On the entropy theory of finitely-generated nilpotent group actions

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(Received 29 March 2000 and accepted in revised form 19 July 2001)

Abstract. The entropy for actions of a finitely-generated nilpotent group $G$ is investigated. The Pinsker algebra of such actions is described explicitly. The systems with completely positive entropy are shown to have a sort of ‘asymptotic independence property’, just as in the case of $\mathbb{Z}^d$-actions. The invariant partitions are used to prove that the property of completely positive entropy is equivalent to the property of the K-system (the property of the existence of special ‘good’ partitions). A complete spectral characterization of K-systems is given. A construction of examples of completely positive non-Bernoullian actions of general countable nilpotent groups and a class of solvable groups is presented.

0. Introduction

Kolmogorov distinguished a class of dynamical systems which possess a partition with several good properties of an algebraic nature [9]. Those systems were later called K-systems after Kolmogorov by Rokhlin and Sinai [15]. They considered, in particular, the dynamical systems with trivial Pinsker algebra (systems with completely positive entropy (c.p.e.)), and proved that this property is equivalent to the K-property. In order to do that, a technique related to a class of invariant partitions was developed.

A description of Pinsker algebras for actions of $\mathbb{Z}^d$, $d < \infty$, was obtained by Conze [1], who also considered the notion of completely positive entropy and the K-property for actions of the above groups. The equivalence of the two latter properties was proved by Kamiński [6], who developed the Rokhlin–Sinai [15] theory of invariant partitions in the context of $\mathbb{Z}^d$. An alternate approach to studying the c.p.e. actions of $\mathbb{Z}^d$, $1 \leq d \leq \infty$, was suggested by Kamiński and Liardet [7]. Recently a new rise of activities has appeared in this sphere. Rudolph and Weiss [16] established that any action of a countable amenable group with completely positive entropy possesses the property of asymptotic independence. In what follows, this property is called the Rudolph–Weiss asymptotic property. Also, Glasner et al [4] discovered some new properties of Pinsker algebras.
related to disjointness and quasi-factors. Also, the above works elaborate new techniques for studying those subjects. Recently these results were supplied with a relativized version by Danilenko, who introduced a notion of entropy for cocycles \([2]\). In our work \([3]\) a reverse version of the Rudolph–Weiss asymptotic property was proved and used to produce a construction of non-Bernoullian actions for a class of Abelian groups.

On the other hand, there are still plenty of open problems in this sphere. In particular, these include the problem of describing a structure of Pinsker algebra for actions of amenable (especially, non-Abelian) groups, the properties of c.p.e. systems and their relationship to K-systems, the spectral properties of c.p.e. systems, and the problem of the existence of c.p.e. actions which are non-Bernoullian.

In this work, we approach the above class of problems by considering the case of finitely-generated nilpotent groups. This class of groups, modulo passage to a normal finite index subgroup, just consists of upper triangular unipotent matricial groups with integer entries and their subgroups \([8]\). We prove an analogue of the well-known Pinsker formula (4.2) together with the associated asymptotic relations, which makes it possible to describe explicitly the Pinsker algebra (Theorem 2.6). The next step is to consider the systems with trivial Pinsker algebra (c.p.e. systems, see Definition A in §2) and to establish that they are exactly the K-systems (Definition E, §6), the latter being introduced in a similar way as in \([1, 6]\). To do this, we develop the approaches of Rokhlin and Sinai \([15]\) and Kamiński \([6]\) so as to apply the techniques related to invariant partitions in our setting.

On the other hand, the c.p.e. systems also possess a property of 'asymptotic independence' (see Proposition 2.8), just as in the case of \(\mathbb{Z}^d\). A complete spectral characterization of K-systems is presented (see §5). A construction of c.p.e. non-Bernoullian actions for general countable nilpotent groups and a class of solvable groups is produced (see §7).

Our attention was attracted to this problem by J.-P. Thouvenot.

In some cases, we simplify our exposition by considering the case of the Heizenberg group in detail (see §§2 and 3), and then show how to transfer our techniques onto the general case.

1. **Preliminaries**

Recall the basic definitions. For a partition \(\alpha\) of a Lebesgue space \((X, A, \mu)\) with at most a countable number of elements \(A_i\), define the entropy of \(\alpha\) by

\[
H(\alpha) = -\sum_i \mu(A_i) \log \mu(A_i).
\]

Let \(\mathcal{L}\) denote the collection of partitions \(\alpha\) as above with \(H(\alpha) < \infty\). Given any measurable partition \(\beta\), the conditional entropy \(H(\alpha|\beta)\) is defined in a standard way. In particular, \(H(\alpha|\beta) = 0\) is equivalent to \(\alpha \leq \beta\); \(H(\alpha|\beta)\) is increasing in \(\alpha\) and decreasing in \(\beta\). The distance \(d\) on \(\mathcal{L}\) is given by

\[
d(\alpha, \beta) = H(\alpha|\beta) + H(\beta|\alpha).
\]

With \(d\), \(\mathcal{L}\) becomes a complete metric space. The basic property of the entropy for partitions is

\[
\text{for all } \alpha, \beta \in \mathcal{L}, \text{ for all } \gamma, \quad H(\alpha\beta|\gamma) = H(\alpha|\gamma) + H(\beta|\alpha \vee \gamma). \quad (1.1)
\]
and hence the subadditivity of $H$ is

$$H(\alpha \cup \beta | \gamma) \leq H(\alpha | \gamma) + H(\beta | \gamma).$$

If $\{\gamma_n\}$ is an increasing sequence of partitions with $\gamma = \bigvee_n \gamma_n$, then for all $\alpha \in \mathcal{L}$ one has

$$\lim_{n \to \infty} H(\alpha | \gamma_n) = H(\alpha | \gamma).$$

In a similar way, for a decreasing sequence of partitions $\{\gamma_n\}$ with $\gamma = \bigwedge_n \gamma_n$ one has, for all $\alpha \in \mathcal{L}$,

$$\lim_{n \to \infty} H(\alpha | \gamma_n) = H(\alpha | \gamma).$$

Let $T$ be an automorphism of $(X, \mathcal{A}, \mu)$. For any partition $\alpha$, we use the notation

$$\alpha^n_T = \bigvee_{k=0}^{n-1} T^k \alpha, \quad \alpha^-_T = \bigvee_{k=1}^\infty T^{-k} \alpha, \quad \alpha_T = \bigvee_{k=-\infty}^\infty T^k \alpha.$$

The sequence $(1/n)H(\alpha^n_T)$ has a limit which is denoted by $h(\alpha, T)$; it can also be represented as a conditional entropy $H(\alpha | \alpha^-_T)$. More generally, if $\gamma$ is a $T$-invariant measurable partition, the sequence $(1/n)H(\alpha^n_T | \gamma)$ has a limit equal to $H(\alpha | \alpha^-_T \cup \gamma)$, which is denoted by $h(\alpha, T, \gamma)$. In particular, $h(\alpha, T) = h(\alpha, T, \nu)$, with $\nu$ being the trivial partition.

The entropy of the automorphism $T$ is defined by

$$h(T) = \sup_{\alpha \in \mathcal{L}} h(\alpha, T).$$

Recall a very important Pinsker formula

$$h(\alpha \cup \beta, T) = h(\alpha, T) + H(\beta | \alpha^-_T \cup \beta^-_T),$$

and, in a more general setting,

$$h(\alpha \cup \beta, T, \gamma) = h(\alpha, T, \gamma) + H(\beta | \alpha^-_T \cup \beta^-_T \cup \gamma)$$

for any $T$-invariant measurable partition $\gamma$.

2. Entropy and Pinsker algebras for matricial nilpotent groups

We start our exposition by considering the simplest case of the Heisenberg group, to be extended later on to a larger class of nilpotent groups. Let $G$ be the two-step nilpotent countable matrix group

$$G = \left\{ \begin{pmatrix} 1 & n_3 & n_1 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} : n_i \in \mathbb{Z} \right\}.$$ 

We fix the generators

$$T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad T_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$
Here $T_1$ generates the center $Z$ of $G$. Define the linear order relation on the above generators by setting $T_3 > T_2 > T_1$, together with the associated lexicographic linear order relation on $G$: $T_3^{i_3}T_2^{i_2}T_1^{i_1} < T_3^{j_3}T_2^{j_2}T_1^{j_1}$ iff $(j_3, j_2, j_1)$ is lexicographically less than $(k_3, k_2, k_1)$. This order relation is invariant with respect to the left translations of $G$, and so we have the notion of the ‘past’ in $G$ defined as a subset of all elements of $G$ which are less than the identity.

Among the finite subsets of $G$, we distinguish the rectangles $T_3^{3}T_2^{2}T_1^{1} : m_1 \leq i_1 \leq M_1, m_2 \leq i_2 \leq M_2, m_3 \leq i_3 \leq M_3$. Given a sequence of such rectangles $\{\rho_n\}$, we say that the modulus of $\rho_n$ tends to infinity if

$$\lim_{n \to \infty} (M_{k}(n) - m_{k}(n)) = \infty$$

$\lim_{n \to \infty} (M_{i}(n) - m_{i}(n)) = \infty$, $\lim_{n \to \infty} (M_{j}(n) - m_{j}(n)) = \infty$ as $n \to \infty$.

**PROPOSITION 2.1.** Let $\{\rho_n\}$ be a sequence of rectangles in $G$ whose modulus tends to infinity. Then $\{\rho_n\}$ is a Følner sequence of sets in $G$.

**Proof.** It follows from the commutation relation

$$(T_3^{i_3}T_2^{i_2}T_1^{i_1})^{-1}T_3^{k_1}T_2^{k_2}T_1^{k_1} = T_3^{k_1-i_3}T_2^{k_2-i_2}T_1^{k_1+i_3(k_3-i_3)-i_1}$$

that for any $g = (T_3^{i_3}T_2^{i_2}T_1^{i_1})^{-1}$ and $\rho$ given by $0 \leq k_j \leq p_j$, $j = 1, 2, 3$,

$$S(g\rho_n \Delta \rho_n) \leq 2|i_3|p_1p_2 + 2|i_2|p_1p_3 + 2(|i_1| + |i_2i_3| + |i_2|p_3)p_2p_3,$$

and so

$$\frac{S(g\rho_n \Delta \rho_n)}{S(\rho_n)} \leq \frac{2|i_3|}{p_3} + \frac{2|i_2|}{p_2} + \frac{2(|i_1| + |i_2i_3| + |i_2|p_3)}{p_1} \to 0$$

as $n \to \infty$. \qed

It follows from Proposition 2.1 and [13, Theorem 1] that for any sequence $\{\rho_n\}$ of rectangles in $G$ whose modulus tends to infinity and any $\alpha \in \mathcal{L}$, there exists a limit $1/S(\rho_n)H(\alpha_\rho_n)$. This limit is independent of the choice of $\rho_n$. It is called the mean entropy of $\alpha$ with respect to $G$, and is denoted by $h(\alpha, G)$. However, it admits different expressions.

Now introduce the past for a partition $\alpha$ with respect to the $G$-action. Specifically, we set up $\alpha^-_G = \sqrt{T_3^{k_1}T_2^{k_2}T_1^{k_3}\alpha}$, where the join is taken over all triples $(k_3, k_2, k_1) \in \mathbb{Z}^3$ which are lexicographically less than $(0, 0, 0)$. In more detail, $\alpha^-_G = T_1^{-\rho_1}(\alpha_{T_1})^{-1}(\alpha_{T_1T_2})^{-1}(\alpha_{T_1T_2T_3})^{-1}$.

It is known from [13, 17] that, for any partition $\alpha \in \mathcal{L}$, $h(\alpha, G) = H(\alpha|\alpha^-_G)$.

We now turn to a more general case of $UT_n(\mathbb{Z})$, the group of integral unipotent upper triangular matrices. It turns out, and will be used in the following, that this group admits a Følner sequence of subsets which are also fundamental domains for a decreasing sequence of finite index subgroups with trivial intersection. This seems to be inaccessible for general finitely-generated, torsion-free, nilpotent groups.

