

# ALGEBRAS, REPRESENTATIONS AND THEIR DERIVED CATEGORIES OVER FINITE FIELDS

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ABSTRACT. We present a survey on the latest development in the representation theory of algebras with Frobenius morphisms. This includes a generalization of the graph and root system folding relation to folding relations at the representation level, at the category level, and further at the derived category level. As an application, we use it to approach Kac's theory for Kac-Moody algebras with symmetrizable Cartan matrices.

## INTRODUCTION

Let  $\mathbb{F}_q$  be the finite field of  $q$  elements and let  $k = \overline{\mathbb{F}_q}$  be its algebraic closure. A *Frobenius map* on a vector space over  $k$  is an abelian automorphism  $F : V \rightarrow V$  satisfying

(F1)  $F(\lambda v) = \lambda^q F(v)$  for all  $v \in V$  and  $\lambda \in k$ ;

(F2) for any  $v \in V$ ,  $F^n(v) = v$  for some  $n > 0$ .

In case  $V$  is finite dimensional, by Lang-Steinberg's theorem [29, 10.1], (F2) follows from (F1) ( see, e.g., [7, Lemma 2.2] for a proof). Note that the existence of a Frobenius map  $F$  on  $V$  implies that  $V^F = \{v \in V \mid F(v) = v\}$  is an  $\mathbb{F}_q$ -structure of  $V$ , i.e.,  $V = V^F \otimes_{\mathbb{F}_q} k$ , and vice versa (see [9, 3.5]). This special case of Galois Descent Theory plays a fundamental role in our consideration.

A *Frobenius morphism* on a  $k$ -algebra  $A$  (with 1) is a Frobenius map  $F = F_A$  on the underlying vector space satisfying  $F(ab) = F(a)F(b)$  for all  $a, b \in A$ . If  $M$  is an  $A$ -module, then we call a Frobenius map  $F_M$  on the space  $M$  a *module Frobenius map* (relative to  $F_A$ ) if  $F_M(am) = F(a)F_M(m)$  for all  $a \in A$  and  $m \in M$ . In this case, the fixed point space  $A^F = \{a \in A \mid F(a) = a\}$  is an  $\mathbb{F}_q$ -algebra; while  $M^{F_M}$  is naturally an  $A^F$ -module.

The notion of Frobenius morphisms on  $k$ -algebras is a natural extension of Frobenius morphisms on algebraic varieties/groups over  $k$ . The latter is fundamental in the theory of algebraic groups and their representations. In [25], G. Lusztig investigated Frobenius maps on representation varieties of a quiver with automorphism in connection with the geometric construction of quantized enveloping algebras of symmetrizable Kac-Moody algebras. Inspired by this work, we obtain in [6, 7] a general theory which relates representation theories of  $k$ -algebras with Frobenius morphisms and their fixed point algebras. One of the highlights of the theory is to

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generalize the folding relation between quivers and valued quivers, which induces a folding relation on their root systems, to a “folding” relation between the representations of quivers and  $\mathbb{F}_q$ -species. As further applications of the theory and partially built on the work of Hubery [20, 21], connections between Kac’s theories including Kac’s polynomials and Kac’s theorem for symmetric Kac-Moody algebras and symmetrizable ones have been established. Moreover, such a folding relation has also been extended nicely to the derived/root category level in [7].

This is a *going-down* approach in which the study of representations of a  $k$ -algebra  $A$  with a Frobenius morphism  $F$  completely determines the representations of the fixed-point  $\mathbb{F}_q$ -algebra  $A^F$ . Since every  $\mathbb{F}_q$ -algebra is isomorphic to such a fixed-point algebra, this approach becomes a powerful approach to representations of  $\mathbb{F}_q$ -algebras. Moreover, going-down from  $A$  to  $A^F = B$  and from ( $F$ -stable)  $A$ -module  $M$  to  $A^F$ -module  $M^F$ , one sees easily which properties or theories that hold for  $A$  continue to hold for  $B$  (see [7, §9]).

