PHYSICALISM IN MATHEMATICS

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NOMINALISM, REALISM & PHYSICALISM IN MATHEMATICS:
An Introduction to the Issues†

1. A Recent Renaissance in Philosophy of Mathematics

The past decade and a half has witnessed a renaissance in the philosophy of mathematics not seen since the days of Hilbert, Russell and Brouwer in the early part of this century. Much of this exciting new work has arisen in response to the kind of concerns made prominent by Paul Benacerraf in his influential 1973 article, "Mathematical Truth". In the article Benacerraf discusses the difficulty of developing an acceptable semantics for mathematics. He emphasizes that traditionally there has been a difficulty in providing a semantics which is capable of being integrated within both a comprehensive theory of truth and a scientific epistemology. As he points out,

two quite distinct kinds of concerns have separately motivated accounts of the nature of mathematical truth: (1) the concern for having a homogeneous

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† See Benacerraf [1973]. This article has motivated much of the contemporary discussion in much the same way that Benacerraf [1965] set the tone of debate a decade earlier. Just as Quine [1968], Field [1974], White [1974], Kitcher [1978], Maddy [1981], Hazen [1983] and others all dealt with themes introduced in Benacerraf's [1965], recent work such as Steiner [1973] and [1975], Kim [1981] and [1982], Grandy [1977], Maddy [1980] and [1984], Resnik [1981] and [1982], Field [1980] and others can all be interpreted as responses to Benacerraf's [1973].

semantical theory in which semantics for the propositions of mathematics parallel
the semantics for the rest of the language, and (2) the concern that the account of
mathematical truth mesh with a reasonable epistemology. ... almost all accounts
of the concept of mathematical truth can be identified with serving one or another
of these masters at the expense of the other.\footnote{Benacerraf [1973], p. 661; reprinted in Benacerraf & Putnam [1983], p. 403.}

For example, Benacerraf points out that given the prominence of
causal theories of reference and knowledge, it is not at all clear how non-
physical mathematical objects (if they exist) could ever be known. It is one
thing to postulate the existence of such entities; it is something else again to
be able to show that we have knowledge of them. Benacerraf puts it as
follows:

If, for example, numbers are the kind of entities they are normally taken to be,
then the connection between the truth conditions for the statements of number
theory and any relevant events connected with people who are supposed to have
mathematical knowledge cannot be made out. It will be impossible to account for
how anyone knows any properly number-theoretical propositions.\footnote{Benacerraf [1973], p. 673; reprinted in Benacerraf & Putnam [1983], p. 414. Also see
Benacerraf & Putnam [1983], pp. 30ff.}

Mark Steiner makes much the same point. Steiner argues that mathematical
entities, as traditionally conceived, will be outside of the causal nexus.
Hence, such entities (if they exist) will be necessarily imperceptible and so
unknowable. In Steiner's words,

The objection is that, if mathematical entities really exist, they are unknowable—
accepting mathematical truths are unknowable. There cannot be a science treating of
objects that make no causal impression on daily affairs. ... Since numbers, et al.
are outside all causal chains, outside time and space, they are inscrutable. Thus
the mathematician faces a dilemma: either his axioms are not true (supposing
mathematical entities not to exist), or they are unknowable.\footnote{Steiner [1973], p. 58.}

Arguments such as these are motivated in large measure by the desire
to integrate mathematics into a physicalistically acceptable world view. The
desire is therefore, on the one hand, to develop a naturalized epistemology
for mathematics and, on the other hand, to eliminate any ontological
commitments not recognized or required by the natural sciences. The result
has been not only a flurry of new work in the philosophy of mathematics, but a re-drawing of the once traditional boundaries within the discipline.\footnote{In one sense, though, this debate about the difficulty of reconciling semantics with epistemology in the case of mathematics is not new. Because of its ostensibly \textit{a priori} and necessary nature, mathematics has often been seen as a major stumbling block for empiricist based epistemologies. Thus, within the twentieth century especially, the empiricist tradition has emphasized mathematics as being of special significance and interest. Two quotations from earlier in the century serve to emphasize this point. The first is from Hahn [1930]:

\begin{quote}
Indeed, the understanding of logic and mathematics has always been the main crux of empiricism; for any general proposition that has its origin in experience will always carry with it an element of uncertainty, whereas in the propositions of logic and mathematics we find no such uncertainty. \textit{... Only the elucidation of the place of logic and mathematics \ldots [can make] a consistent empiricism possible.} (p. 21)
\end{quote}