Let $I$ be the identity matrix and $u_{ij}$ a matrix unit (the matrix whose entry with indices $k, p$ is $\delta_{ik}\delta_{jp}$). Consider the set of generators for $UT_n(\mathbb{Z})$ formed by the matrices
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$T_{ij} = I + u_{ij}, i < j, 1 \leq i, j \leq n$. Impose a linear order relation on these generators by setting $T_{ij} < T_{i'j'}$ if $j - i > j' - i'$ or $j - i = j' - i'$ and $i < i'$. This order relation is then extended to the entire group $UT_n(\mathbb{Z})$ by the lexicographic rule. In this setting, $UT_n(\mathbb{Z})$ becomes an ordered group [17]. In fact, the $T_{ij}$ work as a Maltsev basis [8] for $UT_n(\mathbb{Z})$.

Now let $p$ be a prime number. For each $m \in \mathbb{N}$ consider a subgroup $\Gamma_m$ generated by $T_{ij}^{p_m(j-i)}$. Clearly, each $\Gamma_m$ is a finite index subgroup in $UT_n(\mathbb{Z})$, $\Gamma_{m+1} \subset \Gamma_m$, and $\bigcap_m \Gamma_m = \{I\}$.

For any two elements $a, b$ of $UT_n(\mathbb{Z})$, the commutator $aba^{-1}b^{-1}$ will be denoted by $[a, b]$. The commutation relation in $UT_n(\mathbb{Z})$ could be written

$$[T_{ij}, T_{i'j}] = \begin{cases} T_{ij'}, & \text{if } j = i', \\ T_{j'i'}, & \text{if } i = i', \\ I, & \text{otherwise.} \end{cases}$$

(2.1)

It follows from these relations that the fundamental domain for $\Gamma_m$ can be written as a rectangle

$$\rho_m = \{ T_{n-1,n}^{k_{n-1,n}} \cdots T_{1,n}^{k_{1,n}} : l_{n-1,n}(m) \leq k_{n-1,n} \leq L_{n-1,n}(m), \ldots, l_{1,n}(m) \leq k_{1,n} \leq L_{1,n}(m) \},$$

with

$$l_{i,j}(m) = \left\lfloor \frac{p_m(j-i)}{2} \right\rfloor, \quad L_{i,j}(m) = \left\lceil \frac{p_m(j-i) + 1}{2} \right\rceil,$$

$[x]$ being the integral part of $x$.

It follows from the choice of generators that the multiplication law in $UT_n(\mathbb{Z})$ can be described by the relation

$$T_{n-1,n}^{k_{n-1,n}} \cdots T_{1,n}^{k_{1,n}} = T_{n-1,n}^{k_{n-1,n} + f_{n-1,n} + f_{n-1,n}} \cdots T_{1,n}^{k_{1,n} + q_{1,n} + f_{1,n}},$$

(2.2)

where $f_{n-1,n} = 0$ and for $1 \leq i < j \leq n$ the integer-valued polynomial function (see [8]) $f_{i,j} = f_{i,j}(k_{i,j}, q_{i,j} : T_{i,j}^{-1} T_{i',j'}^{-1} T_{i,j} T_{i',j'} \geq T_{i,j})$ is independent of $k_{i,j}, q_{i,j}$ with $T_{i',j'}, T_{i',j'}^{-1} \leq T_{i,j}$.

Let

$$F_{i,j}(m) = \max \{ f_{i,j}(k_{1,n}, \ldots, k_{n-1,n}, q_{1,n}, \ldots, q_{n-1,n}) : l_{i,j}(m) \leq k_{i,j} \leq L_{i,j}(m); \quad -m \leq q_{i,j} \leq m \}, \quad \text{for } 1 \leq i < j \leq n.$$

A routine verification shows that

$$F_{i,j}(m) \leq \text{const}(i, j)mp^{m(j-i-1)},$$

and hence

$$\frac{F_{i,j}(m)}{L_{i,j}(m) - l_{i,j}(m)} \to 0$$

(2.3)

as $m \to \infty$.

We claim that the sequence of rectangles $\rho_m$ forms a right Følner sequence of sets. Let $g = T_{n-1,n}^{q_{n-1,n}} \cdots T_{1,n}^{q_{1,n}}$ be fixed. To verify that $\#(\rho_n g \Delta \rho_n)/\#(\rho_n) \to 0$, 
it suffices to observe that $T_{n-1,1}^{k_{n-1,1}} \cdots T_{1,1}^{k_{1,1}} = T_{n-1,1}^{k_{n-1,1}+q_{n-1,1}+e_{n-1,1}(k_{1,1}, \ldots, k_{n-1,1})}$

Now our statement is an obvious consequence of the condition (2.3) as above. A similar argument shows that the $\rho_m$ also form a left Følner sequence of sets.

Let $G$ be an arbitrary countable amenable group, $(X, \mu)$ a $G$-space with invariant probability $\mu$, and $\alpha$ a partition of $X$ with finite entropy. The mean entropy $h(\alpha, G)$ is defined as

$$h(\alpha, G) = \lim_{n \to \infty} \frac{1}{\#(\rho_n)} H \left( \bigvee_{g \in \rho_n} g\alpha \right),$$

with $\rho_n$ being a right Følner sequence in $G$. It follows from the results of [13, 17] that, in the case when $G = UT_n(\mathbb{Z})$, $h(\alpha, UT_n(\mathbb{Z}))$ can be written in terms of the ‘past’ in $UT_n(\mathbb{Z})$ with respect to the left invariant order relation:

$$h(\alpha, UT_n(\mathbb{Z})) = H(\alpha|\alpha^\ominus_{UT_n(\mathbb{Z})}),$$

with $\alpha^\ominus_{UT_n(\mathbb{Z})} = \bigvee_{g < e} g\alpha$.

Now we are in a position to prove the following Lemma 2.2, Proposition 2.3, and Corollary 2.4. Despite our context in this section, they are valid for an arbitrary countable amenable group $G$ and will be used in the following.

Given a subset $E \subset G$ and a partition $\alpha$, we use the notation $\alpha_E$ for the partition $\bigvee_{g \in E} g\alpha$.

**Lemma 2.2.** [3, Lemma 4] Let $G_r$ be a subgroup of $G$ of a finite index $r$, $\delta_r$ a (finite) subset of $G$ which meets each $G_r$-coset exactly once, and $\alpha_r = \alpha_{\delta_r}$. Then

$$h(\alpha^r, G_r) = rh(\alpha, G).$$

**Proposition 2.3.** For $\alpha, \beta \in \mathcal{L}$,

$$h(\beta, G) \leq h(\alpha, G) + H(\beta|\alpha G). \quad (2.4)$$

**Proof.** Let $\rho_n$ be a right Følner sequence in $G$. One has

$$H(\beta_{\rho_n}) \leq H(\beta_{\rho_n}, \alpha_{\rho_n} \rho_p) = H(\alpha_{\rho_n} \rho_p) + H(\beta_{\rho_n}|\alpha_{\rho_n} \rho_p)$$

$$\leq H(\alpha_{\rho_n} \rho_p) + \sum_{f \in \rho_n} H(f \beta|\alpha_{\rho_n} \rho_p) \leq H(\alpha_{\rho_n} \rho_p) + \sum_{f \in \rho_n} H(f \beta|\alpha_{\rho_n} \rho_p)$$

$$= H(\alpha_{\rho_n} \rho_p) + \#(\rho_n) H(\beta|\alpha_{\rho_p}).$$

It follows from the Følner property that $\#(\rho_n \rho_p) / \#(\rho_n) \to 1$ as $n \to \infty$. Thus, after dividing out by $\#(\rho_n)$ and sending $n$ to infinity, we get for any fixed $p$

$$h(\beta, G) \leq h(\alpha, G) + H(\beta|\alpha_{\rho_p}).$$

Now it remains to send $p$ to infinity and to observe that $H(\beta|\alpha_{\rho_p}) \to H(\beta|\alpha_G)$ in order to get the desired relation. □
Corollary 2.4. For \( \alpha, \beta \in \mathcal{L} \),
\[
h(\alpha \lor \beta, G) \leq h(\alpha, G) + H(\beta \mid \alpha G).
\] (2.5)

Proof. This is an easy consequence of Proposition 2.3 when \( \beta \) is replaced by \( \alpha \lor \beta \) and then (1.1) applied.

Again we restrict our attention to the case of \( G = UT_n(\mathbb{Z}) \).

Theorem 2.5. (Pinsker formula) If \( \alpha, \beta \in \mathcal{L} \) then
\[
h(\alpha \lor \beta, UT_n(\mathbb{Z})) = h(\alpha, UT_n(\mathbb{Z})) + H(\beta \mid \beta_{UT_n(\mathbb{Z})}^+ \lor \alpha_{UT_n(\mathbb{Z})}).
\] (2.6)

Proof. Note that in the special case of the Heisenberg group, the proof of this formula is just the same as that found in [1]. It is based on Lemma 2.2 and Corollary 2.4. We demonstrate an extension of this argument to \( UT_n(\mathbb{Z}) \). The Følner sequence of subsets \( \delta_m \) is a fundamental domain for a finite index subgroup \( \Gamma_m \). These subgroups form a decreasing sequence with trivial intersection. An application of Lemma 2.2 and Corollary 2.4 yields
\[
h(\alpha \lor \beta, UT_n(\mathbb{Z})) = \frac{1}{\# \delta_m} h(\alpha_{\delta_m} \lor \beta_{\delta_m}, \Gamma_m) \leq \frac{1}{\# \delta_m} h(\alpha_{\delta_m}, \Gamma_m) + \frac{1}{\# \delta_m} H(\beta_{\delta_m} \mid (\alpha_{\delta_m})_{\Gamma_m})
\]
\[= h(\alpha, UT_n(\mathbb{Z})) + \frac{1}{\# \delta_m} H(\beta_{\delta_m} \mid \alpha_{UT_n(\mathbb{Z})}).
\]

Now apply a relativized version of the definition of mean entropy to observe that
\[
\frac{1}{\# \delta_m} H(\beta_{\delta_m} \mid \alpha_{UT_n(\mathbb{Z})}) \rightarrow h(\beta, UT_n(\mathbb{Z}), \alpha_{UT_n(\mathbb{Z})}) = H(\beta \mid \beta_{UT_n(\mathbb{Z})}^+ \lor \alpha_{UT_n(\mathbb{Z})}).
\]

Thus we obtain
\[
h(\alpha \lor \beta, UT_n(\mathbb{Z})) \leq h(\alpha, UT_n(\mathbb{Z})) + H(\beta \mid \beta_{UT_n(\mathbb{Z})}^+ \lor \alpha_{UT_n(\mathbb{Z})}).
\]

To get the converse inequality, consider the relation
\[
\frac{1}{\# \delta_m} H(\alpha_{\delta_m} \lor \beta_{\delta_m}) = \frac{1}{\# \delta_m} H(\alpha_{\delta_m}) + \frac{1}{\# \delta_m} H(\beta_{\delta_m} \mid \alpha_{\delta_m})
\]
\[\geq \frac{1}{\# \delta_m} H(\alpha_{\delta_m}) + \frac{1}{\# \delta_m} H(\beta_{\delta_m} \mid \alpha_{UT_n(\mathbb{Z})}).
\]
and send \( m \) to infinity, which completes the proof.

Turn to a description of Pinsker algebras in the special case of the Heisenberg group \( G \). Our argument admits a natural generalization, to be expounded below.

