UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS
MATH1131/1141 Mathematics 1A Algebra S1 2014
TEST 1 VERSION 3a

This sheet must be filled in and stapled to the front of your answers

Student's Family Name
Initials

Student Number

Tutorial Code
Tutor's Name

Mark

Note: The use of a calculator is NOT permitted in this test
Show all your working. All answers should be given in the appropriately SIMPLIFIED form.

QUESTIONS  (Time allowed: 25 minutes)

1. (2 marks)
   For the points $A(3,2,1)$ and $B(6,3,-2)$
   (i) Find a parametric vector equation for the line $AB$.
   (ii) Find Cartesian equations for the line $AB$.

2. (2 marks)
   Find a parametric vector equation for the plane in $\mathbb{R}^3$ with cartesian equation
   
   $7x_1 + 2x_2 - x_3 = 1$

   Hence give two non-parallel, non-zero vectors which are parallel to the plane.

3. (3 marks)
   For the points $A(1, 4, 1)$, $B(3, 5, -2)$ and $C(5, 1, 2)$,
   (i) Find $\cos(\angle BAC)$.
   (ii) Find $\text{proj}_{AC}(AB)$.

4. (3 marks)
   In the plane with a cartesian co-ordinate system, let $OACB$ be a parallelogram, with $O$ the origin and $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, where $\mathbf{a} \parallel \mathbf{b}$.
   (i) Write down (and label as such), parametric vector equations of the lines $OC$ and $AB$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
   (ii) Find the co-ordinates of the point $P$ of intersection of lines $OC$ and $AB$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
   (iii) Show that $|\overrightarrow{OP}| = |\overrightarrow{PC}|$ and $|\overrightarrow{PA}| = |\overrightarrow{PB}|$.

Please write your answers on lined A4 paper and staple to this cover sheet.
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QUESTIONS  (Time allowed: 25 minutes)

1. (3 marks)
A block of wood is subject to 3 forces: \( \mathbf{F}_1 = 3 \text{N due north, } \mathbf{F}_2 = 2 \text{N due East and } \mathbf{F}_3 = 5 \text{N due south-west.} \) (N = newtons, a unit of force).

Let \( \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \) be the resultant force on the block.

(i) On a scaled diagram draw the 3 forces \( \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3 \) and the resultant force \( \mathbf{F} \) which shows geometrically how the forces add.

(ii) Using co-ordinates, find simplified expressions for \( |\mathbf{F}| \) and \( \tan(\theta) \) where the resultant force is in the direction \( W \theta S \), i.e. \( \theta \) south of west: \( W \theta S \).

2. (3 marks)
Consider the line \( \ell \) and plane \( \Pi \) in \( \mathbb{R}^3 \) with cartesian equations:

\[
\ell: \quad \frac{x-2}{3} = \frac{y+1}{4} = \frac{z+3}{1}
\]
\[
\Pi: \quad 3x - 2y - 4z = 11.
\]

(i) Find a parametric equation of the line \( \ell \).

(ii) Find the co-ordinates of the point \( P \) where \( \ell \) meets \( \Pi \).

3. (4 marks)
For the points \( A(1, 2, 3), B(-2, 5, 3), C(2, 4, 5). \)

(i) Calculate \( \text{proj}_{\mathbf{AC}} \left( \mathbf{AB} \right) \).

(ii) Calculate the length of the altitude in \( \triangle ABC \) through \( A \) and perpendicular to \( BC \).

(iii) Find the co-ordinates \( \mathbf{m} \) of the point \( M \) on \( BC \) where \( AM \) is this altitude through \( A \) and perpendicular to \( BC \).