The second is from Ayer [1936]:

\begin{quote}
Where the empiricist does encounter difficulty is in connection with the truths of formal logic and mathematics. \ldots if empiricism is correct no proposition which has a factual content can be necessary or certain. Accordingly, the empiricist must deal with the truths of logic and mathematics in one of the two following ways: he must say either that they are not necessary truths, in which case he must account for the universal conviction that they are; or he must say that they have no factual content, and then he must explain how a proposition which is empty of all factual content can be true and useful and surprising. (pp. 72f)
\end{quote}

As with the contemporary debate, these quotations emphasize a dilemma regarding epistemological foundations: either empiricism provides a satisfactory account of the nature of mathematical knowledge but, in doing so, fails to reconcile itself to a natural account of mathematical truth; or it accepts a natural account of mathematical truth but does so only at the expense of empiricist scruples. Thus, although the exact statement of the dilemma has changed over the past sixty years, in its essentials it still remains a question of reconciling a satisfactory epistemology with an adequate semantics; a question of proof versus truth.}

Benacerraf's article was followed closely by Steiner's 1975 \textit{Mathematical Knowledge} and then by articles by Penelope Maddy\footnote{See Maddy [1980] and [1984].} and Michael Resnik.\footnote{See Resnik [1981] and [1982].} All three of these authors emphasized the first of Benacerraf's two \textit{desiderata}, namely that of a "homogeneous semantical theory". As a result, they all champion one form or another of platonism. Having accepted the benefits of a unified semantics, they then attempt to develop a scientifically acceptable account of mathematical knowledge which is consistent with their semantics.
At the same time, several ambitious nominalization programmes began to appear. Of these, the most prominent has been that which Hartry Field initiated in 1980 with his now well known *Science Without Numbers.*\(^8\) Programmes such as Field's emphasized the second of Benacerraf's two desiderata, that of a non-fideistic epistemology. By the construction of nominalistic alternatives to the realist's interpretation of mathematics, authors such as Field in effect argued that a scientifically acceptable epistemology need not be abandoned with respect to mathematics. Instead, Field claims that mathematics can be interpreted instrumentally and so remains physically acceptable.

Since then, the literature has expanded rapidly. In 1980 Michael Resnik's *Frege and the Philosophy of Mathematics* appeared. Then, in 1983, Philip Kitcher's *The Nature of Mathematical Knowledge* and Crispin Wright's *Frege's Conception of Numbers* were both well received. These were followed by Michael Hallett's *Cantorian Set Theory and Limitation of Size* in 1984 and Michael Detlefsen's *Hilbert's Program* in 1986.


2. Traditional Difficulties with Nominalism

Arguments such as Benacerraf's and Steiner's, which provided the motivation for nominalist programmes such as Field's, have much to recommend them. They emphasize that any viable epistemology must not only be capable of accounting for mathematical knowledge, it must be consistent with our best theory of scientific knowledge as well. As a result of this emphasis, all such arguments require the acceptance of some form of naturalized epistemology. They also insist that any plausible account of knowledge should be based (at least initially) on what we understand as paradigm examples of knowledge, viz. cases in which we claim to know something just because it is possible to trace out a causal chain of the

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\(^8\) See Field [1980], [1982] and [1984]. Also see Daniel Bonevac's [1980] and Dale Gottlieb's [1980].
appropriate sort. As a result, the two immediate alternatives with regard to mathematics appear to be either to accept nominalism (and thereby avoid the traditional epistemological difficulties of the platonist), or to accept some form of realism (while at the same time having to defend an account of mathematical knowledge consistent with the involvement of mathematical entities within the causal nexus).

Of these two alternatives, it is the nominalist option which may first appear preferable. After all, mathematical entities have traditionally been viewed as existing outside of space and time and therefore as causally inert. Once this is recognized, nominalist accounts may take on a prima facie appeal. Nevertheless, projects such as Field's have been criticized both for their unintuitive consequences and for technical inadequacies. As an example of the latter it has often been pointed out that the second-order resources which Field allows himself are just too rich for the nominalistic purposes at hand. And even if they are allowed, it is not clear that all of mathematical physics can be captured in this type of reconstruction.9

As an example of the former one can note that if Field is correct in claiming that mathematical terms fail to refer, and hence that mathematical statements (i.e. statements which quantify over mathematical terms), although useful, are generally false, then sentences such as

