2

“All” statements

The results proved in Chapter 1 were particular statements. A particular statement either asserts that a property, or predicate, is true of a subject (e.g. “seven is prime”; “Socrates is mortal”), or asserts some relation between two or more things (e.g., \( \sqrt[9]{8!} < \sqrt[9]{9!}; \frac{1}{1,000} - \frac{1}{1,001} < \frac{1}{1,000,000} \)). In any science, but especially in mathematics, we are interested in generalising from particular facts. When we saw that,

\[
\frac{1}{1,000} - \frac{1}{1,001} < \frac{1}{1,000,000}
\]

we realised that something similar would have happened whenever we subtracted the reciprocals of two consecutive whole numbers. A similar proof would have shown, for example, that,

\[
\frac{1}{203} - \frac{1}{204} < \frac{1}{(203)^2}
\]

The particular result that,

\[
\frac{1}{1,000} - \frac{1}{1,001} < \frac{1}{1,000,000}
\]

is an instance of the general pattern: for all whole numbers \( n \),

\[
\frac{1}{n} - \frac{1}{n + 1} < \frac{1}{n^2}
\]

This is an example of an “all” statement, or a “universal generalisation” in the jargon of logic. (Unfortunately, logic is a subject especially given to producing jargon; a precise technical language is sometimes needed, but should not be overdone.)

Some equivalent ways to express the same result in English are:

For every whole number \( n \),

\[
\frac{1}{n} - \frac{1}{n + 1} < \frac{1}{n^2}
\]

For any whole number \( n \),

\[
\frac{1}{n} - \frac{1}{n + 1} < \frac{1}{n^2}
\]

If \( n \) is a whole number, then,

\[
\frac{1}{n} - \frac{1}{n + 1} < \frac{1}{n^2}
\]

The difference of the reciprocals of any two consecutive whole numbers is less than the reciprocal of the square of the smaller number.

Some other examples of “all” statements are:

- All men are mortal. (This example has been traditional since the time of the ancient Greeks.)
- All multiples of ten are multiples of five.
- For all whole numbers \( n \),

\[
\sqrt[n]{n!} < \sqrt[n]{(n + 1)!}
\]

(this is the natural generalisation of the particular result in Chapter 1, Example 3: \( \sqrt[9]{8!} < \sqrt[9]{9!} \)).
- Every fourth power is a square (i.e. if a whole number is the fourth power of some other whole number, it is the square of some whole number). For example, 81 is a fourth power because it is \( 3^2 \) and is also a square because it is \( 9^2 \).

The general form of an “all” statement is,

\[
\text{All As are Bs}
\]

This states that anything that has the property of being an A also has the property of being a B. While some of the above examples cannot be put easily or naturally into the form, “All As are Bs”. (“All whole numbers \( n \) are such that, \( 1/n - 1/(n + 1) < 1/n^2 \); for example, is a little awkward.) The form, “All As are Bs”, is nevertheless very useful for explaining the general features of “all” statements.

To prove an “All As are Bs” statement, we must show that anything that is an A is also a B. So the proof of, “All As are Bs”, should look like this:

Let \( x \) be an A

\[
\begin{cases}
\text{ } \\
\text{therefore, } x \text{ is a B.}
\end{cases}
\]

**Example 1**

Prove that for all whole numbers \( n \),

\[
\frac{1}{n} - \frac{1}{n + 1} < \frac{1}{n^2}
\]

**Proof**

Let \( n \) be a whole number. Then,
\[
\frac{1}{n} - \frac{1}{n+1} = \frac{n+1 - n}{n(n+1)} = \frac{1}{n^2 + n}
\]

but,
\[
n^2 + n > n^2
\]

so,
\[
\frac{1}{n^2 + n} < \frac{1}{n^2}
\]

therefore,
\[
\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}
\]

### Notes
1. Although the result,
\[
\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}
\]
is much more general than,
\[
\frac{1}{1,000} - \frac{1}{1,001} < \frac{1}{1,000,000}
\]
its proof is no harder. If anything, it is easier, since there is no danger of being blinded by any irrelevant facts about the particular numbers chosen (e.g. their large size, which might easily lead to mistakes in comparing them).