Suppose \( G \) acts on a Lebesgue space \((X, \mathcal{A}, \mu)\) with a finite invariant measure \( \mu \). Associate to any measurable partition \( \alpha \) the partition \( \pi(\alpha) \) which is defined to be the least upper bound of all those partitions \( \beta \in \mathcal{L} \) where \( \beta \leq \alpha G \) and \( h(\beta, G) = 0 \). In particular, with \( \alpha = \varepsilon \), one has \( \pi(\varepsilon) = \pi(G) \), the Pinsker partition of the dynamical system \((X, G)\). We are going to describe \( \pi(\alpha) \) more explicitly.
For \( \alpha \in \mathcal{L} \) we set up
\[
\hat{\alpha} = \bigwedge_n (T_1^{-n} \alpha_{\hat{r}_1} \lor T_2^{-n}(\alpha_{T_1})_{\hat{r}_2} \lor T_3^{-n}(\alpha_{T_1T_2})_{\hat{r}_3}).
\]
To make this partition invariant, set up \( \alpha_\infty = \bigvee_n \hat{\alpha}_{\rho_n} \) with \( \rho_n \) being a sequence of rectangles in \( G \) as discussed above, so it forms a right Følner sequence of sets in \( G \).

**Theorem 2.6.** For any partition \( \alpha \in \mathcal{L} \), \( \pi(\alpha) = \alpha_\infty \).

We need the following.

**Lemma 2.7.** If \( \alpha, \beta \in \mathcal{L} \), then
\[
\lim_{n \to \infty} H(\alpha|T_3^{-n}(\beta_G)|\alpha_G) = H(\alpha|\alpha_G).
\]

**Proof.** Apply the Pinsker formula (2.6) to \( \alpha \) and \( T_3^{-n} \beta \):
\[
h(\alpha \lor T_3^{-n} \beta, G) = h(\alpha, G) + H(T_3^{-n} \beta)(\beta_G \lor \alpha_G) = h(\alpha, G) + H(\beta|\beta_G \lor \alpha_G),
\]
with the latter equality being due to the obvious relation \( T_3^{-n} \beta G = T_3^{-n}(\beta G) \).

On the other hand, an application of (1.1) yields
\[
h(\alpha \lor T_3^{-n} \beta, G) = H(\alpha \lor T_3^{-n} \beta|\alpha_G \lor (T_3^{-n} \beta)_{\hat{r}_3})
\]
\[
= H(\alpha|\alpha_G \lor T_3^{-n}(\beta_G)) + h(T_3^{-n} \beta|\alpha \lor \alpha_G \lor T_3^{-n}(\beta_G))
\]
\[
= H(\alpha|\alpha_G \lor T_3^{-n}(\beta_G)) + H(\beta|T_3^{n}(\alpha \lor \alpha_G) \lor \beta_G).
\]
It is easy to see that \( T_3^n(\alpha \lor \alpha_G) \) is an increasing sequence of partitions whose join is \( \alpha_G \), and hence \( H(\beta|T_3^n(\alpha \lor \alpha_G) \lor \beta_G) \to H(\beta|\alpha_G \lor \beta_G) \) as \( n \to \infty \). Now the comparison of the two expressions for \( h(\alpha \lor T_3^{-n} \beta, G) \) proves our statement. \( \Box \)

**Proof of Theorem 2.6.** First we suppose that \( \beta \in \mathcal{L} \), and \( \beta \leq \alpha_\infty \). Note that the set of partitions which are less than some \( \hat{\alpha}_{\rho_n} \) is dense in the set of partitions which are less than \( \alpha_\infty \). In this context, it suffices to show that \( h(\beta, G) = 0 \) for \( \beta \in \mathcal{L}, \beta \leq \hat{\alpha}_{\rho_n} \). Actually, we consider here the case \( \beta \leq \hat{\alpha} \); the general case can be treated similarly. One has for all \( k_1, k_2, k_3 \)
\[
H(\beta|T_1^{-k_1}(\alpha_{T_1})_{\hat{r}_1} \lor T_2^{-k_2}(\alpha_{T_1T_2})_{\hat{r}_2} \lor T_3^{-k_3}(\alpha_{T_1T_2T_3})_{\hat{r}_3}) = 0.
\]
Let \( k_1 \to \infty \), then an application of the limit relation [1, equation (21')] for a single transformation \( T_1 \) and a \( T_1 \)-invariant partition \( \sigma = T_2^{-k_2}(\alpha_{T_1})_{\hat{r}_2} \lor T_3^{-k_3}(\alpha_{T_1T_2})_{\hat{r}_3} \lor (\beta_{T_1})_{\hat{r}_3} \lor (\beta_{T_1T_2})_{\hat{r}_3} \) yields
\[
H(\beta|T_2^{-k_2}(\alpha_{T_1})_{\hat{r}_2} \lor T_3^{-k_3}(\alpha_{T_1T_2})_{\hat{r}_3} \lor (\beta_{T_1})_{\hat{r}_3} \lor (\beta_{T_1T_2})_{\hat{r}_3}) = 0.
\]
and hence
\[
H(\beta|T_2^{-k_2}(\alpha_{T_1})_{\hat{r}_2} \lor (\alpha_{T_1})_{\hat{r}_1} \lor (T_1^{-k_1}(\alpha_{T_1})_{\hat{r}_2} \lor T_3^{-k_3}(\alpha_{T_1T_2})_{\hat{r}_3} \lor (\beta_{T_1})_{\hat{r}_3} \lor (\beta_{T_1T_2})_{\hat{r}_3})) = 0.
\]
Apply again an analogue of Lemma 2.7 for \( G = \mathbb{Z}^2 \) [1] with \( \sigma = T_3^{-k_3}(\alpha_{T_1T_2})_{\hat{r}_3} \lor (\beta_{T_1})_{\hat{r}_3} \lor (\beta_{T_1T_2})_{\hat{r}_3} \) and the generators \( T_1, T_2 \) to get
\[
H(\beta|T_3^{-k_3}(\alpha_{T_1T_2})_{\hat{r}_3} \lor (\beta_{T_1})_{\hat{r}_3} \lor (\beta_{T_1T_2})_{\hat{r}_3}) = 0.
\]
which also implies
\[ H(\beta | T^{-k} \alpha) \leq (\alpha \nu \alpha) \leq (\alpha \nu \alpha) \leq (\beta \nu \alpha) \leq (\beta \nu \alpha). \]

Finally, an application of Lemma 2.7 assures \( H(\beta | \beta_G) = 0 \), which was to be proved.

Conversely, let \( \beta \in \mathcal{L}, \beta \leq \alpha_G \), and \( h(\beta, G) = 0 \). Recall that for any subgroup \( G_p \) of a finite index, \( h(\beta, G_p) \leq h(\beta^p, G_p) = ph(\beta, G) = 0 \). Given any partition \( \gamma \in \mathcal{L} \), the double application of the Pinsker formula (2.6) yields
\[ H(\gamma | \gamma_{G_p}) + H(\beta | \beta_{G_p} \vee \gamma_{G_p}) = H(\beta | \beta_{G_p} \vee \gamma_{G_p}) + H(\gamma | \gamma_{G_p} \vee \beta_{G_p}), \]

and since \( H(\beta | \beta_{G_p} \vee \gamma_{G_p}) = H(\beta | \beta_{G_p}) = 0 \), we obtain
\[ H(\gamma | \gamma_{G_p}) = H(\gamma | \gamma_{G_p} \vee \beta_{G_p}), \]

hence \( H(\gamma | \gamma_{G_p}) = H(\gamma | \gamma_{G_p} \vee \beta) \).

Let \( G_p \) be generated by \( T_1^n, T_2^n, T_3^n \), then \( \gamma_{G_p} \preceq T_1^{-n} \gamma_{T_1} \vee T_2^{-n} (\gamma_{T_1})_{T_2} \vee T_3^{-n} (\gamma_{T_1})_{T_3} \).

Thus one has
\[ H(\gamma | \beta) \geq H(\gamma | \gamma_{G_p} \vee \beta) = H(\gamma | \gamma_{G_p}) \geq H(\gamma | T_1^{-n} \gamma_{T_1} \vee T_2^{-n} (\gamma_{T_1})_{T_2} \vee T_3^{-n} (\gamma_{T_1})_{T_3}). \]

Consider the limit of this as \( n \to \infty \) to get \( H(\gamma | \beta) \geq H(\gamma | \hat{\beta}) \). In the case when \( \gamma \leq \alpha_\beta \) for some rectangle \( \beta \) we also deduce that
\[ H(\gamma | \beta) \geq H(\gamma | \hat{\beta}) \geq H(\gamma | \hat{\beta}_\rho) \geq H(\gamma | \hat{\alpha}_\infty). \]

It follows from the condition \( \beta \leq \alpha_G \) that \( \beta \) can be represented as a limit in the metric \( d \) of some sequence of partitions \( \gamma_n \) with \( \gamma_n \leq \alpha_\rho_n \) for some sequence of rectangles \( \rho_n \) whose union is \( G \). Thus considering the limit of the above inequality gives \( 0 = H(\beta | \beta) = H(\beta | \alpha_\infty) \), that is \( \beta \leq \alpha_\infty \), and hence \( \pi(\alpha) \leq \alpha_\infty \).

Definition A. The dynamical system \((X, G)\) is said to have a c.p.e. if \( \pi(G) = \pi(\nu) = \nu \); equivalently, for any partition \( \beta \in \mathcal{L} \), \( h(\beta, G) = 0 \) implies \( \beta = \nu \).

Fix the sequence of finite index subgroups of \( G \) that was mentioned in the proof of Theorem 2.6: for each \( n \in \mathbb{N} \), the subgroup \( \Gamma_n \) is generated by \( T_1^n, T_2^n, T_3^n \). This is used in the formulation of the following proposition.

Proposition 2.8. A dynamical system \((X, G)\) has c.p.e. if and only if for any partition \( \alpha \in \mathcal{L} \) one has \( H(\alpha) = \sup_{\Gamma_n} h(\alpha, \Gamma_n) \).

Proof. Suppose \((X, G)\) has c.p.e., and we are given a finite partition \( \alpha \). We claim that \( \bigwedge_{\Gamma_n} \alpha_{\Gamma_n} \leq \pi(G) \), that is, under our assumptions, \( \alpha_{\Gamma_n} \to \nu \) as \( n \to \infty \). In fact, suppose \( \gamma \leq \bigwedge_{\Gamma_n} \alpha_{\Gamma_n} \), that is, \( H(\gamma | \bigwedge_{\Gamma_n} \alpha_{\Gamma_n}) = 0 \). Since
\[ \bigwedge_{\Gamma_n} \alpha_{\Gamma_n} \preceq T_1^{-n} \alpha_{T_1} \vee T_2^{-n} (\alpha_{T_1})_{T_2} \vee T_3^{-n} (\alpha_{T_1})_{T_3}, \]

one also has
\[ H(\gamma | T_1^{-n} \alpha_{T_1} \vee T_2^{-n} (\alpha_{T_1})_{T_2} \vee T_3^{-n} (\alpha_{T_1})_{T_3}) = 0. \]
Tending to a limit as \( n \to \infty \), we get \( H(\gamma | \hat{\alpha}) = 0 \), that is \( \gamma \leq \hat{\alpha} \leq \alpha_\infty \leq \pi(\alpha) \leq \pi(G) \), and so our statement is proved. This implies

\[
H(\alpha) \geq h(\alpha, \Gamma_n) = H(\alpha | \alpha_\Gamma_n^-) \to H(\alpha) \quad \text{as } n \to \infty
\]

which was to be proved.

Conversely, if \( h(\alpha, G) = 0 \), it follows from Lemma 2.2 that \( h(\alpha, G_p) = 0 \) for all finite index subgroups \( G_p \), that is, \( H(\alpha) = 0 \), and hence a complete positivity. \( \square \)

3. \textit{K-systems and invariant partitions for the Heisenberg group}

This section is intended to reproduce in an appropriate form the theory of Conze \cite{1} and Kamiński \cite{6} in the case of actions of a finitely-generated, torsion-free, nilpotent group. For the sake of simplicity, we expound the arguments in the case of the Heisenberg group as in \S 2. As before, an obvious modification of our technique also works in the general case.