(1) Russell knows that $3^3 = 27$,

although generally thought to be true, will be false. Similarly, sentences such as

(2) There does not exist a prime greater than 100,

although generally thought to be false, will be true. In the case of (1), if it is true that Russell knows that $3^3 = 27$, it will follow from the entailment thesis (viz. that knowledge entails truth) that "$3^3 = 27$" is true, which Field denies. In the case of (2), it will turn out that "There does not exist a prime greater than 100" is true since, on Field's account, there do not exist any numbers whatsoever.10

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10 I am indebted to John Bacon for these particular examples.
Despite this, even if these two types of objection were met, it is not clear that mathematical nominalism can escape the well-known objections made concerning traditional nominalism, let alone the demands of Benacerraf's dilemma. Traditional objections to nominalism with respect to universals appear to be equally effective against contemporary nominalistic programmes in mathematics.

As an example, consider the sentence

\[(3) \text{ The number of F's is at least 3.}\]

Such a sentence is regularly translated into first-order logic without difficulty. For example, as Bigelow points out,\(^{11}\) one such translation might be

\[(4) \forall x \forall y \exists z (x \neq z \& y \neq z \& Fz).\]

What is striking about this and other similar translations is that (4), unlike (3), does not refer either to a specific number, such as 3, or even to numbers in general. Thus, we are led to notice an analogy between nominalism in mathematics and traditional nominalism with respect to universals.\(^{12}\) That is, we are led to notice the similarity between (3) and (4) and two additional sentences

\[(5) \text{ The colour of the book is red,}\(^{13}\)

and

\[(6) \text{ The book is red.}\]

Like (4), sentence (6) is often taken to be a translation which is especially perspicacious in that, unlike (5), it disavows reference to anything over and above individuals. Both (4) and (6) therefore go some distance towards justifying the belief that nothing over and above particulars need to be

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\(^{11}\) Bigelow [1988], p. 48. This sentence is equivalent to the more cumbersome sentence: \(\exists x \forall y \exists z (x \neq y \& x \neq z \& y \neq z \& Fx \& Fy \& Fz)\).

\(^{12}\) Of course, nothing special hangs on the requirement that the properties and relations in question be universals. Sicutian abstract particulars or D. C. Williams's tropes (i.e., properties and relations which themselves may be considered as particulars) would remain "abstract" enough to require elimination according to most nominalists and so would do just as well for current purposes. (For an interesting discussion of abstract particulars and tropes, see Armstrong [1978], Ch. 8.)

\(^{13}\) Or better yet, (5') Red is the colour of the book.
postulated as existing. If, in Quine's famous phrase, to be is to be the value of a variable, (4) and (6) become ontologically pure in a way in which (3) and (5) are not.

Should nominalists such as Field find comfort in this analogy between (3) and its more perspicacious translation (4), and (5) and its more perspicacious translation (6)? Do paraphrases such as these, if carried out in a systematic and comprehensive fashion, carry the day for either traditional or mathematical nominalism?

To answer this question, we need only develop the analogy a little further in order to see that the same well-known difficulties which apply to traditional nominalism may also apply to mathematical nominalism. Consider, for example, sentences such as

(7) The number of planets is equal to the number of orifices in the human head,

or

(8) There are three natural numbers greater than one but less than five,

or

(9) Five is a number.

How might the nominalist reconstruct these sentences? Immediate difficulties bring to mind, not the analogy between (3) and (5), and between their nominalistically acceptable translations (4) and (6) but, rather, an analogy between (7) and, say,

(10) Red resembles orange more than it resembles blue,

or an analogy between (9) and, say,

(11) Red is a colour.

The difficulties in construing (10) and (11) nominalistically are well known and, at least to date, unresolvable.14 And given our analogy, such difficulties do not bode well for nominalism with respect to (7) and (9). For example, the sentence

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14 See Pap [1959], Jackson [1977], and Armstrong [1978], pp. 58ff from which these examples are taken.
(12) For any particulars, $x$, $y$ and $z$, if $x$ is red and $y$ is orange and $z$ is blue, then $x$ resembles $y$ more than it resembles $z$,

which refers only to individuals (and not to properties) will not do as an equivalent for (10) since (12) is simply not true, even though (10) is. (It is always possible, and indeed likely, that some particular $x$ will resemble $z$ more than it resembles $y$, but on grounds other than colour.)