It should be pointed out that Frobenius maps on representations of quivers over finite fields have often been used in the context of Galois group actions; see [22, §3] and [24, §5]. In this approach, one starts with an  $\mathbb{F}_q$ -algebra  $B$  and considers  $A = B \otimes_{\mathbb{F}_q} K$  where  $K$  is a finite Galois extension of  $\mathbb{F}_q$ . Then one studies those  $A$ -modules  $X \otimes_{\mathbb{F}_q} K$  arising from  $B$ -modules  $X$ . This has been a standard approach used in the literature (see for example [5, 24, 20, 21]). We call it the *going-up* approach. In this approach, there is a natural Frobenius morphism  $F$  on  $A = B \otimes K$  taking  $b \otimes \lambda \mapsto b \otimes \lambda^q$ . Then  $B = A^F$  and for each  $B$ -module  $X$ , the  $A$ -module  $X \otimes_{\mathbb{F}_q} K$  admits a natural Frobenius map  $F$  on  $X \otimes_{\mathbb{F}_q} K$  taking  $m \otimes \lambda \mapsto m \otimes \lambda^q$  such that  $(X \otimes K)^F = X$ . Thus, results similar to 1.5, 2.1 and 3.2 have been obtained in this going-up approach.

This paper presents a brief account of the main achievements on representations of algebras with Frobenius morphisms. In §1, we define the Frobenius (twist) functor in two versions. We then give a criterion for the existence of a module Frobenius map on an  $A$ -module and embed naturally the category of  $A^F$ -modules as the subcategory of  $F$ -stable  $A$ -modules. In §2, we apply the theory to quivers with automorphisms and prove that every  $\mathbb{F}_q$ -species is isomorphic to some  $A^F$  where  $A$  is the path algebra of a quiver  $Q$  and  $F$  is the Frobenius morphism on  $A$  induced from an automorphism of  $Q$ . We further show that we may realize every finite dimensional basic  $k$ -algebra in terms of a quiver with automorphism and certain relations. Several important topics in representations of algebras are discussed in §3. They include Morita equivalence, representation type, hereditary algebras and the Auslander-Reiten theory. In §4, we look at some applications to Lie Theory, especially to Kac’s theory. Thus, we prove that counting number of representations or indecomposable ones of any  $\mathbb{F}_q$ -species results in certain polynomials. We also present a generalization of Kac’s theorem. Finally, we discuss Frobenius functors on the associated homotopy, derived and root categories and establish a triangulated category equivalence between the bounded derived categories  $\mathcal{D}^b(A^F)$  of  $A^F$ -**mod** and  $\mathcal{D}^b(A)^F$  of  $F$ -stable objects in  $\mathcal{D}^b(A)$ .

Our theory has provided a simple and convenient approach to representations of finite dimensional algebras over finite fields, and has been used in three recent PhD theses [3, 31, 32] to study a certain elliptic Lie algebra and the structure of Hall algebras of valued quivers; see §5 for more details.

Throughout we assume that all  $k$ -algebras  $A$  and  $A$ -modules are finite dimensional, and all modules are left modules.

## 1. FROBENIUS (TWIST) FUNCTORS

We first define the Frobenius twist of an  $A$ -module. The definition has two versions: the absolute version and the relative version.

- The absolute version

Let  $\mathfrak{f} : k \rightarrow k$  be the field automorphism sending  $\lambda$  to  $\lambda^q$ . For each  $k$ -space  $V$  and  $r \geq 1$ , let  $V^{(r)}$  be the new vector space obtained from  $V$  by base change via  $\mathfrak{f}^r$ :

$$V^{(r)} = V \otimes_{\mathfrak{f}^r} k.$$

Thus, for  $v \in V$  and  $\lambda \in k$ , we have  $\lambda v \otimes 1 = v \otimes \lambda^{q^r}$ . In other words, putting  $v^{(r)} = v \otimes 1$ , we have

$$(u + v)^{(r)} = u^{(r)} + v^{(r)}, \quad (\lambda v)^{(r)} = \lambda^{q^r} v^{(r)}.$$

Note that  $V^{(r)}$  may be identified as  $V$  with a twisted scalar multiplication

$$\lambda \cdot v = \sqrt[q^r]{\lambda} v.$$

Further, for a  $k$ -linear map  $\phi : U \rightarrow V$ , the map  $\phi^{(r)} := \phi \otimes 1 : U^{(r)} \rightarrow V^{(r)}$  is again a  $k$ -linear map. In this way, we obtain an exact additive functor  $(\ )^{(r)}$  from the category of  $k$ -vector spaces onto itself (see [12]).

Let  $\tau_{V,r} : V \rightarrow V^{(r)}$  be the  $\mathbb{F}_q$ -linear isomorphism sending  $v$  to  $v^{(r)}$  and let  $\tau_V = \tau_{V,1}$ . If  $A$  is a  $k$ -algebra, then  $A^{(1)}$  is also a  $k$ -algebra, and  $\tau_A : A \rightarrow A^{(1)}$  becomes an  $\mathbb{F}_q$ -algebra isomorphism. Clearly, a map  $F : A \rightarrow A$  is a Frobenius morphism if and only if  $\tau_A \circ F^{-1} : A \rightarrow A^{(1)}$  is a  $k$ -algebra isomorphism.