Let \( G \) be the Heisenberg group. We need a notion of \( \sigma \)-relative entropy for a \( G \)-invariant measurable partition \( \sigma \) which is defined by \( h(\sigma, G, \sigma) = \sup_{\alpha \in \mathcal{L}} h(\alpha, G, \sigma) \). Here

\[
h(\alpha, G, \sigma) = \lim_{n \to \infty} \frac{1}{\#(\rho_n)} H\left( \bigvee_{g \in \rho_n} g\alpha \bigg| \sigma \right)
\]

is the relative mean entropy of the partition \( \alpha \). In our case of an ordered group \( G \), \( h(\alpha, G, \sigma) = H(\alpha | \alpha_\sigma^- \vee \sigma) \).

Also, by the Pinsker closure \( \overline{\sigma} \) of \( \sigma \) with respect to \( \sigma \) we mean the join of all partitions \( \alpha \in \mathcal{L} \) such that \( h(\alpha, G, \sigma) = 0 \). Clearly, with \( \sigma \) the trivial partition, \( \overline{\sigma} = \pi(G) \). We list here some straightforward properties of \( \overline{\sigma} \):

(1) \( \sigma \leq \overline{\sigma} \);
(2) \( \overline{\sigma} \) is \( G \)-invariant;
(3) \( \overline{\overline{\sigma}} = \overline{\sigma} \);
(4) for any automorphism \( T \) commuting with \( G \) one has \( T\overline{\sigma} = T\overline{\sigma} \).

There exists a \( \sigma \)-relative version of the Rokhlin–Sinai Theorem concerning the existence of perfect partitions for a single automorphism \( S \) \cite{15} which is generalized below. Specifically, we have (see \cite{6, Lemma 1}) the following lemma.

**LEMMA 3.1.** There exists a measurable partition \( \zeta \) such that:

(i) \( \sigma \leq S^{-1}\zeta \leq \zeta \);
(ii) \( \bigvee_{n=0}^\infty S^n\zeta = \varepsilon \);
(iii) \( \bigwedge_{n=0}^\infty S^{-n}\zeta = \overline{\sigma} \);
(iv) \( h(\zeta, S, \sigma) = h(S, \sigma) \).

**LEMMA 3.2.** For every partition \( \alpha \in \mathcal{L} \) and the \( G \)-invariant measurable partition \( \sigma_1, \sigma_2 \), one has \( h(\alpha, G, \overline{\sigma_1 \vee \sigma_2}) = h(\alpha, G, \sigma_1 \vee \sigma_2) \).

**Proof.** This is just the same as that of \cite[Lemma 2]{6}. \( \square \)

Define some special invariance properties related to the lexicographic order relation in \( G \). From now on we replace the ordinary notion of \( G \)-invariance (\( g\zeta = \zeta \) for all \( g \in G \)) by the term \textit{total invariance}. 

A measurable partition $\gamma$ is said to be $G$-invariant if $T_1^{-1} \gamma \leq \gamma$, $T_2^{-1} \gamma \leq \gamma$, and $T_3^{-1} \gamma \leq \gamma$. Note that $\gamma$ is $G$-invariant if $T_1^{k_1}T_2^{k_2}T_3^{k_3} \gamma \leq \gamma$ for those triples $(k_1, k_2, k_3)$ which are lexicographically less than $(0, 0, 0)$. Another important observation is that for $\gamma$ $G$-invariant, $\gamma_G = T_1^{-1} \gamma$.

A measurable partition $\gamma$ is said to be strongly invariant iff $T_1^{-1} \gamma \leq \gamma$, $\bigwedge_{n=0}^{\infty} T_1^{-n} \gamma = T_2^{-1} \gamma \gamma_1$, and $\bigwedge_{n=0}^{\infty} T_2^{-n} \gamma_1 = T_3^{-1} \gamma \gamma_2$. Of course this notion is stronger than invariance.

A measurable partition $\gamma$ is said to be $G$-exhaustive if $\gamma$ is strongly invariant and $\gamma_G = \varepsilon$.

**Lemma 3.3.** Suppose that a measurable partition $\gamma$ is strongly invariant, $\alpha \in \mathcal{L}$, $\alpha \leq T_3^p \gamma \gamma_1 \gamma_2$ for some positive integer $p$, and $h(\alpha, G) = 0$. Then $\alpha \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \gamma \gamma_1 \gamma_2$.

**Proof.** Let us first prove our statement under the assumption $\alpha \in \mathcal{L}$, $\alpha \leq T_3^p \gamma \gamma_1 \gamma_2$ for some integer $q$. It follows from $h(\alpha, G) = 0$ and Lemma 2.2 that for the finite index subgroup $G_\ell$ generated by $T_1^k$, $T_2^k$, $T_3$ one has $h(\alpha, G_\ell) = 0$. An application of a relativized version of the Pinsker formula (2.6) for the case $G = \mathbb{Z}$ generated by $T_1^k$ [1] yields for any partition $\gamma \in \mathcal{L}$

$$
H(\alpha \vee \gamma | \alpha_1^{-1} \gamma_1 T_1^2 \gamma_2^{-1} \vee (\alpha_1 T_1^2) \gamma_2^{-1} \vee (\gamma T_1^2) \gamma_2^{-1} \vee T_3^p \gamma_1) = H(\alpha | \alpha_1^{-1} \gamma_1 T_1^2 \gamma_2^{-1} \vee (\alpha_1 T_1^2) \gamma_2^{-1} \vee (\gamma T_1^2) \gamma_2^{-1} \vee T_3^p \gamma_1) + H(\gamma | \alpha_1^{-1} \gamma_1 T_1^2 \gamma_2^{-1} \vee (\alpha_1 T_1^2) \gamma_2^{-1} \vee (\gamma T_1^2) \gamma_2^{-1} \vee T_3^p \gamma_1)
$$

It follows from our assumptions on $\alpha$ and the strong invariance of $\gamma$ that $(\alpha T_1^2 T_2 \gamma_1 T_3)$ $\gamma_2 \leq T_3^{-1} \gamma \gamma_1 \gamma_2$, and so

$$
H(\alpha | \alpha_1^{-1} \gamma_1 T_1^2 \gamma_2^{-1} \vee (\alpha_1 T_1^2) \gamma_2^{-1} \vee (\gamma T_1^2) \gamma_2^{-1} \vee T_3^p \gamma_1) = H(\alpha | \alpha_1^{-1} \gamma_1 T_1^2 \gamma_2^{-1} \vee (\alpha_1 T_1^2) \gamma_2^{-1} \vee (\gamma T_1^2) \gamma_2^{-1} \vee T_3^p \gamma_1) = 0,
$$

and hence

$$
H(\gamma | \gamma_1^{-1} \vee (\alpha_1 T_1^2) \gamma_2^{-1} \vee (\gamma T_1^2) \gamma_2^{-1} \vee T_3^p \gamma_1) = H(\gamma | \gamma_1^{-1} \vee (\alpha_1 T_1^2) \gamma_2^{-1} \vee (\gamma T_1^2) \gamma_2^{-1} \vee T_3^p \gamma_1) = 0
$$

for all $k$. If we assume $\gamma \leq T_3^r T_2^q T_1^s \gamma$ for some integer $r$ we get

$$
H(\gamma | \alpha \vee T_3^p \gamma_1 \gamma_2) = H(\gamma | \gamma \alpha_1^{-1} \gamma_1 T_1^2 \gamma_2^{-1} \vee (\alpha_1 T_1^2) \gamma_2^{-1} \vee (\gamma T_1^2) \gamma_2^{-1} \vee T_3^p \gamma_1) \gamma_2
$$

$$
= H(\gamma | \gamma_1^{-1} \vee (\alpha_1 T_1^2) \gamma_2^{-1} \vee (\gamma T_1^2) \gamma_2^{-1} \vee T_3^p \gamma_1)
$$

$$
\geq H(\gamma | T_3^p T_2^q T_1^r \gamma \gamma_1 \gamma_2 \vee T_3^p T_2^q \gamma_1 \gamma_2)
$$

If we let $k$ go to infinity in this relation and apply the strong invariance of $\gamma$, we obtain

$$
H(\gamma | \alpha \vee T_3^p \gamma_1 \gamma_2) = H(\gamma | T_3^p T_2^q \gamma_1 \gamma_2 \vee T_3^p \gamma_1 \gamma_2) = H(\gamma | T_3^p T_2^q \gamma_1 \gamma_2).
$$
Since this is valid for a class of partitions $\gamma$ which is $d$-dense in the class of partitions $\alpha \in \mathcal{L}$, $\alpha \leq T_3^p T_2^q \xi_1$, we can replace $\gamma$ with $\alpha$ in the latter inequality to get

$$H(\alpha | T_3^p T_2^q \xi_1) = 0,$$

and thus we have deduced $\alpha \leq T_3^p T_2^q \xi_1$ from $\alpha \leq T_3^p T_2^q \xi_1$. If we proceed in this way infinitely many times, we obtain $\alpha \leq \bigwedge \{ T_3^p T_2^q \xi_1 = T_3^p \xi_1 T_2 \}$. Now we can get rid of our more subtle assumption $\alpha \leq T_3^p T_2^q \xi_1$ as compared with that written in the statement of the lemma via an approximation argument since $\bigvee \{ T_3^p T_2^q \xi_1 = T_3^p \xi_1 T_2 \}$.

Thus we have deduced that $\alpha \leq T_3^p \xi_1 T_2$. Now proceed by induction in $p$ to get the desired statement. $\square$

A relativized version of Lemma 3.3 is also valid. Specifically we have the following.

**Lemma 3.4.** Let $\sigma$ be a $G$-totally invariant partition. Suppose also that a measurable partition $\xi$ is strongly invariant, $\xi \geq \sigma$, $\alpha \in \mathcal{L}$, $\alpha \leq \bigwedge T_3^p \xi_1 T_2$ for some positive integer $p$, and $h_\sigma(\alpha, G) = 0$. Then $\alpha \leq \bigwedge \{ T_3^p \xi_1 T_2 \}$.

**Proof.** One can reproduce the proof of Lemma 3.3 with the functionals $H(\cdot | \cdot)$ replaced by $H(\cdot | \cdot \lor \sigma)$ and $\pi(G)$ by $\bar{\mathcal{L}}$. $\square$

**Lemma 3.5.** For every $G$-exhaustive partition $\xi$, $\bigwedge \{ T_3^p \xi_1 T_2 \} \geq \pi(G)$.

**Proof.** Let $\alpha \in \mathcal{L}$ and $\alpha \leq \pi(G)$, that is $h(\alpha, G) = 0$, and assume also $\beta \in \mathcal{L}$ with $\beta \leq T_3^p \xi_1 T_2$ for some positive integer $p$. Also denote by $G_k$ the finite index subgroup generated by $T_1, T_2, T_3$, then certainly $h(\alpha, G_k) = 0$. Since this subgroup has quite the same structure as $G$ itself, we can apply the Pinsker formula (2.6) as follows:

$$h(\beta \lor G_k) = h(\alpha \lor G_k) + H(\beta | \beta_{G_k} \lor \alpha_{G_k}) = h(\beta, G_k) + H(\alpha | \beta_{G_k} \lor \alpha_{G_k}).$$

Since $h(\alpha, G_k) = H(\alpha | \beta_{G_k} \lor \alpha_{G_k}) = 0$, we deduce that $h(\beta, G_k) = H(\beta | \beta_{G_k} \lor \alpha_{G_k})$ for all $k$, and hence

$$H(\beta | \alpha) \geq H(\beta | \beta_{G_k} \lor \alpha_{G_k})$$

$$= h(\beta, G_k) \geq H(\beta | T_3^{k-1} \beta_{T_1} \lor T_2^{k-1} (\beta_{T_1} T_2) \lor T_3^{k-1} (\beta_{T_1} T_2) T_2).$$

If we let $k$ go to infinity, we get $H(\beta | \alpha) \geq H(\beta | \hat{\beta})$ with $\hat{\beta}$ as in the definition of $\pi(\beta)$.

Now our purpose is to prove that $\hat{\beta} \leq \bigwedge \{ T_3^{-n} \xi_1 T_2 \}$.

