that a real number is rational if and only if its decimal expansion is terminating or (eventually) repeating.

**Linear algebra**

**19.** Show that a vector in \( \mathbb{R}^3 \) is a linear combination of \((1, 1, 2)\) and \((2, 2, 3)\) if and only if it lies on the plane, \(x = y\).

**Calculus**

**20.** Show that \( y_1 \) and \( y_2 \) are solutions of,
   \[ \frac{dy}{dx} + A \frac{dy}{dx} + By = 0 \]
   if and only if \( y_1 + y_2 \) and \( y_1 - y_2 \) are solutions.