Similarly, one might at first think that (11) could be taken as equivalent to the nominalistically acceptable

(13) For any particular, $x$, if $x$ is red then $x$ is coloured.

But if (11) is equivalent to (13) then surely

(14) For any particular, $x$, if $x$ is red then $x$ is extended,

would be equivalent to

(15) Red is an extension.

But since (14), which is true, does not even imply (15), which is false, the two can hardly be equivalent. So (11) cannot be equivalent to (13).

Thus, even though (4) may well be a nominalistically acceptable translation of (3), just as (6) may be a nominalistically acceptable translation of (5), if our analogy holds, sentences such as (7), (8) and (9) will be at least as difficult to nominalize as sentences such as (10) and (11). After all, (7), (8) and (9), like (10) and (11), do not appear to be easily reducible to talk of particulars. For example, one might try to nominalize (9) by equating it with

(16) For any collection of particulars, $X$, if the members of $X$ are five then the members of $X$ are numbered.

But if (9) is equivalent to (16) then surely

(17) For any collection of particulars, $X$, if the members of $X$ are five then the members of $X$ are distinct,
would be equivalent to

(18) Five is distinctness.

But since (17), which is true, does not even imply (18), which is false, the two can hardly be equivalent. So (9) cannot be equivalent to (16).

The moral here is that mathematical sentences in general appear to be no more easily nominalized than traditional ones and, therefore, they may not be capable of nominalization at all.

Nevertheless, even if a programme such as Field's did prove possible in the sense that it could provide a nominalistically acceptable translation for every such sentence, it might still be claimed with some justification that this will not decide the issue. After all, as Bigelow points out, a sentence such as

(19) John is a (full) sibling of Mark,

might well be thought to be an equivalent to

(20) There are two distinct people, each being a parent of John, and each being a parent of Mark,

which did away with reference to parents. Yet no one should think of (19) as being an adequate ground for disbelief in parents. Elaborate nominalization strategies must be viewed as only a necessary condition for the invocation of Ockham's Razor, not a sufficient one.

3. The Realist Alternative

Realism, when juxtaposed to a standard referential semantics, has a number of prima facie advantages over nominalism. Not the least of these is that the realist's account of mathematical truth is consistent with the pre-philosophical, intuitive semantical views of many working mathematicians.

However, for anyone wishing to affirm the existence of mathematical entities, the difficulty of reconciling such entities with a satisfactory episte-

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15 Bigelow [1988], p. 51. Also see Papineau [1989].
mology must be resolved. Once a referential account of mathematical truth is introduced, the following question immediately arises: how can knowledge of the facts which make statements of mathematics true ever be obtained? The realist is faced with the problem of relating a feasible interpretation of the referential apparatus of mathematical theories to an explanation of how a minimum number of basic, pre-formal, non-inferential statements of mathematics can be known.

In discussing this issue, Steiner raises the distinction between ontological and epistemological platonism. He defines ontological platonism as the view that "the truths of mathematics describe infinitely many real mathematical objects". Moreover, it is often argued that these objects, since necessarily infinite in number, must be distinct from those of the physical world. Statements of mathematics, in turn, are true if and only if they correctly describe these objects. In Steiner's words: "It follows, then, that whether (ontological) platonism is tenable is the same question as whether the axioms of mathematics are true. This conclusion, of course, puts ontological platonism in a very favorable light".

In contrast, epistemological platonism is "the doctrine that we come to know facts about mathematical entities through a faculty akin to sense perception, or at least that some people do". It is here that difficulties arise for the platonist. If ontological platonism is correct, and mathematical entities exist which are distinct from those of the physical world, it would seem that epistemological platonism can have little chance of being true. It is unlikely, to say the least, that perceptions of any kind could result in the observation of objects that play no causal role within the natural world of space and time. Physicalism, together with the demands of a naturalized epistemology, raises immediate concerns for the platonist.

