REVIEW OF BARRY MAZUR'S IMAGINING NUMBERS (PARTICULARLY THE SQUARE ROOT OF MINUS FIFTEEN) AND GISBERT WÜSTHOLZ'S A PANORAMA OF NUMBER THEORY, OR THE VIEW FROM BAKER'S GARDEN

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SINCE 1995, when Andrew Wiles finally proved Fermat's Last Theorem, number theory has enjoyed a higher profile in the world's imagination. And it has brought in its train a number of philosophical questions that once were the province of philosophers of mathematics alone, but are now felt by those who, under normal circumstances, might never have worried about such matters; questions such as: what are numbers? how ought we to conceive of them, ontologically and, perhaps even more pressingly, epistemologically? and how do they fit into our total scientific world view?

Of course this issue is made vastly more complicated by the way in which mathematics has embraced number systems that go beyond the natural numbers or integers. The negative numbers and zero may once have been regarded with suspicion (and Cardano in the 16th Century called them *fictae*), but we have surely adjusted to them. Schoolchildren learn to think of negative numbers as either debts or points on a line on the other side of zero. And it surely helps them to think of them as having some kind of equal status with positive numbers that the zero point can be entered at an arbitrary point on the graph paper: so positive and negative begins to look as arbitrary as 'right and left of here'.

But people in large numbers still baulk at imaginary numbers like $\sqrt{-1}$: schoolchildren, philosophers, and even mathematicians. For most people these are certainly *fictae*—creatures of the imagination alone.

Of course it doesn't help that students are taught one year after the next that negative numbers can have no square roots. This only makes it certain that when complex numbers are eventually introduced that they are greeted with howls of protest—almost as a betrayal of trust. But what then are we to make of the fact that complex numbers play a central rôle in much of physics, most notably quantum mechanics? Do we take their central, and seemingly ineliminable, rôle in the mathematics of our best physical theories to indicate that they should be thought of as more than mere *fictae* after all; or do we take their use in physics as a sign of the instrumental nature of that physics? (An interesting question that I think has never been addressed is whether the instrumental Copenhagen interpretation of quantum mechanics wasn't made to seem more reasonable than it was by the rôle that was being played by complex numbers—borrowing their unreality, as it were.)

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Barry Mazur's aim is to help non-mathematicians 'imagine' numbers like $\sqrt{-1}$—though what 'imagine' means here is an interesting question in its own right. Mazur is, of course, a very distinguished Harvard mathematician, and by the contents of this book, a man of rich culture. (His wife is a novelist, and biophysicist, and his circle of friends is clearly broad and sophisticated—in many respects his book looks like a letter addressed to them.) His aim, in a sense, is to draw analogies between the mathematician's acts of imagining and the poet's, showing that they are at root the same.

This aspect of Mazur's book is to my mind the least convincing. For all the effort directed its way it is not plausible to see the act of creating or appreciating a metaphor as the same as the three hundred years or so of collective effort that was required to make complex numbers seem reasonable. From Cardano onwards, mathematicians were *using* complex numbers and in the process discovering that they could think about them, and make them accessible to rationality. The more applications were found to problems that didn't seem to involve complex numbers at all, the more they seemed like a natural extension. Long before complex numbers were found to carry some of the explanatory load of quantum mechanics, Jacques Hadamard said, 'The shortest path between two truths in the real domain passes through the complex domain.'

This is fundamentally different to the process of writing or creating in language. A better analogy would have been that of the religious person, trying to understand and internalise some conception of God. The effort is to take something that runs counter to experience and common sense and bend the mind towards it by a slow process of assimilation.

The meta-lesson to be learned from complex numbers was that experience and commonsense were not to be relied on in mathematics. Just by working on some present problem—in this case solutions to the cubic—one could be led into unfamiliar, and, to present rationality, 'impossible' terrain. (Remember that the '*i*' of imaginary numbers once stood also for 'impossible', not just 'imaginary'.) As this lesson was assimilated in the early 19th Century, by Gauss above all, it inaugurated an explosion of creativity in mathematics that continues to this day.

Mazur's account doesn't quite reach Gauss; rather it covers the period from Cardano to Legendre, with a heavy emphasis on the immediate successors to the former, namely Tartaglia and Bombelli. The history here is quite fascinating, and Mazur tells it well. The only real disappointment is that Wallis' discovery that $\sqrt{-1}$ can be thought of as the length of a perpendicular to the ordinary number line—that it is the geometric mean, or mean proportional of -1 and 1—is omitted. Arguably this was the pivotal discovery in the 'geometrising' of complex numbers, and since that is largely the story that Mazur wants to tell, its omission is hard to understand.

Mazur is not himself much interested in the ontological issue: what numbers, even complex numbers, are. But his presentation lends itself to a form of realism. This has rankled with some mathematical reviewers, committed to realism about no more than the real numbers $\mathbb{R}$. So for them $\sqrt{-1}$ is still a *fictae* and we can get complex numbers, of the form $a+bi$, out of $\mathbb{R}\times\mathbb{R}$, plus the addition of some axioms that will *mimic* the presence of $\sqrt{-1}$.