Let $\gamma \in \mathcal{L}$ and $\gamma \leq \hat{\beta}$. It was shown in the proof of Theorem 2.6 that $\hat{\beta} \leq \pi(G)$, that is $h(\gamma, G) = 0$. Since $\beta \leq T_3^p \xi_1 T_2$ and $\xi$ is $G$-invariant, we have $\hat{\beta} \leq T_3^p \xi_1 T_2$ and hence $\gamma \leq T_3^p \xi_1 T_2$. Now by Lemma 3.3 we deduce that $\gamma \leq \bigwedge \{ T_3^{-n} \xi_1 T_2 \}$, and hence our statement.

This, together with our previous observations imply that

$$H(\beta | \alpha) \geq H(\beta | \bigwedge \{ T_3^{-n} \xi_1 T_2 \}).$$
Since \( \xi_G = \epsilon \), we observe that the latter inequality is valid for \( \beta \) in a dense subset of \( L \), and hence it is also valid for \( \alpha \). Thus we obtain

\[
0 = H(\alpha|\alpha) \geq H \left( \alpha \left| \bigwedge_{n=0}^{\infty} T_3^{-n} \xi_{T_{1}T_{2}} \right. \right),
\]

that is, \( \alpha \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \xi_{T_{1}T_{2}} \), which was to be proved.

We also need a relativized version of Lemma 3.5.

**LEMMA 3.6.** Let \( \sigma \) be a \( G \)-totally invariant partition. For every \( G \)-exhaustive partition \( \xi \), \( \xi \geq \sigma \), one has \( \bigwedge_{n=0}^{\infty} T_3^{-n} \xi_{T_{1}T_{2}} \geq \sigma \).

**Proof.** This is just the same as that of Lemma 3.5, with some minor changes as indicated in the proof of Lemma 3.4. \( \square \)

**LEMMA 3.7.** There exists a measurable partition \( \eta \) with the properties:

(a) \( T_1^{-1} \eta \leq \eta \), \( T_2^{-1} \eta_{T_1} \leq \eta \), \( T_3^{-1} \eta_{T_1T_2} \leq \eta \);

(b) \( \eta_G = \epsilon \);

(c) \( \bigwedge_{n=0}^{\infty} T_3^{-n} \eta_{T_1T_2} \leq \pi(G) \);

(d) \( h(G) = H(\eta|\eta_G) = H(\eta|T_1^{-1} \eta) \).

**Proof.** Let \( \alpha_k \in L \) be an increasing sequence whose join is \( \xi \). We claim that there exists an increasing sequence of positive integers \( n_1 < n_2 < \cdots \) such that for \( \xi_p = \bigvee_{k=1}^{p} T_3^{-n_k} \alpha_k \) one has

\[
H(\xi_p|\xi_{p-1})_G - H(\xi_p|\xi_{p+s+1})_G \leq \frac{1}{p}, \quad s \geq 0.
\]

To form such sequence of \( n_k \), it suffices to ensure the following property is satisfied:

\[
H(\xi_p|\xi_{q-1})_G - H(\xi_p|\xi_{q})_G \leq \frac{1}{p} 2^{-p}, \quad p < q.
\]

In fact, if we take a sum in (3.2), we get \( H(\xi_p|\xi_q)_G - H(\xi_p|\xi_{q-1})_G < 1/p \) as desired. In order to obtain (3.2), we have to proceed by induction. Specifically, suppose \( n_1, \ldots, n_{q-1} \) have already been chosen, then \( n_q \) should be selected to be so large that (3.2) is valid with \( p = 1, \ldots, q - 1 \). This is possible by virtue of Lemma 2.7.

Let \( \xi = \bigvee_{p=1}^{\infty} \xi_p \), and \( \eta = \xi \vee \xi_G \). It is easy to verify that (a) and (b) are satisfied for \( \eta \).

Let us prove that (c) holds for \( \eta \). Since \( (\xi_p)_G \) is an increasing sequence of partitions whose join is \( \xi_G = \eta_G \), we can tend to a limit in (3.1) as \( s \to \infty \) to get

\[
H(\xi_p|\xi_{q})_G - H(\xi_p|\xi_{p+1})_G \leq \frac{1}{p}, \quad p \geq 1.
\]

Now let \( \alpha \in L \) and \( \alpha \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \eta_{T_1T_2} \). Since the latter partition is totally invariant with respect to \( G \), we have

\[
\alpha_G = \bigwedge_{n=0}^{\infty} T_3^{-n} \eta_{T_1T_2} = \bigwedge_{n=0}^{\infty} T_3^{-n} \xi_{T_1T_2}.
\]

Hence \( \alpha_G \leq \eta_{T_1T_2} \) for all \( n \geq 1 \), and so \( \alpha_G \leq (\eta_{T_1T_2})_{T_3} \).
It follows from the Pinsker formula (2.6) that
\[ h(\alpha \vee \xi_p, G) = h(\xi_p, G) + H(\alpha|a_G^{\sim} \vee (\xi_p)G) = h(\alpha, G) + H(\xi_p|a_G^{\sim} \vee (\xi_p)G), \]
and hence
\[ h(\alpha, G) = h(\xi_p, G) - H(\xi_p|a_G^{\sim} \vee (\xi_p)G) + H(\alpha|a_G^{\sim} \vee (\xi_p)G). \] (3.4)

It follows from Lemma 3.2 that
\[ H(\xi_p|a_G^{\sim} \vee (\xi_p)G) \geq H(\xi_p|a_G^{\sim} \vee (\eta_{T_1T_2}^{T_1})) = H(\xi_p|a_G^{\sim} \vee (\eta_{T_1T_2}^{T_1})) \]
\[ \geq H(\xi_p|a_G^{\sim} \vee (\eta_{G})) = H(\xi_p|a_G^{\sim}). \]

and so (3.4) implies
\[ h(\alpha, G) \leq h(\xi_p, G) - H(\xi_p|\eta_{G}^{\sim}) + H(\alpha|a_G^{\sim} \vee (\xi_p)G) \leq \frac{1}{p} + H(\alpha|a_G^{\sim} \vee (\xi_p)G). \]

Since \( \sqrt[p]{\sum_{n=1}^{\infty}(\xi_p)G} = \varepsilon \), we can tend to a limit in the latter inequality as \( p \to \infty \) to get
\[ h(\alpha, G) = 0, \text{ that is } \alpha \leq \pi(G). \]

To prove (d), observe that since \( \alpha_k \) increases up to \( \varepsilon \), \( H(\xi_p|a_G^{\sim}) = h(\xi_p, G) = h(\alpha, G) \to h(G). \) On the other hand, since \( \xi_p \) is increasing up to \( \xi \), one also has \( H(\xi_p|\eta_G^{\sim}) \to H(\xi|\eta_G^{\sim}). \) Now it follows from (3.3) that \( H(\xi_p|\eta_G^{\sim}) \) and \( H(\xi_p|\eta_{G}^{\sim}) \) have the same limit, and so \( h(G) = H(\xi|\eta_G^{\sim}) = H(\xi|\eta_{G}^{\sim}) = H(\xi|\eta_{G}^{\sim}). \)

Definition B. A measurable partition \( \xi \) is said to be \( G \)-perfect if:
(i) \( T_1^{-1}\xi \leq \xi, T_2^{-1}\xi T_1 \leq \xi, T_3^{-1}\xi T_1T_2 \leq \xi; \)
(ii) \( \xi G = \varepsilon; \)
(iii) \( \bigwedge_{n=0}^{\infty} T_1^{-n}\xi T_1 = T_2^{-1}\xi T_1, \bigwedge_{n=0}^{\infty} T_2^{-n}\xi T_1 = T_3^{-1}\xi T_1T_2; \)
(iv) \( \bigwedge_{n=0}^{\infty} T_3^{-n}\xi T_1T_2 = \pi(G); \)
(v) \( h(G) = H(\xi|\xi G) = H(\xi|\xi_{G}^{-1}). \)

THEOREM 3.8. For any \( G \)-action there exists a \( G \)-perfect partition.

Proof. Let \( \eta \) be a measurable partition which satisfies the properties (a)–(d) of Lemma 3.7. The property (a) implies \( T_3^{-1}\eta_{T_1T_2} \leq \eta_{T_1T_2}. \) Now a relativized version of this theorem for the case of \( \mathbb{Z}^2 \) generated by \( T_1, T_2 \) on the quotient space \( X/\eta_{T_1T_2} \) implies that there exists a measurable partition \( \xi \) with the properties
\[ T_3^{-1}\eta_{T_1T_2} \leq \xi \leq \eta_{T_1T_2}, \] (3.5)
\[ T_1^{-1}\xi \leq \xi, T_2^{-1}\xi T_1 \leq \xi, \] (3.6)
\[ \xi T_1T_2 = \pi_{T_1T_2}, \] (3.7)
\[ \bigwedge_{n=0}^{\infty} T_1^{-n}\xi = T_2^{-1}\xi T_1, \bigwedge_{n=0}^{\infty} T_2^{-n}\xi T_1 = T_3^{-1}\xi T_1T_2, \] (3.8)
\[ h_{T_3^{-1}\eta_{T_1T_2}}(\xi, (T_1, T_2)) = h_{T_3^{-1}\eta_{T_1T_2}}(T_1, T_2) = H(\xi|T_1^{-1}\xi \vee T_3^{-1}\eta_{T_1T_2}). \] (3.9)

To prove this, one has to repeat the proof of [6, Theorem 4], with some slight changes being introduced as in the proof of Lemma 3.4 and \( \sigma = T_3^{-1}\eta_{T_1T_2}. \)
Let $\zeta$ is an easy consequence of Theorem 3.8. ✷

(i)

Alternatively, $G$-exhaustive, and so Lemma 3.5 implies $\bigcap_{n=0}^{\infty} T^{-n}_{1} \eta T_{1} = G$. It also follows from (c) that $\bigcap_{n=0}^{\infty} T^{-n}_{3} \eta T_{1} = \pi(G)$, so in view of (3.7) we get (iv). Now it remains to verify (v).

Let $\alpha_{k} \in L$ be an increasing sequence of partitions with $\bigvee_{k} \alpha_{k} = \zeta$. Hence $(\bigvee_{k} \alpha_{k})_{G} = \zeta_{G}^{-1} = T_{1}^{-1} \zeta$, and so

$$H(\alpha_{k}|T_{1}^{-1} \zeta) \leq H(\alpha_{k}|(\alpha_{k})_{G}) = h(\alpha_{k}, G) \leq h(G), \quad k \geq 1.$$ 

Tend to a limit as $k \to \infty$ to get $H(\zeta|T_{1}^{-1} \zeta) \leq h(G)$. Rewrite (3.9) in the form

$$H(\zeta|T_{1}^{-1} \zeta) \vee T^{-1}_{3} \eta T_{1} = \sup \{H(\alpha|\alpha(T_{1}^{-1} \zeta) \vee T^{-1}_{3} \eta T_{1}) \}.$$ 

In view of (3.5) one has $T^{-1}_{3} \eta T_{1} \leq \eta(T_{1}^{-1} \zeta)$, and so

$$H(\zeta|T_{1}^{-1} \zeta) \leq \eta(T_{1}^{-1} \zeta) = H(\zeta).$$ 

(11)

It follows from Lemma 3.2 that $H(\alpha|\alpha(T_{1}^{-1} \zeta) \vee T^{-1}_{3} \eta T_{1}) = H(\alpha|T_{1}^{-1} \zeta T_{1})$, and so

$$\sup \{H(\alpha|\alpha(T_{1}^{-1} \zeta) \vee T^{-1}_{3} \eta T_{1}) \} \geq \sup \{H(\alpha|T_{1}^{-1} \zeta T_{1}) \} \geq \sup \{H(\alpha|\alpha(T_{1}^{-1} \zeta) \vee T^{-1}_{3} \eta T_{1}) \} \geq \sup \{H(\alpha|T_{1}^{-1} \eta T_{1}) \} = H(\eta|T_{1}^{-1} \eta).$$

A comparison of this result with (d), (3.10) and (3.11) yields $h(G) = H(\eta|T_{1}^{-1} \eta) \leq H(\zeta|T_{1}^{-1} \zeta)$, which completes the proof.