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By the coordinate system as pictured, the coordinates of the three forces are
\[
\begin{align*}
F &= (0, 3), \quad F_2 = (2, 0), \quad F_3 = (5 \cos 45^\circ, -5 \sin 45^\circ)
\end{align*}
\]
Hence, \[ F = (2 - 5 \frac{\sqrt{2}}{2}, \ 3 - 5 \frac{\sqrt{2}}{2}) \]

\[
|F| = \sqrt{(2 - 5 \frac{\sqrt{2}}{2})^2 + (3 - 5 \frac{\sqrt{2}}{2})^2}
\]
\[
= \sqrt{4 - 10\sqrt{2} + \frac{25}{2} + 9 - 15\sqrt{2} + \frac{25}{2}}
\]
\[
= \sqrt{38 - 20\sqrt{2}} \quad (\text{not simplified})
\]
\[
\tan \theta = \frac{3 - \frac{5\sqrt{2}}{2}}{2 - \frac{5\sqrt{2}}{2}} = \frac{3\sqrt{2} - 5}{2\sqrt{2} - 5} \quad \text{[1]}
\]

(i) \[ \lambda = \frac{x-2}{3} \]
\[ \Rightarrow \ x = \left( \frac{2 + 3\lambda}{1} \right) + \lambda \left( \frac{3}{1} \right) \quad \text{[1]}
\]

(ii) A point on \( l \) has coordinates \[ x = \left( \frac{2 + 3\lambda}{1} \right) \text{ for some } \lambda \in \mathbb{R} \]

If the point lies in \( \Pi \), then \( l \) satisfies the plane equation
\[ 3(2 + 3\lambda) - 2(1 + 4\lambda) - 4(-1 + \lambda) = 11 \]
\[ \Rightarrow \ -3\lambda = 11 - 20 = -9 \quad \lambda = 3 \]

Hence, the intersection has coordinates \[ x = \left( \frac{11}{1} \right) \]

(i) \( \overrightarrow{AB} = \left( \begin{array}{c} 2 \\ 2 \\ 3 \end{array} \right) \), \( \overrightarrow{AC} = \left( \begin{array}{c} 1 \\ 2 \\ 2 \end{array} \right) \)

\[ \text{proj}_{\overrightarrow{AC}} \overrightarrow{AB} = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{\| \overrightarrow{AC} \|^2} \overrightarrow{AC} = \frac{6 - 3}{9} \left( \begin{array}{c} 1 \\ 2 \\ 2 \end{array} \right) = \left( \begin{array}{c} 1 \\ 2 \\ 2 \end{array} \right) \quad \text{[1]}
\]

(ii) \( \overrightarrow{BC} = \left( \begin{array}{c} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{array} \right) \)

Line through \( B, C \) has eq \[ x = \left( \frac{2}{3} \right) + \lambda \left( \frac{4}{3} \right) \quad \text{[4]}
\]

The plane through \( A \) with normal \( \overrightarrow{BC} \) has (pt-normal) equation \( \left( \frac{1}{3} \right) x + \left( \frac{1}{3} \right) y + \left( \frac{1}{3} \right) z = 8 \quad \text{[5]}
\]

Insert \([4]\) into \([5]\): \[ \left( \frac{2}{3} + \frac{2}{3} \lambda \right) - \left( 5 - \lambda \right) + 2 \left( \frac{2}{3} + \lambda \right) = 8 \quad \lambda = 1 \]

Hence, \[ \lambda = \left( \frac{2 + 2\lambda}{3} \right) = \left( \frac{6}{3} \right) \quad \text{[6]}
\]

\[ \text{altitude} = |\overrightarrow{AC}| = \left( \frac{1}{3} \right) \left( \frac{16}{1} \right) = \frac{4}{3} \sqrt{5} \quad \text{[7]}
\]

\[ \overrightarrow{AB} = \left( \begin{array}{c} 1 \\ 2 \\ 2 \end{array} \right) \quad \text{[8]}
\]

\[ \overrightarrow{AC} = \left( \begin{array}{c} 1 \\ 2 \\ 2 \end{array} \right) \quad \text{[9]}
\]

\[ \overrightarrow{BC} = \left( \begin{array}{c} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{array} \right) \quad \text{[10]}
\]

\[ \text{altitude} = \sqrt{\frac{16}{9} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}} = \sqrt{\frac{16}{9} \cdot \frac{4}{3}} = \frac{2}{3} \sqrt{5} \quad \text{[11]}
\]