2. It is most important to understand that, to prove an, "All As are Bs" statement, it is **not** sufficient to look at a number of particular As and check that they are Bs. There is no "proof by example". Thus, it would be incorrect to argue,
\[
\frac{1}{1} - \frac{1}{2} < \frac{1}{1^2}
\]
and,
\[
\frac{1}{2} - \frac{1}{3} < \frac{1}{2^2}
\]
and,
\[
\frac{1}{3} - \frac{1}{4} < \frac{1}{3^2}
\]
so,
\[
\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}
\]
for all n. Checking some examples **supports** the "all" statement, but **does not prove** it; the statement could be true for the examples checked, but false for some others. (This joke told by mathematicians about physicists illustrates the point. A physicist argued that he could show that all odd numbers are prime: "Three and five are prime," he said, "and so is seven. Nine—well, experimental error—but eleven and thirteen are prime, so all odd numbers are prime." Underlying this joke is a real difference between mathematics and experimental sciences such as physics; in physics the "all" statements can only ultimately be confirmed by a sufficiently large number of experiments, while in mathematics they can be **proved** true.)

### Example 2
Show that every whole number that is a fourth power is a square.

**Proof**

We should begin the proof by taking a number that is a fourth power, and writing in symbols what this means:

Let \( x \) be a whole number that is a fourth power. That is, \( x = a^4 \) for some number \( a \).

We wish to conclude that \( x \) is in fact a square, so we ask how we can write \( a^4 \) as the square of something. As \( a^4 \) is \( a \cdot a \cdot a \cdot a \), it can be written \( (a \cdot a)^2 \), that is \( (a^2)^2 \). So the complete proof is:

Let \( x \) be a whole number which is a fourth power. So,

\[ x = a^4 \]

for some whole number \( a \)

\[= a \cdot a \cdot a \cdot a \]

\[= (a^2)^2 \]

So \( x \) is a square.

Every fourth power is therefore a square.

### Notes
1. As already pointed out, it is not sufficient to check that the result is true for some particular \( x \)s that are fourth powers, for example, 16 and 81. The proof must be general, that is, it must apply to any \( x \) that is a fourth power.
2. The step after, "\( x \) is a whole number which is a fourth power" expresses in symbols what it means to be a fourth power. It often happens that in a proof of, "All As are Bs", the second line (after, "Let \( x \) be an A") expresses the defining characteristic of As, that is, what it means to be an A. Similarly, the second last line (before, "therefore, \( x \) is a B") is often an expression in symbols which shows that \( x \) is a B. Check this in the example above.
3. The sentence, "Let \( x \) be an A", is sometimes expressed as, "Take a general (or arbitrary) A".

This expression is possibly misleading, since all actual As are particular. It can be useful, however, as a reminder that in the proof of "All As are Bs", we must use only facts that are **true of all As**.

Watch for these points in the following example.

### Example 3
Prove that the square of an odd number is odd. (Obviously, "of an odd number" means "of any odd number".)
Finding the proof

We are asked to prove something about all odd numbers, namely, that their squares are odd. So the proof must look like:

Let \( x \) be an odd number.

\[
\therefore x^2 \text{ is odd.}
\]

To complete the proof it is necessary to express "is odd" in symbols. A number is odd if it is of the form \( 2k + 1 \) for some \( k \); this is the general form of an odd number. So the proof should look like this:

Let \( x \) be an odd number.

So, \( x = 2k + 1 \) for some number \( k \),

\[
\therefore x^2 = 2(\text{something}) + 1
\]

therefore, \( x^2 \) is odd.

To fill in the steps still missing, we ask how to get from \( x = 2k + 1 \) to something about \( x^2 \). Clearly we should square both sides. The complete proof is as follows:

Proof

Let \( x \) be an odd number.

So,

\[
x = 2k + 1
\]

for some number \( k \),

\[
x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1
\]

therefore, \( x^2 \) is odd.

Note

As occasionally happens, this proof has actually proved a stronger result than was intended. We intended to show that the square of any odd number is of the form \( 2k + 1 \); we actually showed it was of the form \( 4k + 1 \). This is a stronger result, in the sense that the statement

"The square of any odd number is of the form \( 4k + 1 \)"

implies the statement,

"The square of any odd number is odd"
In practice, however, almost all the “if . . . then” statements used in mathematics can be thought of in the form “if \( x \) is an \( A \) then \( x \) is a \( B \)”.

We conclude this chapter with some examples from the theory of linear equations, linear algebra and calculus. They illustrate how important “all” statements are in every branch of mathematics.

**Example 4 (Linear equations)**

Show that every system of linear equations of the form,

\[
\begin{align*}
    x + Ay &= B \\
    2x + Cy &= D
\end{align*}
\]

such that \( C \neq 2A \) can be solved.

**Proof**

Taking the second equation minus twice the first gives,

\[
(C - 2A)y = D - 2B
\]

since \( C \neq 2A \) and so \( C - 2A \neq 0 \) we may divide by \( C - 2A \):

\[
y = \frac{D - 2B}{C - 2A}
\]

Then from the first equation,

\[
x = B - Ay = B - \frac{A(D - 2B)}{C - 2A}
\]

So the system has been solved.