**Lemma 3.9.** The following conditions are equivalent:

(a) $h(G) = 0$;

(b) the only $G$-exhaustive partition is $\zeta$;

(c) each $G$-strongly invariant partitions is totally invariant.

**Proof.** This can be reproduced from that of [6, Theorem 5].

**Definition C.** A $G$-action is said to be a $K$-system (or merely $G$ is said to be a $K$-group if the action of $G$ is implicit) if there exists a measurable partition $\zeta$ such that:

(i) $T_{1}^{-1} \zeta \leq \zeta, T_{1}^{-1} \xi T_{1} \leq \zeta, T_{1}^{-1} \eta T_{1} \leq \zeta$;

(ii) $\xi G = \zeta$;

(iii) $\bigcap_{n=0}^{\infty} T^{-n}_{1} \eta T_{1} = T_{1}^{-1} \zeta T_{1}, \bigcap_{n=0}^{\infty} T^{-n}_{2} \xi T_{1} = T_{1}^{-1} \xi T_{1}$;

(iv) $\bigcap_{n=0}^{\infty} T^{-n}_{3} \eta T_{1} = T_{1}^{-1} \eta T_{1}$.

**Theorem 3.10.** $G$ is a $K$-group iff it has c.p.e.

**Proof.** Let $G$ be a $K$-group and $\zeta$ be a partition as in the definition above. In particular, $\zeta$ is $G$-exhaustive, and thus by Lemma 3.5 $\pi(G) = \zeta$, that is $G$ has c.p.e. The converse is an easy consequence of Theorem 3.8.

\[ \square \]
4. **Entropy and Pinsker algebras for general torsion-free, nilpotent groups**

We turn to sketching the ways of extending the results of the previous sections onto general finitely-generated, torsion-free, nilpotent groups. Let $G$ be a finitely-generated, torsion-free, nilpotent group. Consider the upper central series of $G$. Specifically, let $Z_1$ be the center of $G$. Define the subgroup $Z_2$ in such a way that $Z_2/Z_1 = \text{center}(G/Z_1)$. Continue this procedure by choosing at step $i$ the subgroup $Z_i$ so that $Z_i/Z_{i-1} = \text{center}(G/Z_{i-1})$. After finitely many steps we get $Z_N = G$, with $N$ being the order of nilpotence [10]. Thus we have a finite sequence of subgroups

$$G = Z_N \supset Z_{N-1} \supset Z_{N-2} \supset \cdots \supset Z_2 \supset Z_1 \supset Z_0 = \{e\}.$$  

It is shown in [8] that there exists a sequence of generators of $G$ with some special properties, called a Maltsev basis. Specifically, the sequence of generators in question

$$T_1, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_{n_2}, T_{n_2+1}, \ldots, T_{n_{N-1}}, T_{n_{N-1}+1}, \ldots, T_{n_N}$$

can be chosen so that $n_0 = 0$, $T_{n_i+1}, \ldots, T_{n_i+1} \in Z_{i+1}$, and $T_{n_i} \not\in Z_i$ for $k > n_i$, $j \in \mathbb{Z} \setminus \{0\}$.

Define the linear order relation on the above generators by setting $T_1 < T_2 < \cdots < T_M$ ($n_M = M$), together with the associated lexicographic linear order relation on $G$: $T_{i_1}^{j_1} \cdots T_{i_m}^{j_m} < T_{k_1}^{j_1} \cdots T_{k_m}^{j_m}$ iff $(j_1, \ldots, j_m)$ is lexicographically less than $(k_1, \ldots, k_m)$. It is easy to verify that this order relation is invariant with respect to the left translations on $G$. So we have the notion of the ‘past’ in $G$ as a subset of all elements of $G$ which are less than the identity. A Maltsev basis is used in [8] to prove that any finitely-generated, torsion-free, nilpotent group is embeddable into the group $UT_n(\mathbb{Z})$ of integral unipotent upper triangular matrices.

It follows from the choice of generators that the multiplication law in $G$ can be described by the relation

$$T_1^{f_1} \cdots T_{M}^{f_M} = T_1^{k_1+q_1+f_1} \cdots T_{M}^{k_m+q_m+f_m},$$

where $f_M = 0$ and for $j = 1, \ldots, M-1$ the integer-valued polynomial function (see [8]) $f_j = f_j(k_{j+1}, \ldots, k_{M}, q_{j+1}, \ldots, q_{M})$ is independent of $k_1, \ldots, k_{j}, q_1, \ldots, q_{j}$.

Let $G_p$ be a subgroup of $G$ of index $p$. We consider only the subgroups generated by $T_1^{p_1} \cdots T_M^{p_M}$ with the powers $p_i$ being chosen so that $T_1^{p_1} \cdots T_M^{p_M}$ never generate the same subgroup when $p_i < p_i$ for at least one $i$. Consider the space $G_p \setminus G$ of left $G_p$-cosets. Let $\delta_p \subset G$ be a ‘fundamental domain’ (section) for this homogeneous space which contains the identity of $G$. This ‘fundamental domain’ $\delta_p$ can be chosen to be a rectangle and can be written explicitly in the form \{ $T_M^{q_M} \cdots T_1^{q_1} : 0 \leq q_1 < p_1, \ldots, 0 \leq q_M < p_M$ $\}$. In this case the index of $G_p$ is $p = p_1 \cdots p_M$. One can also verify that $\delta_p$ is also a section for the space $G/G_p$ of right $G_p$-cosets. As before, given a partition $\alpha$, we denote by $\alpha^p$ the partition $\alpha^p = \bigvee_{g \in \delta_p} g\alpha$.

As was noted above, given any finitely-generated, torsion-free, nilpotent group $G$, its Maltsev basis generates a left-invariant order relation on $G$. So, if $(X, \mu)$ is a $G$-space with invariant probability $\mu$, and $\alpha$ a partition of $X$ with finite entropy, we can apply the results of [13, 17] as follows. Define the mean entropy $h(\alpha, G)$ as in §2. It follows from
the results of [13, 17] that \(h(\alpha, G)\) can be written in terms of the ‘past’ in \(G\) with respect to the left-invariant order relation:

\[ h(\alpha, G) = H(\alpha|a_G), \]

with \(a_G = \bigvee_{g \in G} g a\). An obvious modification of these definitions leads to the notion of the relative entropy \(h(\alpha,G,\sigma)\) for \(\sigma\) a \(G\)-invariant measurable partition.

Turn now to extending the Pinsker formula (2.6) onto general finitely-generated, nilpotent groups, to be used in describing Pinsker algebras.

**Theorem 4.1.** If \(\alpha, \beta \in L\) and \(G\) is a finitely-generated nilpotent group, then

\[ h(\alpha \vee \beta, G) = h(\alpha, G) + H(\beta|\beta \vee \alpha G). \]  

(4.2)

The reader is referred to [13, 17] for a definition of the ordered group \(G\). This definition involves the notion of ‘past’, a collection of the elements of \(G\) which are less than the identity.

**Proposition 4.2.** Suppose that for an amenable group \(G\) with a ‘past’ (4.2) is valid for some specific choice of the ‘past’ in \(G\) (equivalently, a specific left-invariant order relation); then (4.2) is also valid for any other ‘past’ in \(G\).

**Proof.** Note that two of the three terms which constitute (4.2), \(h(\alpha \vee \beta, G)\) and \(h(\alpha, G)\), are defined independently of the very notion of ‘past’. If one looks at the last term \(H(\beta|\beta \vee \alpha G)\), it is easy to observe that a relativized version of the results of [13, 17] allows one to treat it as a relative mean entropy \(h(\beta,G,\alpha G) = \lim_{n \to \infty} [\frac{1}{\#F_n} H(\beta F_n \mid \alpha G)]\) with respect to the \(G\)-invariant partition \(\alpha G\) (\(F_n\) is a right Følner sequence of sets). This makes clear its independence of the choice of ‘past’. □

Turn to the proof of (4.2) in the case of a general finitely-generated, torsion-free, nilpotent group \(G\). Observe that such a group admits an embedding \(G \hookrightarrow \text{UT}_n(\mathbb{Z})\) [8], so we may treat \(G\) as a subgroup of \(\text{UT}_n(\mathbb{Z})\).

Let \((X, \mu)\) be the space of the \(G\)-action. Denote by \(\sigma : G \setminus \text{UT}_n(\mathbb{Z}) \to \text{UT}_n(\mathbb{Z})\) a section for the (right) homogeneous space with respect to \(G\), which possesses the property \(\sigma([I]) = I\).

Form the product space \(Y = X^{G \setminus \text{UT}_n(\mathbb{Z})}\) with the associated product measure \(\nu\) and introduce an action of \(\text{UT}_n(\mathbb{Z})\) on \(Y\) by

\[ (gy)_y = \sigma(y)g \sigma(\gamma g)^{-1} y_{\gamma g}, \quad y = (y_{\gamma}) \in Y, \quad \gamma \in G \setminus \text{UT}_n(\mathbb{Z}), \quad g \in \text{UT}_n(\mathbb{Z}). \]

An easy verification shows that this action is well-defined (in particular, \(\sigma(y)g \sigma(\gamma g)^{-1} \in G\) and \((gy)|_{[I]} = g y|_{[I]}\) for \(g \in G\)).

Now, given finite entropy partitions \(\alpha\) and \(\beta\) on \(X\), form partitions \(\overline{\alpha}\) and \(\overline{\beta}\) on \(Y\). Specifically, we say that \(y' = (y'_{\gamma})\) and \(y'' = (y''_{\gamma})\) are in the same element of the partition \(\overline{\alpha}\) if \(y'|_{[I]}\) and \(y''|_{[I]}\) are in the same element of the partition \(\alpha\); the partition \(\overline{\beta}\) is defined in a similar way.

An application of Theorem 2.5 and Proposition 4.2 to \(\overline{\alpha}, \overline{\beta}\), and the \(\text{UT}_n(\mathbb{Z})\)-action on \(Y\) yields

\[ h(\overline{\alpha} \vee \overline{\beta}, \text{UT}_n(\mathbb{Z})) = h(\overline{\alpha}, \text{UT}_n(\mathbb{Z})) + H(\overline{\beta}|\overline{\beta} \vee \overline{\alpha})\text{UT}_n(\mathbb{Z}), \]
with the past in $UT_n(\mathbb{Z})$ being defined just as in §2. More precisely, one has

$$ H(\alpha \triangledown \beta) = \bigvee_{g \in G} g \alpha \triangledown g \beta = \bigvee_{g \in UT_n(\mathbb{Z}) \setminus G} g \alpha \triangledown \bigvee_{g \in G} g \beta $$

Observe that, by virtue of our construction, for any subsets $A, B \subseteq G$, $C, D \subseteq UT_n(\mathbb{Z}) \setminus G$, the partitions $\overline{\pi}_A \triangledown \overline{\pi}_B$ and $\overline{\pi}_C \triangledown \overline{\pi}_D$ are independent. Note also that the intersection of the ‘past’ in $UT_n(\mathbb{Z})$ with $G$ is a ‘past’ in $G$. So, if one takes into account Proposition 4.2, the above relation reduces to

$$ H(\alpha \triangledown \beta) = H(\alpha \triangledown \beta) $$

with the ‘past’ in $G$ being determined by some Maltsev basis. Since everything is now concentrated in a single direct multiple of the space $Y$ with index $[I]$, the latter relation can be rewritten in the form

$$ H(\alpha \triangledown \beta) = H(\alpha \triangledown \beta) $$

which is exactly our statement.