**Definition 1.1.** Let  $A$  be a  $k$ -algebra with Frobenius morphism  $F$  and let  $M$  be an  $A$ -module defined by the  $k$ -algebra homomorphism  $\pi : A \rightarrow \text{End}_k(M)$ . This gives a  $k$ -algebra homomorphism  $\pi^{(1)} : A^{(1)} \rightarrow \text{End}_k(M)^{(1)}$ . Thus, the composition of the following maps

$$A \xrightarrow{F^{-1}} A \xrightarrow{\tau_A} A^{(1)} \xrightarrow{\pi^{(1)}} \text{End}_k(M)^{(1)} \cong \text{End}_k(M^{(1)})$$

defines an  $A$ -module structure on  $M^{(1)}$  with the following new action

$$(1.1.1) \quad a \bullet (m^{(1)}) = (F^{-1}(a)m)^{(1)}, \quad \forall a \in A, m \in M.$$

We denote this module by  $M^{[1]}$  and call it the *Frobenius twist* of  $M$ . If  $\pi^{[1]}$  denotes the corresponding representation of  $M^{[1]}$ , then

$$(1.1.2) \quad \pi^{[1]}(a) = \tau_M \circ \pi(F^{-1}(a)) \circ \tau_M^{-1} \text{ for all } a \in A.$$

If  $f : M \rightarrow N$  is an  $A$ -module homomorphism, then the  $k$ -linear map  $f^{(1)} : M^{(1)} \rightarrow N^{(1)}$  becomes an  $A$ -module homomorphism  $M^{[1]} \rightarrow N^{[1]}$  which is denoted by  $f^{[1]}$  in the sequel. Thus, we obtain a functor

$$(1.1.3) \quad (\ )^{[1]} = (\ )_{\mathbf{mod}\text{-}A}^{[1]} : \mathbf{mod}\text{-}A \rightarrow \mathbf{mod}\text{-}A.$$

This functor will be called the *Frobenius (twist) functor* on  $\mathbf{mod}\text{-}A$ . Clearly, it is a category equivalence.

Inductively, we can define the  $s$ -fold Frobenius twist  $M^{[s]} := (M^{[s-1]})^{[1]}$  of  $M$  and  $f^{[s]} = (f^{[s-1]})^{[1]}$  for  $s \geq 1$ , where  $M^{[0]} = M$  and  $f^{[0]} = f$  by convention. Thus, the corresponding representations  $\pi^{[s]} : A \rightarrow \text{End}_k(M^{[s]})$  of  $M^{[s]}$  is given by

$$(1.1.4) \quad \pi^{[s]}(a) = \tau_{M,s} \circ \pi(F^{-s}(a)) \circ \tau_{M,s}^{-1} \quad \text{for all } a \in A.$$

Further, we can define  $M^{[-1]}$  to be the  $A$ -module  $N$  such that  $M = N^{[1]}$  and similarly for  $f^{[-1]}$ . Thus,  $M^{[s]}$  and  $f^{[s]}$  are well-defined for all  $s \in \mathbb{Z}$ .

- The relative version

We now define a Frobenius twist relative to a given Frobenius map.

Let  $M$  be an  $A$ -module and let  $F_M : M \rightarrow M$  be any given Frobenius map (not necessarily a module Frobenius map). We define  $M^{[F_M]}$  to be the  $A$ -module such that  $M^{[F_M]} = M$  as a vector space with  $F$ -twisted action

$$(1.1.5) \quad a * m := F_M(F^{-1}(a)F_M^{-1}(m)) \quad \text{for all } a \in A, m \in M.$$

In other words, if  $\pi : A \rightarrow \text{End}_k(M)$  and  $\pi^{[F_M]} : A \rightarrow \text{End}_k(M^{[F_M]})$  denote the corresponding representations, then

$$(1.1.6) \quad \pi^{[F_M]}(a) = F_M \circ \pi(F^{-1}(a)) \circ F_M^{-1} \quad \text{for all } a \in A.$$