However, mimicing is not being, and, arguably, unless there really is a number $\sqrt{-1}$ the axioms needed to mimic complex multiplication are entirely *ad hoc*. Why these and not some others? Why not none at all? The obvious explanation is that they are mimicing something which genuinely exists in nature. Moreover, Euler's identity,

$$e^{i\pi}+1=0,$$

which Richard Feynman called the most remarkable formula in all mathematics, seems to be tying together *natural* constants. There appears to be no room in here for *fictae*.

It is also worth mentioning that the standard reductionist programmes, such as Frege's Logicism, do not seem to have any chance of working here. Nor those programmes, so beloved of philosophers, of reducing numbers to heaps, aggregates, or sets of material objects. And if not these then how much less, vague suggestions of 'social construction.'

The standard argument among philosophers these days for realism about mathematical entities is called the Quine-Putnam indispensability argument. Roughly, mathematical entities are posits that, if they are indispensible in scientific explanations, deserve to be taken with due ontological seriousness. But by that measure it is not the natural numbers or the reals that should command our acceptance, but the complex numbers—for physical explanation requires (complex) Hilbert spaces and their self-adjoint operators. Non-commutativity is intimately related to the use of these numbers.

A striking example of the unity of the pure mathematical world and the world of physics is given by the discovery that the eigenvalues of a random Hermitian matrix (such as might be found in certain quantum mechanical problems) have the same spacing properties as the non trivial zeroes of the Riemann zeta function—which are not at all random! This is the Montgomery-Odlyzko Law, and now fairly widely confirmed. But it implies that there is a deep connection between quantum theory and the distribution of prime numbers.

This suggests that the relation between even quite recondite areas of pure mathematics (that are usually thought of as *a priori*) and physical theories (which are usually taken to be *a posteriori*) is much closer than has traditionally been thought. We might formulate this as the *interpenetration thesis*—a stronger alternative to the indispensability thesis.

But, importantly for present purposes, it also provides a strong argument for realism concerning the complex numbers, finding them in properties of the natural numbers—which, it is useful to recall, are the usual target of Logicist constructions.

The boundary objects between the traditional *a priori* and *a posteriori* are the operator algebras, invented by von Neumann (and thought by G.H. Hardy to be not really maths at all). It is now known that all viable physical theories, classical or quantum, can be expressed as C^*-algebras—a surprising fact. On the one hand these provide the framework for quantum entanglement and quantum information, and on the other, via the Vaughan Jones index theorem, provide a polynomial invariant for knots—an area of topology seemingly unrelated to physics.

Of course, the random Hermitian matrix mentioned above, is an example of an operator in such an operator algebra. And the Riemann zeta function, though it is about the spacing of natural number primes, is a *complex* function. But then, as Hadamard intimated, complex numbers are ubiquitous throughout the theory of natural numbers.

Alan Baker has had a distinguished career (he won the Fields Medal in 1970) as a number theorist in areas that—and Hardy surely would have approved—are far removed from the applied sciences. His area is principally transcendental number theory, and the application of logarithmic forms thereto. In the year 1999, on the occasion of his 60th birthday, a conference was held in Zurich, Switzerland, to celebrate his work and achievements. The proceedings represents work at the leading edge of number theory.

The papers range from reports on developments in logarithmic forms to the *abc*-conjecture of Richard Mason (a student of Baker's). The papers that have the greatest bearing on applied science issues are those that concern transcendence and elliptic integrals. As Gisbert Wüstholz mentions in an early paper, a theorem of his own (the Analytic Subgroup Theorem on abelian varieties) is strongly related to an unsolved question of Leibniz on celestial mechanics.

But even if we just consider irrational and transcendental numbers there is much food for philosophical thought—though long use has staled their power to inspire the right degree of awe. For example, we know by the continuum problem that we do not even know just how many irrational numbers there are—the number could be very large indeed. And if we reflect on what an irrational number is we can see that, on the one hand, it requires us to accord reality to an *infinite* form, namely the decimal expansion of the number; and on the other, it suggests the defeat of a naive attempt to reduce number, via a kind of operationalism, to proportional lengths expressed as whole numbers. We may not be able to get from this the Pythagorean catch-cry: 'All is number,' but we do get a strong argument for realism about numbers—an argument that seems to have been too little considered by philosophers.

We can repeat this argument for eluding a reductionist or operationalist definition with transcendental numbers. If we were to have thought that nature is written in the language of polynomial equations (with integer coefficients) and that all numbers are just roots thereof, then transcendental numbers show, once again, that some numbers escape the net.

But there is an interesting connection with the reality of $\sqrt{-1}$ even here. This is because, though it is notoriously difficult to prove that a given number is transcendental (though in 1965 Alan Baker produced an improved method that showed that a large class of numbers are transcendental), it was proven in 1929 by A.O. Gel'fond that e^{π} is transcendental. But by the Euler identity, mentioned above,

$$e^{\pi} = (-1)^{-i}.$$

Thus the RHS is transcendental too—and the mysteries close about us from a different direction.

I believe, as philosophers we have some way to go to appreciate these mysteries—and promissory notes of reductionist programmes never really begun will likely only distract us from the task. As Nikolai Lobatchevsky once said, 'there is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.' This should serve as a warning on thinking that *indispensability* has only narrow scope.

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