Finally, observe that any finitely-generated, nilpotent group $G$ contains a normal finite-index, torsion-free subgroup $G_0$ [8] (suppose its index is $p$ and $\alpha^p$ is the same as in the statement of Lemma 2.2). Application of the above result, Lemma 2.2, and its relativized version gives

$$ h(\alpha \triangledown \beta, G_0) = \frac{1}{p} h(\alpha^p \triangledown \beta^p, G_0) + \frac{1}{p} h(\beta^p \triangledown (\beta^p)_{G_0} \triangledown (\alpha^p)_{G_0}) $$

$$ = h(\alpha, G_0) + \frac{1}{p} h(\beta^p, G_0, \alpha_G) = h(\alpha, G) + h(\beta, G, \alpha_G) $$

$$ = h(\alpha, G) + H(\beta\alpha_G \triangledown \alpha_G). $$

**Lemma 4.3.** If $\alpha, \beta \in L$, $G$ is a finitely-generated, torsion-free, nilpotent group, and $T_1 < T_2 < \cdots < T_M$ is its Maltsev basis, then

$$ \lim_{n \to \infty} H(\alpha | T_M^{-n}(\beta_G) \alpha_G) = H(\alpha | \alpha_G). $$

**Proof.** This is the same as that of Lemma 2.7, with $T_1$ being replaced by $T_M$. 

Certainly, the arguments used in §2 to describe Pinsker algebras for actions of the Heisenberg group admit an obvious extension to actions of a general finitely-generated, torsion-free, nilpotent group with a fixed Maltsev basis $T_1, T_2, \ldots, T_M$. Note only that in Proposition 2.8 one should involve a sequence of finite index subgroups $\Gamma_p$ generated by
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where \( p \) is prime. With \( p \) larger than the nilpotence order of \( G \), each such subgroup \( \Gamma_p \) contains only \( p \)th powers of elements of \( G \), and \( \bigwedge \Gamma_p \) is the identity subgroup \([5]\).

As for a general finitely-generated, nilpotent group \( G \), recall that it contains a normal finite-index, torsion-free subgroup \( G_0 \) \([8]\). We claim that the Pinsker algebra of a \( G \)-action is the same as that of its restriction to \( G_0 \). In fact, suppose \( \alpha \) is a finite partition and \( h(\alpha, G) = 0 \). Then by Lemma 2.2

\[
h(\alpha, G_0) \leq h(\alpha^p, G_0) = ph(\alpha, G) = 0,
\]

where \( p \) is the index of \( G_0 \) in \( G \).

Conversely, let \( \delta_p \) be a section in \( G \) for the right coset space \( G_0 \backslash G \) and \( F_n \) a right Følner sequence of sets in \( G_0 \). Suppose \( h(\alpha, G_0) = 0 \). Then it follows from the normality of \( G_0 \) in \( G \) that

\[
h(\alpha, G) = \frac{1}{p} h(\alpha^p, G_p) = \lim_{n \to \infty} \frac{1}{p\# F_n} H \left( \bigvee_{g \in \delta} \alpha_{F_n g} \right) \leq \frac{1}{p} \sum_{g \in \delta_p} \lim_{n \to \infty} \frac{1}{\# F_n} H(\alpha_{g^{-1} F_n g})
\]

\[
= \frac{1}{p} \sum_{g \in \delta_p} \lim_{n \to \infty} \frac{1}{\# F_n} H(\alpha_{g^{-1} F_n g}) = h(\alpha, G_0) = 0.
\]

5. Relation between entropy and spectral properties of nilpotent group actions

To simplify the argument, we restrict ourselves to the special case of the Heisenberg group \( G \). It will be evident that everything is extendable to a general case of a finitely-generated, torsion-free nilpotent group \( G \).

Remember that a representation of \( G \) is said to have a countable Lebesgue spectrum if it is equivalent to a countable direct sum of the regular representation of \( G \).

**Lemma 5.1.** \([14]\) Let \( \xi \) be a measurable partition such that \( T_1^{-1} \xi \not\leq \xi \) and \( (X/\xi, \mu_\xi) \) is non-atomic, then \( \dim(L^2(\xi) \oplus L^2(T_1^{-1} \xi)) = \infty \).

**Theorem 5.2.** If \( h(G) > 0 \) then \( G \) has a countable Lebesgue spectrum in \( L^2(X, \mu) \oplus L^2(\pi(G)) \).

**Proof.** Let \( \xi \) be a \( G \)-perfect partition. It follows from \( \xi_G = \epsilon \) and \( \bigwedge_n T_3^n \xi_{(T_1, T_2)} = \pi(G) \) that

\[
L^2(X, \mu) \oplus L^2(\pi(G)) = \bigoplus_{n=\infty}^{+\infty} L^2(T_3^n \xi_{(T_1, T_2)}) \oplus L^2(T_3^{-n} \xi_{(T_1, T_2)})
\]

\[
= \bigoplus_{n=\infty}^{+\infty} U_{T_3}^n (L^2(\xi_{(T_1, T_2)}) \oplus L^2(T_3^{-1} \xi_{(T_1, T_2)})).
\]

In a similar way, the relation \( \bigwedge_n T_2^n \xi_{T_1} = T_3^{-1} \xi_{(T_1, T_2)} \) implies

\[
L^2(\xi_{(T_1, T_2)}) \oplus L^2(T_3^{-1} \xi_{(T_1, T_2)}) = \bigoplus_{n=\infty}^{+\infty} L^2(T_2^n \xi_{T_1}) \oplus L^2(T_2^{-n} \xi_{T_1})
\]

\[
= \bigoplus_{n=\infty}^{+\infty} U_{T_2}^n (L^2(\xi_{T_1}) \oplus L^2(T_2^{-1} \xi_{T_1})).
\]
and, finally,
\[ L^2(\xi T_1) \ominus L^2(T_2^{-1} \xi T_1) = \bigoplus_{n=-\infty}^{+\infty} L^2(T^n_1 \xi) \ominus L^2(T^{n-1}_1 \xi) = \bigoplus_{n=-\infty}^{+\infty} U^n_{T_1}(L^2(\xi) \ominus L^2(T^{n-1}_1 \xi)). \]

Thus we obtain
\[ L^2(X, \mu) \ominus L^2(\pi(G)) = \bigoplus_{(k,m,n) \in \mathbb{Z}^3} U^k_{T_1} U^m_{T_2} U^n_{T_1}(L^2(\xi) \ominus L^2(T^{n-1}_1 \xi)). \]

We claim that \( T^{-1}_1 \xi \leq \xi \). In fact, if one assumes the contrary, i.e. \( T^{-1}_1 \xi = \xi \), one can deduce from the properties (ii), (iii), and (iv) of a perfect partition that \( \pi(G) = \varepsilon \), that is \( h(G) = 0 \), which contradicts our assumptions on the \( G \)-action. Thus \( T^{-1}_1 \xi \leq \xi \). Now the properties (i) and (ii) imply that the space \((X/\xi, \mu_\xi)\) is non-atomic, and so by Lemma 5.1 one has \( \dim L^2(\xi) \ominus L^2(T^{-1}_1 \xi)) = \infty \). Let \( f_1, f_2, \ldots \) be a basis in \( L^2(\xi) \ominus L^2(T^{-1}_1 \xi) \).

It follows from (5.1) that \( \int g f_i = U_g f_i, \ i \in \mathbb{N}, \ g \in G \), form an orthonormal basis in \( L^2(X, \mu) \ominus L^2(\pi(G)) \), and \( U_g f_{h,i} = f_{gh,i} \). This means that \( G \) has a countable Lebesgue spectrum in \( L^2(X, \mu) \ominus L^2(\pi(G)) \).

As a consequence of Theorems 3.10 and 5.2 we obtain the following.

**Corollary 5.3.** If \( G \) is a \( K \)-group then it has a countable Lebesgue spectrum.

**Corollary 5.4.** If \( G \) has a singular spectrum or a spectrum with finite multiplicity then \( h(G) = 0 \). In particular, every group \( G \) with discrete spectrum has zero entropy.

6. **Perfect partitions and \( K \)-systems: the general case**

Describe briefly how to extend the results of §3 onto actions of a finitely-generated, torsion-free, nilpotent group \( G \). The reader is referred to §4 for a notation. We restrict ourselves to those statements only which could not be reproduced literally.

A measurable partition \( \xi \) is said to be \( G \)-invariant if \( T^{-1}_1 \xi \leq \xi, \ldots, T^{-1}_M \xi T_1 \cdots T_{M-1} \xi \leq \xi \).

A measurable partition \( \xi \) is said to be strongly invariant iff \( T^{-1}_1 \xi \leq \xi, \wedge_{n=0}\infty T^{-n}_1 \xi = T^{-1}_2 \xi T_1, \ldots, \wedge_{n=0}\infty T^{-n}_M \xi T_1 \cdots T_{M-1} = T^{-1}_M \xi T_1 \cdots T_{M-1} \).

**Lemma 6.1.** (An analogue of Lemma 3.3) Suppose that a measurable partition \( \xi \) is strongly invariant, \( \alpha \in \mathbb{L}, \alpha \leq T^p_1 \xi T_2 \) for some integer \( p \), and \( h(\alpha, G) = 0 \). Then \( \alpha \leq \wedge_{n=0}\infty T^{-n}_M \xi T_1 \cdots T_{M-1} \).

**Lemma 6.2.** (An analogue of Lemma 3.5) For every \( G \)-exhaustive partition \( \xi \) one has \( \wedge_{n=0}\infty T^{-n}_M \xi T_1 \cdots T_{M-1} \geq \pi(G) \).

**Lemma 6.3.** (An analogue of Lemma 3.7) There exists a measurable partition \( \eta \) with the properties:

(a) \( T^{-1}_1 \eta \leq \eta, \ldots, T^{-1}_M \eta T_1 \cdots T_{M-1} \leq \eta \);
(b) \( \eta G = \varepsilon \);
(c) \( \wedge_{n=0}\infty T^{-n}_M \eta T_1 \cdots T_{M-1} \leq \pi(G) \);
(d) \( h(G) = H(\eta|\eta G) = H(\eta|T^{-1}_1 \eta) \).
Definition D. (An analogue of Definition B) A measurable partition $\xi$ is said to be $G$-perfect if:

(i) $T_1^{-1}\xi \leq \xi, \ldots, T_M^{-1}\xi_{T_1\cdots T_{M-1}} \leq \xi$;

(ii) $\xi_G = \varepsilon$;

(iii) $\bigwedge_{n=0}^{\infty} T_1^{-n}\xi = T_2^{-1}\xi_{T_1}, \ldots, \bigwedge_{n=0}^{\infty} T_M^{-n}\xi_{T_1\cdots T_{M-1}} = T_M^{-1}\xi_{T_1\cdots T_{M-1}}$;

(iv) $\bigwedge_{n=0}^{\infty} T_M^{-n}\xi_{T_1\cdots T_{M-1}} = \pi(G)$;

(v) $\mu(G) = H(\xi|\xi_G) = H(\xi|T_1^{-1}\xi)$.

Definition E. (An analogue of definition C) $G$ is said to be a K-group if there exists a measurable partition $\xi$ such that:

(i) $T_1^{-1}\xi \leq \xi, \ldots, T_M^{-1}\xi_{T_1\cdots T_{M-1}} \leq \xi$;

(ii) $\xi_G = \varepsilon$;

(iii) $\bigwedge_{n=0}^{\infty} T_1^{-n}\xi = T_2^{-1}\xi_{T_1}, \ldots, \bigwedge_{n=0}^{\infty} T_M^{-n}\xi_{T_1\cdots T_{M-1}} = T_M^{-1}\xi_{T_1\cdots T_{M-1}}$;

(iv) $\bigwedge_{n=0}^{\infty} T_M^{-n}\xi_{T_1\cdots T_{M-1}} = \nu$.

With the above background, the results of §5 concerning the spectral properties can literally be reformulated in the case of a general finitely-generated, torsion-free, nilpotent group.