The  $A$ -module  $M^{[F_M]}$  is called the  $F_M$ -twist of  $M$ . Similarly, for each  $s \in \mathbb{Z}$ , we define the  $s$ -fold  $F_M$ -twist  $\pi^{[F_M^s]} : A \rightarrow \text{End}_k(M^{[F_M^s]})$  by

$$(1.1.7) \quad \pi^{[F_M^s]}(a) = F_M^s \circ \pi(F^{-s}(a)) \circ F_M^{-s} \quad \text{for all } a \in A.$$

**Lemma 1.2.** *For each  $s \in \mathbb{Z}$ , we have  $M^{[s]} \cong M^{[F_M^s]}$ . In particular, the  $s$ -fold  $F_M$ -twist  $M^{[F_M^s]}$  of  $M$  is independent of the selection of  $F_M$ , up to isomorphism.*

*Proof.* By (1.1.4) and (1.1.7), the  $k$ -linear isomorphism  $\varphi_M = \tau_{M,s} \circ F_M^{-s} : M^{[F_M^s]} \rightarrow M^{[s]}$  satisfies  $\pi^{[s]} = \varphi_M \circ \pi^{[F_M^s]} \circ \varphi_M^{-1}$ , that is,  $\varphi_M$  is an  $A$ -module homomorphism.  $\square$

Note that, if  $F_M$  happens to be a module Frobenius map, then  $a * m = am$  for all  $a \in A$  and  $m \in M$ . Thus, in this case,  $M^{[F_M]} = M$  as  $A$ -modules.

Let  $M$  and  $N$  be two  $A$ -modules with Frobenius maps  $F_M$  and  $F_N$ , respectively. If  $f : M \rightarrow N$  is an  $A$ -module homomorphism, then

$$(1.2.1) \quad f^{[F]} := F_N \circ f \circ F_M^{-1} : M^{[F_M]} \longrightarrow N^{[F_N]}$$

is also an  $A$ -module homomorphism.

- Module Frobenius maps and  $F$ -periods

Recall that a module Frobenius map on an  $A$ -module  $M$  is a Frobenius map  $F_M$  on  $M$  satisfying  $F_M(am) = F(a)F_M(m)$ . Not every  $A$ -module  $M$  admits a module Frobenius map, as seen from the following criterion (see [7, Prop. 2.8]).

**Lemma 1.3.** *Let  $M$  be an  $A$ -module. Then  $M \cong M^{[r]}$  if and only if there exists a Frobenius map  $F_M$  on  $M$  such that*

$$F_M^r(am) = F^r(a)F_M^r(m), \quad \forall a \in A, m \in M$$

*(or equivalently,  $M^{[F_M^r]} = M$  as  $A$ -modules by (1.1.7)). Moreover,  $M$  admits a module Frobenius map (relative to  $F$ ) if and only if  $M \cong M^{[1]}$ .*

*Proof.* By (1.1.4) and (1.1.7), we observe that a Frobenius map  $F_M$  on  $M$  satisfies  $F_M^r(am) = F^r(a)F_M^r(m)$  for all  $a \in A, m \in M$  if and only if  $F_M^r \circ \tau_{M,r}^{-1} : M^{[r]} \rightarrow M$  is an  $A$ -module isomorphism. This clearly implies the sufficiency of the first assertion. For the necessity, take an  $A$ -module isomorphism  $\phi : M^{[r]} \xrightarrow{\sim} M$ . Then the composition  $F' := \phi \circ \tau_{M,r}$  is a Frobenius map on  $M$  with respect to  $q^r$  or  $\mathbb{F}_{q^r}$ . By choosing a basis for the  $\mathbb{F}_{q^r}$ -structure  $M^{F'}$  of  $M$ , we may define a Frobenius map  $F_M : M \rightarrow M$  such that  $F_M^r = F' = \phi \circ \tau_{M,r}$ , that is,  $F_M^r \circ \tau_{M,r}^{-1} = \phi$  is an  $A$ -module isomorphism  $M^{[r]} \rightarrow M$ .  $\square$

Let  $p(M) = p_F(M)$  be the minimal number  $r$  satisfying  $M \cong M^{[r]}$ . We call it the  $F$ -period of  $M$  and call  $M$   $F$ -stable if  $p_F(M) = 1$ . The lemma above shows that if  $p_F(M) > 1$  then  $M$  does not admit a module Frobenius map relative to  $F$ . However, we have the following.

**Corollary 1.4.** *Let  $M$  be an  $A$ -module with  $F$ -period  $r$ . Then  $\tilde{M} := M \oplus M^{[1]} \oplus \dots \oplus M^{[r-1]}$  admits a module Frobenius map  $\tilde{F}_M$  defined by*

$$\tilde{F}_M(x_0, x_1, \dots, x_{r-1}) = (F_M(x_{r-1}), F_M(x_0), \dots, F_M(x_{r-2})),$$

where  $F_M$  is a Frobenius map on  $M$  satisfying  $M^{[F_M]} = M$  as  $A$ -modules..