**Example 5 (Linear algebra)**

Show that all linear combinations of the vectors \( (1, 1, 2) \) and \( (2, 3, 5) \) in \( \mathbb{R}^3 \) lie on the plane,

\[
x + y - z = 0
\]

**Proof**

Any linear combination of \( (1, 1, 2) \) and \( (2, 3, 5) \) is of the form, \( a(1, 1, 2) + b(2, 3, 5) \)

for some \( a, b \in \mathbb{R} \). However, this is \( (a + 2b, a + 3b, 2a + 5b) \), and this point satisfies the equation,

\[
x + y - z = 0
\]

since

\[
(a + 2b) + (a + 3b) - (2a + 5b) = a + a - 2a + 2b + 3b - 5b = 0
\]

So every linear combination of \( (1, 1, 2) \) and \( (2, 3, 5) \) lies on the plane,

\[
x + y - z = 0
\]

**Example 6 (Calculus)**

Show that, for all \( x > 0 \), \( x \gg \ln x + 1 \)

**Proof**

We use calculus methods to compare the graphs of,

\[
y = x
\]

and

\[
y = \ln x + 1
\]

The graph of,

\[
y = x
\]

is the straight line through \((0, 0)\) with gradient 1. Now for \( x = 1 \),

\[
x = \ln x + 1 = 1
\]

so the two graphs meet at \((1, 1)\). Also,

\[
\frac{d}{dx}(\ln x + 1) = \frac{1}{x}
\]

and for all \( x > 1 \),

\[
\frac{d}{dx}(\ln x + 1) = \frac{1}{x} < 1
\]

(and indeed \( 1/x \to 0 \) as \( x \to \infty \)).

So for \( x > 1 \), \( y = \ln x + 1 \) has a gradient less than \( y = x \). Thus for \( x \gg 1 \),

\[
x \gg \ln x + 1
\]

For \( 0 < x < 1 \), \( 1/x \) (the gradient of \( y = \ln x + 1 \)) is greater than 1 (the gradient of \( y = x \)). So here again

\[
x > \ln x + 1
\]
So for all $x > 0$,

$$x \ln x + 1$$

(See Figure 2.2, which shows these graphs.)

Figure 2.2

**Exercises**

(Grading of exercises: * easy, ** moderate, *** difficult.)

*1. Prove that the square of any even number is even.

*2. Prove that the sum of any two consecutive numbers is odd.

*3. Prove that the product of any two odd numbers is an odd number.

*4. Give an "all" statement relating "cows" and "mammals" which illustrates that, "All A's are B's" is not logically equivalent to, "All B's are A's".

*5. Decide whether the statement:

$$x^2 - 3x + 2 < 0$$

for $1 < x < 2$ (i.e. for all $x$ between 1 and 2) is true or false and prove your answer.

*6. (a) Rewrite the all statement, "All A's are B's" in the forms

(i) "... only if ..."
(ii) "... then ..."

(b) A father told his son, "Only if you pass will you get a bike". The son passed but he did not get a bike. Did the father break his promise? Explain.

*7. Show that if $m$ and $n$ are odd integers, then $m + n$ is even.

*8. Comment on the reasoning:

(a) It takes one person two hours to mow this lawn. So it would take 6,000 people $\frac{2}{6,000}$ hours.

(b) If it takes four ships three days to cross the Tasman, how long will it take seven ships?

*9. Give one example of an "all" statement (universal generalisation) in the form, "All A's are B's". Then rewrite it in different forms.

*10. Show that for any whole number $n$, $\frac{n}{n+1} < \frac{n+1}{n+2}$

*11. Prove that $\sqrt{n} < \frac{n}{\sqrt{n+1}}$ for any whole number $n$ (imitate the proof in Chapter 1).

*12. Prove that the product of three consecutive whole numbers, of which the middle one is odd, is divisible by 24.

*13. Let $\alpha$ and $\beta$ be the roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$.

Show that:

$$\alpha + \beta = -\frac{b}{a}$$

and,

$$\alpha \beta = \frac{c}{a}$$

(Do not use the quadratic formula.)

*14. Show that:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\ldots \left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

*15. Prove that if $0 < x < 1$ then, $0 < x^2 < x < 1$

*16. Show that for any non-zero real numbers $x$ and $y$ such that $x + y = 1$,$$
\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right) = 1$$

*17. Prove that $x^2 - 4x + 5 > 0$ for all real numbers $x$.