7. Non-Bernoullian systems with completely positive entropy

Our intention here is to produce an action of a countable, torsion-free, nilpotent group $G$ which is c.p.e. but not Bernoullian. It is based on an example of a single automorphism $Q$ with these properties, which can be found in [11]. We demonstrate here a generalization of the construction used in [3] to produce such actions for countable, torsion-free, Abelian groups. This requires some new ideas related to ordered groups, to be expounded below.

Recall [13, 17] that a countable amenable group is said to be ordered if it admits a left invariant linear order relation. This property is equivalent to the existence of a ‘past’ (a subset of elements that are less than the identity which possess some special properties).

Now assume $A$ to be a countable, torsion-free, Abelian group. We observe that it could be made ordered. In fact, such an $A$ is embedable into a direct sum of a countable collection of copies of the group $\mathbb{Q}$ of rationals [8]. Clearly, an invariant order relation on a group induces an invariant order relation on its subgroup. Hence, to get an invariant order relation, we may assume $A = \bigoplus_{i=1}^{\infty} \mathbb{Q}$. On this group an invariant order relation is defined lexicographically: $[r_i] < [s_i]$ if for $n = \max\{j : r_j \neq s_j\}$ one has $r_n < s_n$.

Let $G$ be a finitely-generated, torsion-free, nilpotent group. As in §4, we fix the upper central series of normal subgroups

$$G = Z_N \supset Z_{N-1} \supset Z_{N-2} \supset \cdots \supset Z_2 \supset Z_1 \supset Z_0 = \{e\}. \quad (7.1)$$

We start by observing that the property of being torsion-free is inherited by the quotient groups in (7.1).

LEMMA 7.1. For each $i = 1, \ldots, N$, $n > 0$, $x \in Z_i$, and $x^n \in Z_{i-1}$ implies $x \in Z_{i-1}$.

Proof. We proceed by induction in $i$. In the case $i = 1$ our statement reduces to observing that $Z_0 = \{e\}$ and $Z_1(\subset G)$ is torsion-free.

Now suppose that our proposition is valid for $i - 1$, and prove it for $i$. That is, we assume $x \in Z_i$ and $x^n \in Z_{i-1}$. For the sake of simplicity, we stick to the
case $n = 2$; the general case can be treated in a similar way. Suppose the contrary: $x \notin Z_{i-1}$. Since $Z_{i-1}/Z_{i-2} = \text{center}(G/Z_{i-2})$, we deduce that there is some $y \in G$ with $[x, y] \notin Z_{i-2}$. On the other hand, $[x, y] \in Z_{i-1}$ since $Z_i/Z_{i-1} = \text{center}(G/Z_{i-1})$.

In a similar way one can verify that $[x^2, y] \in Z_{i-2}$. Since also

$$[x^2, y] = xxyx^{-1}x^{-1}y^{-1} = x[x, y]yx^{-1}y^{-1} \in [x, y]^2Z_{i-2},$$

we deduce that $[x, y]^2 \in Z_{i-2}$. Now the induction hypothesis implies $[x, y] \in Z_{i-2}$, which contradicts our choice of $y$. This proves our statement. 

$\square$

**Proposition 7.2.** A countable, torsion-free, nilpotent group admits a left-invariant order relation.

**Proof.** Fix some invariant linear order relation on each (Abelian) quotient group $Z_i/Z_{i-1}$, $i = 1, \ldots, n$. Let $p_i : Z_i \to Z_i/Z_{i-1}$ be natural projections, and $s_i : Z_i/Z_{i-1} \to Z_i$ sections with $p_is_i = \text{id}$, $s_i([Z_{i-1}]) = e$. Now each $g \in G$ can be written uniquely in the form $g = sNg_{N-1} \ldots g_i$, with $g_i \in Z_i$ being defined inductively as $g_N = s_N(p_N(g))$, $g_i = s_i(p_i(g_{i+1} \ldots g_N^{-1}g))$. A left-invariant order relation on $G$ is defined as follows: given two elements $g = sNg_{N-1} \ldots g_1$ and $h = hNh_{N-1} \ldots h_1$ of $G$, we say $g < h$ if for $n = \max\{i : g_i \neq h_i\}$ one has $p_n(g_n) < p_n(h_n)$. 

$\square$

**Theorem 7.3.** Let $G$ be a countable, torsion-free, nilpotent group. There exists a c.p.e. non-Bernoullian $G$-space.

**Proof.** We proceed by induction in the nilpotence order $N$. For $N = 1$ ($G$ is Abelian) this has been proved in [3]. Assume that our statement is true for all $(N - 1)$-step nilpotent groups, and consider the $N$-step nilpotent group $G$, together with its upper central series (7.1). Let us consider first the special case in which the quotient group $G/Z_{N-1}$ is finitely generated, and hence is isomorphic to $\mathbb{Z}^n$ (no torsion subgroup by Lemma 7.1). Let $(X, \mu)$ be a c.p.e. non-Bernoullian $Z_{N-1}$-space.

Denote by $\pi : G \to G/Z_{N-1}$ the natural projection and by $s : G/Z_{N-1} \to G$ a section, which possesses the property $s(Z_{N-1}) = e$.

Form the product space $Y = X^{G/Z_{N-1}}$ with the associated product measure $\nu$ and introduce an action of $G$ on $Y$ by

$$(y^g)_\nu = s(\nu)^gs(\nu + \pi(g))^{-1}y_{\nu + \pi(g)}, \quad y \in Y, \quad \gamma \in G/Z_{N-1} \cong \mathbb{Z}^n, \quad g \in G,$$

with the given action of $Z_{N-1}$ in each direct multiple of $Y$. An easy verification shows that this action is well-defined (in particular, $s(\nu)^gs(\nu + \pi(g)) \in Z_{N-1}$). To see that this action is non-Bernoullian, we need the following simple proposition, which is, regardless of our context, valid for any countable amenable group $G$ and its subgroup $H$.

**Proposition 7.4.** [3] For a Bernoullian action of $G$, the restriction of this action to a subgroup $H$ is also Bernoullian.

Turn back to our construction. It follows from the definition of our $G$-action that the restriction of it to $Z_{N-1}$ splits into a direct product of actions in direct multiples of $Y$, with each of those being automorphically conjugate to the original $Z_{N-1}$-action on $X$. Hence it
possesses that original $Z_{N-1}$-action as a factor, and thus is non-Bernoullian [12]. Now an application of Proposition 7.4 shows that the entire $G$-action is non-Bernoullian.

Prove that the above action of $G$ has c.p.e. It was mentioned that the restriction of the $G$-action on $Y$ to $Z_{N-1}$ splits into the direct product of $Z_{N-1}$-actions on direct multiples of $Y$. Since each of those has c.p.e., the $Z_{N-1}$-action on $Y$ has c.p.e. [4]. That is, given a finite partition $\xi$ on $Y$, the mean entropy $h(\xi, Z_{N-1})$ is positive. Choose a finite subset $Q \subset G/Z_{N-1}$ such that for some finite partition $\pi$ on $XQ$ and the corresponding partition $\eta$ on $Y$ one has $d(\xi, \eta) = H(\xi|\eta) + H(\eta|\xi) < \frac{1}{2}h(\xi, Z_{N-1})$. Consider a rectangle $Q$ centered at $0$ in $G/Z_{N-1} \cong \mathbb{Z}^n$ which contains $Q$. Clearly $Q$ works as a fundamental domain for a finite-index subgroup $D$ in $G/Z_{N-1}$. Note that $D$ is $Q$-spread, and $\pi^{-1}(D)$ has a finite index in $G$. Recall that, since $\pi^{-1}(D)$ is an ordered group by Proposition 7.2 and [13, 17], $h(\eta, \pi^{-1}(D)) = H(\eta|\eta^{-1}(D))$, with $\eta^{-1}(D) = \bigvee_{g \in G} g\eta$. The specific form of the order relation on $\pi^{-1}(D)$ implies that

$$\eta^{-1}(D) = \bigvee_{g \in G} g\eta \vee \bigvee_{\gamma \in D} s(\gamma)\eta Z_{N-1}. $$

It also follows from our construction and the fact that $D$ is $Q$-spread that $\eta$ is independent of $\bigvee_{\gamma \in D} s(\gamma)\eta Z_{N-1}$, and hence

$$h(\eta, \pi^{-1}(D)) = H(\eta|\eta^{-1}(D)) = H\left(\eta \bigvee_{g \in G} g\eta \vee \bigvee_{\gamma \in D} s(\gamma)\eta Z_{N-1}\right) = H\left(\eta \bigvee_{g \in G} g\eta\right) = h(\eta, Z_{N-1}).$$

This implies

$$h(\xi, \pi^{-1}(D)) > h(\eta, \pi^{-1}(D)) - d(\xi, \eta) > h(\xi, Z_{N-1}) - d(\xi, \eta) > \frac{1}{2}h(\xi, Z_{N-1}) > 0.$$ 

Hence, for $S \subset G$ being a (finite) subset which meets each $\pi^{-1}(D)$-coset exactly once, one has by Lemma 2.2 that

$$h(\xi, G) = \frac{1}{\#(G/\pi^{-1}(D))}h(\xi_S, \pi^{-1}(D)) \geq \frac{1}{\#(G/\pi^{-1}(D))}h(\xi, \pi^{-1}(D)) > 0.$$ 

In the general case, represent $G$ as a union of an increasing sequence of subgroups $G = \bigcup_{i=0}^{\infty} G_i$, with each $G_i$ containing $Z_{N-1}$ and $G_i/Z_{N-1}$ being finitely generated. Let $(Y, v)$ be a c.p.e. non-Bernoullian action of $G_0$ as above. Form the $G$-space $M = Y^{G/G_0}$ exactly as was done above. One can use the same argument to demonstrate that this $G$-space is non-Bernoullian.

Observe that if one sets $M_n = X^{G_n/G_0}$, then for each $n \in \mathbb{N}$, $M = M_n^{G/G_n}$ splits into the direct product of $G_n$-spaces. Since $G_n$ is normal and finitely generated, and the $G_n$-space $M_n$ has the same structure as in the above argument, one can demonstrate that $G_n$-action has c.p.e. on $M_n$ and hence on $M$ [4]. Now one can use the same argument as in [3, proof of Corollary 7] to deduce that the $G$-action on $M$ has c.p.e. □
The above construction for non-Bernoullian actions with c.p.e. seems to possess the required properties in a more general context than that of Theorem 7.3. In particular, one has the following.

**THEOREM 7.5.** Let $G$ be a countable, solvable, torsion-free group whose commutant $[G, G]$ is nilpotent. Then $G$ admits a non-Bernoullian action with c.p.e.

**Proof.** It is easy to verify that $G$ appears to be an ordered group. One has to use the same construction as in the proof of Theorem 7.3, with $\mathbb{Z}_{N-1}$ being replaced with $[G, G]$, whose action with the required properties exists due to Theorem 7.3. Of course, the proof of the fact that the action is non-Bernoullian is the same as above.

Also, starting with the special case of $G/[G, G]$ being finitely generated, one has to observe that the only difference from the above situation is that $G/[G, G]$ is isomorphic to $\mathbb{Z}^n \times F$, with $F$ being a finite group. So, one can prove in the same way as above that the finite-index, normal ordered subgroup of $G$, corresponding to the direct multiple $\mathbb{Z}^n$ of the quotient group $G/[G, G]$, has c.p.e. After that, an easy argument based on Lemma 2.2, extends this property to the entire group $G$.

Finally, the same argument as in the proof of Theorem 7.3 works to get rid of the assumption that $G/[G, G]$ is finitely generated. □

**COROLLARY 7.6.** Any countable, solvable, torsion-free subgroup of $GL(n, \mathbb{R})$ admits a non-Bernoullian action with c.p.e.

**Proof.** It follows from the Kolchin–Maltsev Theorem [8] that such a group contains a normal finite-index subgroup whose commutant is nilpotent, and so Theorem 7.5 is applicable. □

**Acknowledgement.** We would like to thank the referee for helpful remarks.

**REFERENCES**


On the entropy theory of finitely-generated nilpotent group actions