- The Frobenius functor  $(\ )_{\mathbf{mod}}^{[1]}$

Let  $A\text{-mod}$  denote the category of finite dimensional (left)  $A$ -modules. Then, Frobenius twisting induces a functor, the *Frobenius functor*

$$(1.4.1) \quad (\ )^{[1]} = (\ )_{\mathbf{mod}}^{[1]} : A\text{-mod} \rightarrow A\text{-mod}.$$

The Frobenius functor determines a new category  $A\text{-mod}^F$  whose objects are  $F$ -stable  $A$ -modules  $M$  with a fixed isomorphism  $\phi_M : M^{[1]} \xrightarrow{\sim} M$  and whose morphisms are compatible with the isomorphisms  $\phi_M$ , i.e.,

$$\text{Hom}_{A\text{-mod}^F}(M, N) = \{f \in \text{Hom}_A(M, N) \mid \phi_N \circ f^{[1]} = f \circ \phi_M\}.$$

Clearly, a selection of different isomorphisms  $\phi_M$  results in an equivalent category.

Some version of the following category equivalence has already been obtained by Hubery [20, Prop. 17] in the going-up approach.

**Theorem 1.5.** *There is a category equivalence*

$$A^F\text{-mod} \cong A\text{-mod}^F.$$

*Proof.* The base change functor sending an  $A^F$ -module  $N$  to  $N_k = N \otimes k$  has an “inverse” which takes an  $F$ -stable module  $(M, F_M)$  to its fixed point module  $M^{F_M}$ .  $\square$

## 2. REPRESENTATIONS OF QUIVERS WITH AUTOMORPHISMS

Let  $Q = (Q_0, Q_1)$  be a finite quiver, where  $Q_0$  (resp.  $Q_1$ ) denotes the set of vertices (resp. arrows) of  $Q$ . For each arrow  $\rho$  in  $Q_1$ , we denote by  $\rho' \xrightarrow{\rho} \rho''$  to indicate the tail  $\rho'$  and the head  $\rho''$  of  $\rho$ . Let  $\sigma$  be an automorphism of  $Q$ , that is,  $\sigma$  is a permutation on the vertices of  $Q$  and on the arrows of  $Q$  satisfying the compatibility conditions:  $\sigma(\rho') = \sigma(\rho)'$  and  $\sigma(\rho'') = \sigma(\rho)''$  for any  $\rho \in Q_1$ .

Let  $A := kQ$  be the path algebra of  $Q$  over  $k = \overline{\mathbb{F}}_q$  which has the identity  $1 = \sum_{i \in Q_0} e_i$  where  $e_i$  is the idempotent (as a length zero path) corresponding to the vertex  $i$ . Then  $\sigma$  induces a Frobenius morphism

$$(2.0.1) \quad F_{Q,\sigma} = F_{Q,\sigma;q} : A \rightarrow A; \quad \sum_s x_s p_s \mapsto \sum_s x_s^q \sigma(p_s),$$

where  $\sum_s x_s p_s$  is a  $k$ -linear combination of paths  $p_s$ , and  $\sigma(p_s) = \sigma(\rho_t) \cdots \sigma(\rho_1)$  if  $p_s = \rho_t \cdots \rho_1$  for arrows  $\rho_1, \dots, \rho_t$  in  $Q_1$ .

We now construct an  $\mathbb{F}_q$ -species<sup>1</sup> (see [14, 27, 10, 11]) from a quiver  $Q$  with automorphism  $\sigma$ . First, associated to  $(Q, \sigma)$ , there is a valued quiver  $\Gamma = \Gamma(Q, \sigma) = (\Gamma_0, \Gamma_1)$  whose vertex set  $\Gamma_0$  (resp. arrow set  $\Gamma_1$ ) is the set of  $\sigma$ -orbits in  $Q_0$  (resp. in  $Q_1$ ), and whose valuation is defined as follows: we associate to each  $\mathbf{k} \in \Gamma_0$  the number  $\varepsilon_{\mathbf{k}} = \#\mathbf{k}$ , and to each arrow  $\rho : \rho' \rightarrow \rho''$  in  $\Gamma$  the pair  $(d''_{\rho}, d'_{\rho})$  of the numbers defined by

$$(2.0.2) \quad \varepsilon_{\rho} = \#\rho, \quad d''_{\rho} = \varepsilon_{\rho} / \varepsilon_{\rho''} \quad \text{and} \quad d'_{\rho} = \varepsilon_{\rho} / \varepsilon_{\rho'}.$$