This seemingly basic incompatibility between ontological and epistemological platonism appears to depend (at least in part) upon one form or another of the causal theory of knowledge. That is, it depends (in part) upon the claim that it is a necessary condition for knowledge to exist that there be at least some causal connection between an epistemic agent and whatever it is that makes the belief true. If there can be in principle no causal connection between the referents of the names, predicates and

17 Steiner [1973], p. 57.
18 Steiner [1973], p. 57.
quantifiers of a statement and the epistemic agent, it is hard to see how any belief of the agent concerning such a statement could ever be justified. As Maddy points out, in the physical sciences we refer to things and kinds of things by virtue of standing at the end of a complex causal chain of usage back to dubbing. But, it seems that no sample of the kind "set" could be vulnerable to such an initial baptism. So, the argument runs, not only are we unable to know facts about sets, but we are also unable to refer to them, so in fact, our theory of sets cannot be about them.\footnote{Maddy [1980], p. 165.}

Thus, at the heart of the recent debate about the tenability of platonism, has been the question of what form of access the mathematician has to a postulated body of independently existing mathematical objects. For the platonist, the question of how non-physical, non- causal platonistic entities relate to experience must be answered. For the platonist, the difficulty is therefore one of attempting to reconcile ontological platonism with a contemporary scientific theory of mathematical perception and reference.

More explicitly, ontological and epistemological platonism can be understood to involve the following six points:

(i) Mathematical entities exist independently of human thought and of our ability or inability to obtain knowledge of them;
(ii) Such entities are non-physical, existing outside space and time;
(iii) Statements of mathematics possess truth-values, again independent of human thought and of our ability or inability to obtain knowledge of them;
(iv) Such statements obtain their truth-values as a result of properties of mathematical entities (and not as a result of, e.g. properties of natural or formal languages, etc.);
(v) It is possible to refer unequivocally to such entities;\footnote{Thus, the platonist argues that reference to mathematical entities is possible even though theoretical description alone fails to pin down or identify such entities uniquely. For example, the platonist holds that it is possible to refer to the \( \omega \)-sequence despite the fact that contemporary limitative metatheorems, such as those of Gödel and Löwenheim, show that for any first-order theory purporting to describe such a sequence there will always exist non-standard models. Beth, among others, considers this \textit{prima facie} theoretical disadvantage to constitute evidence in \textit{favour} of platonism. He argues that, since limitative metatheorems show that theoretical conditions alone fail to isolate a theory’s standard interpretation, extra-theoretical (that is, platonistic) considerations need}
Having isolated these six points, one can now distinguish between mathematical platonism and mathematical realism: realism encompasses both transcendental mathematical realism (i.e. platonism) and immanent mathematical realism (any realism acceptable to a physicalist). More specifically, realism can be identified narrowly with just (i) above or, more broadly, with (i), (iii), (iv), (v) and (vi). Immanent mathematical realism can then be thought of as realism together with the additional claim that mathematical entities are to be naturalistically construed. It is on this point alone that the immanent realist differs from the mathematical platonist. As with platonism, both realism and immanent realism will have their respective ontological and epistemological components.

For the platonist, having accepted (i) through (iv), the difficulty comes in defending (v) and (vi). Three general types of response immediately come to mind. The first is to deny the causal theory of knowledge and to accept that somehow knowledge can be gained of causally inactive entities. This is the approach advocated in different forms by traditional platonists and, more recently, by Steiner in in his [1973] and [1975]. It is also the option taken in this volume by Resnik [22] and Brown.

The second type of response is to accept some version of the causal theory of knowledge [23] but to deny that mathematical entities are causally inert. This is the approach championed most forcefully by Maddy in her

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21 In addition to (i) - (vi), platonism often involves tenets regarding the nature of the mathematical entities it postulates. For example, there is the question of whether such entities are particulars or universals. There is also the question of whether, because of their postulated nonphysical nature, mathematical entities are causally inert. Although here the purpose of outlining the essential features of platonism is not primarily historical, it is convenient to follow Plato and the traditionalists in emphasizing the essential nonphysical aspect of the platonist's ontology at the expense of other such considerations. For example, since there is room for genuine debate over whether Plato identified mathematical entities with the Forms or with some other level of existence intermediate between the Forms and sensible objects (e.g. see Wedberg [1955]), it will be left open whether the platonist need identify mathematical entities with universals.

22 Also compare Resnik [1982].

23 Or to accept some other theory of knowledge, such as reliabilism, that (in part) relies upon a causal theory of perception or reference.
[1980] and in this volume. It is by abandoning platonism in favour of one form or another of immanent realism that Benacerraf’s epistemological challenge is met.

The third type of response is to alter in essential respects the terms of the above debate, for example either by introducing the kind of modal-structuralism advocated in this volume by Hellman or by defending, as does Gauthier, the kind of constructivist views which may quite naturally (at least in certain respects) appeal to the physicalist.