*18. (a) Prove that if $a \neq 0$ then:

$$-\frac{b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$-\frac{b - \sqrt{b^2 - 4ac}}{2a}$$

are solutions of $ax^2 + bx + c = 0$
(b) Prove that if \( a \neq 0 \) and \( x \) is a solution of \( ax^2 + bx + c = 0 \), then,
\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}
\]

**19.** Find a generalisation of:
\[
\frac{1}{1000} - \frac{1}{1002} < \frac{2}{(1000)^2}
\]
and prove it.

**20.** Show that if \( a, b \) are integers such that 7 divides \( a + b \) and \( a^2 + b^2 \), then 7 divides both \( a \) and \( b \).

**21.** Prove that for all whole numbers \( n \), \( (n + 1)(n + 2) \ldots (2n - 1)(2n) = 2\cdot1\cdot3\cdot5\ldots(2n - 1) \)

**22. (a)** What is wrong with the following "proof" of Pythagoras' theorem? (The theorem states that if \( a, b, c \) are the sides of any right-angled triangle with \( c \) the hypotenuse then \( a^2 + b^2 = c^2 \).)

By the cosine rule,
\[
c^2 = a^2 + b^2 - 2ab \cos 90^\circ
\]
but,
\[
\cos 90^\circ = 0 \quad \text{so} \quad c^2 = a^2 + b^2
\]

(b) What about this attempt?
Let \( \theta \) be the angle opposite \( b \).
Then,
\[
a^2 + b^2 = (c \cos \theta)^2 + (c \sin \theta)^2 = c^2 (\cos^2 \theta + \sin^2 \theta) = c^2 \sin \theta + \sin^2 \theta = 1
\]

**23. (a)** Prove that for any positive real numbers \( x, y \),
\[
xy < \left(\frac{x + y}{2}\right)^2
\]

(b) Hence show that of all rectangles with a fixed perimeter, the square has the largest area.

**24. (a)** Prove that the area, \( S \), of a triangle of base \( b \) and altitude \( h \) is given by,
\[
S = \frac{1}{2}bh
\]

*Hint: Draw some rectangles (make sure all shapes of triangles are accounted for). See Figure 2.3.*

(b) Hence prove that the area \( S \) of a triangle is given by,
\[
S = \frac{1}{2}ab \sin \theta
\]

**25.** Prove that the area \( A \) of a right circular cone is given by,
\[
A = \pi r^2 + \pi r l
\]
where \( r \) is the radius of the base and \( l \) is the slant height. (Assume the formula \( \pi r^2 \) for the area of a circle.)

*Hint: See Figure 2.4.*

**26.** Prove Pythagoras' theorem: If \( a, b, c \) are the sides of any right-angled triangle with \( c \) the hypotenuse, then,
\[
a^2 + b^2 = c^2
\]

**27.** Show that for any positive integers \( m \) and \( n \), 2 is between \( \frac{m}{n} \) and \( \frac{m + 2n}{m} \) (inclusive).

**28. (a)** Show that any figure in the XY plane, which is symmetrical about the X axis and symmetrical about the Y axis, is also symmetrical about the origin.
(A figure is said to be symmetrical about an axis if, for any point in the figure, the point opposite it across the axis is also in the figure. Similarly, a figure is symmetrical about a point if, for any point in the figure, the point opposite it across the point of symmetry is also in the figure. Thus a square is symmetrical about both diagonals and also about its centre.)

(b) Find some similar result in three dimensions and prove it.
Linear equations

29. Show that any system of equations,
\[ x + y = A \\
   x - y = B \]
can be solved.

30. Show that if \( A \) or \( B \) is non-zero, any system of equations,
\[ Ax + By = C \\
   Bx - Ay = D \]
can be solved.

31. Show that any system of equations of the form,
\[ x + y + Az = B \\
   x - y + Cz = D \]
has infinitely many solutions.

Linear algebra

32. Show that any linear combination of \((1, 2, 3)\) and \((-2, -4, -6)\) lies on the line,
\[ 6x = 3y = 2z \]

33. Show that if:
\[ x(1, 1, 0) + y(1, 2, 3) + z(3, 4, 3) = (0, 0, 0) \]
then \((x, y, z)\) is a scalar multiple of \((2, 1, -1)\)

Calculus

34. Prove that for all \( x > 0 \), \( x > \sin x \)

35. Show that for all \( x > 4 \), \( 2^x > x^2 \)

36. Show that any solution of \( \frac{dy}{dx} = y \) is a multiple of \( e^x \)

37. Show that for \( x > 10 \), the graph of,
\[ y = \frac{\sin x}{x^2} \]
lies within 0.01 of the x-axis.

38. Is it true that for all \( X > 0 \),
\[ \int_0^X \frac{x \sin x}{x} \, dx > 0 \]?
Prove your answer.