4. The Current Debate

This volume begins with several papers especially pertinent to the kind of epistemological issues canvassed by authors such as Benacerraf and Steiner. John Burgess’s “Epistemology & Nominalism” begins by reconstructing Benacerraf’s original problem concerning causal theories of knowledge. Burgess goes on to examine the doctrine of epistemological nominalism—the claim that it cannot be known that there are numbers, even if they did exist—in detail. His conclusion is that epistemological nominalism has not yet been proven, but that the work of Field and others remains useful on other, more Quinean, grounds.

Peter Simons’s paper “What Is Abstraction & What Is It Good For?” then introduces and examines a second related theme: the role and relevance of abstraction within the realist/nominalist debate. Together, these two papers by Burgess and Simons provide an excellent introduction to the fundamental task at hand, viz. the task of relating a satisfactory epistemology to an acceptable ontology within a physically acceptable world view.

Following these two papers, Michael Resnik, Crispin Wright and Jim Brown all make contributions to the development of an epistemology compatible with platonism. Resnik, in his “Beliefs About Mathematical Objects”, advances several important features of an epistemological account compatible with his well known version of platonism. Then, in his “Field & Fregean Platonism”, Wright further develops an argument first introduced in his Frege’s Conception of Numbers as Objects. The argument is based upon the observation that our theories of singular terms and singular references are central to our resolution of the dispute between the mathematical nominalist and realist. Finally, in his aptly titled “π In The Sky”,
Brown argues that a satisfactory platonist epistemology can be developed, and that causal theories of knowledge should be rejected.

Turning to possible nominalistic solutions to Benacerraf's dilemma, Bob Hale's "Nominalism", Alasdair Urquhart's "The Logic of Physical Theory" and David Papineau's "Knowledge of Mathematical Objects" all consider and evaluate the most prominent of the current nominalistic programmes, namely that developed by Field. Hale and Papineau find the programme wanting philosophically—Hale because he wonders whether Field's particular version of nominalism is philosophically coherent; Papineau because he doubts, even if Field's programme is coherent, that it provides an adequate reason for showing that one ought to be a fictionalist about mathematics. Urquhart finds the programme wanting on technical grounds. Together, these three papers provide a persuasive critique of Field's programme.

The next several papers all consider possible alternatives to nominalization programmes such as Field's. Michael Hallett begins his "Physicalism, Reductionism & Hilbert" by first comparing Field's programme to that of Hilbert. Since Hilbert's "ideal" statements play much the same role as Field's conservative results, both programmes deny that mathematical terms (by and large) refer to the physical world. It appears that both Field and Hilbert attempt to replace semantic notions of truth and reference in mathematics with syntactic alternatives, viz. conservativeness and consistency. Nevertheless, Hallett concludes that such similarities are superficial. He then goes on to outline in detail the philosophical aspects of Hilbert's original programme and to provide a much needed discussion of Hilbert's often misunderstood views concerning mathematics.

Like Hallett, Penelope Maddy also begins her "Physicalistic Platonism" with a comparison, but this time it is a comparison between Field's nominalism and Maddy's own version of platonism. Capitalizing on what the two programmes share, she then outlines how a realistic account of mathematics can be integrated into a physically acceptable account of the world. John Bigelow continues this line of argument in his "Sets Are Universals", arguing that sets both can and should be construed realistically as physically instantiated universals.

Geoffrey Hellman, in his "Modal-Structural Mathematics", then provides a helpful outline of his modal-structural approach to mathematics, an approach which is in some respects an alternative to both the traditional realist and nominalist pictures. Hellman's sophisticated account of mathe-
matics has several advantages in that, if successful, it avoids restricting the mathematician to only actually existing entities, relying instead upon possible structures of the appropriate sort.

Finally, Yvon Gauthier in effect argues in his "Logical & Philosophical Foundations for Arithmetical Logic" that a constructivist approach to mathematics is preferable to any of the other common alternatives. The book then closes with Chandler Davis's "Criticisms of the Usual Rationale for Validity in Mathematics" in which Davis, a working mathematician, discusses the role of foundations generally.

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References


——— [1989] "Sets are Universals", this volume.


Papineau, David [1989] "Knowledge of Mathematical Objects", *this volume*.
——— [1989] "Beliefs About Mathematical Objects", *this volume*. 