Second, using the Frobenius morphism  $F = F_{Q,\sigma}$  on  $A$  defined above, we can attach naturally to  $\Gamma$  an  $\mathbb{F}_q$ -modulation as follows: for each vertex  $\mathbf{i} \in \Gamma_0$  and each arrow  $\rho \in \Gamma_1$ , we fix  $i_0 \in \mathbf{i}$ ,  $\rho_0 \in \rho$ , and consider the  $F$ -stable subspaces of  $A$

$$A_{\mathbf{i}} = \bigoplus_{i \in \mathbf{i}} k e_i = \bigoplus_{s=0}^{\varepsilon_{\mathbf{i}}-1} k e_{\sigma^s(i_0)} \quad \text{and} \quad A_{\rho} = \bigoplus_{\rho \in \rho} k \rho = \bigoplus_{t=0}^{\varepsilon_{\rho}-1} k \sigma^t(\rho_0).$$

Then we have

$$(2.0.3) \quad A_{\mathbf{i}}^F = \left\{ \sum_{s=0}^{\varepsilon_{\mathbf{i}}-1} x^{q^s} e_{\sigma^s(i_0)} \mid x \in k, x^{q^{\varepsilon_{\mathbf{i}}}} = x \right\} \quad \text{and} \quad A_{\rho}^F = \left\{ \sum_{t=0}^{\varepsilon_{\rho}-1} x^{q^t} \sigma^t(\rho_0) \mid x \in k, x^{q^{\varepsilon_{\rho}}} = x \right\}.$$

Further, the algebra structure of  $A$  induces an  $A_{\rho''}^F$ - $A_{\rho'}^F$ -bimodule structure on  $A_{\rho}^F$  for each arrow  $\rho : \rho' \rightarrow \rho''$  in  $\Gamma$ . Thus, we obtain an  $\mathbb{F}_q$ -modulation  $\mathbb{M} = \mathbb{M}(Q, \sigma) := (\{A_{\mathbf{i}}^F\}_{\mathbf{i}}, \{A_{\rho}^F\}_{\rho})$  over the valued quiver  $\Gamma$ . The  $\mathbb{F}_q$ -species  $(\Gamma, \mathbb{M})$  defined above will be denoted by  $\mathfrak{M}_{Q,\sigma} = \mathfrak{M}_{Q,\sigma;q} = (\Gamma, \mathbb{M})$ .

The following result given in [7, 9.3] shows that every  $\mathbb{F}_q$ -species may arise in this way. Thus, by Theorem 1.5, we may regard representations of  $\mathbb{F}_q$ -species as  $F$ -stable representations of the corresponding quiver.

**Theorem 2.1.** *Let  $(Q, \sigma)$  be a finite quiver with automorphism  $\sigma$ , and let  $\mathfrak{M}_{Q,\sigma}$  be the  $\mathbb{F}_q$ -species associated to  $(Q, \sigma)$ .*

- (1) *If  $A = kQ$  is the path algebra of  $Q$ , and  $F = F_{Q,\sigma}$  is the Frobenius morphism on  $A$  induced by  $\sigma$ , then the fixed point algebra  $A^F$  is isomorphic to the tensor (or path) algebra  $T(\mathfrak{M}_{Q,\sigma})$  of  $\mathfrak{M}_{Q,\sigma}$ .*
- (2) *For any given  $\mathbb{F}_q$ -species  $\mathfrak{M}$ , there is a quiver  $Q$  and an automorphism  $\sigma$  of  $Q$  such that the tensor algebra  $T(\mathfrak{M})$  of  $\mathfrak{M}$  is isomorphic to  $(kQ)^{F_{Q,\sigma}}$ .*

**Remark 2.2.** If an  $\mathbb{F}_q$ -species  $\mathfrak{M}$  involves only *natural* bimodules  ${}_{\mathbb{F}_{q^r}}(\mathbb{F}_{q^n})_{\mathbb{F}_{q^s}}$  in the sense that the bimodule structure is induced from the subfield structure, where  $r, s, n \geq 1$ ,  $r|n$  and  $s|n$ , then the corresponding quiver  $Q$  and its automorphism  $\sigma$  have been constructed in [20, Lemma 21] (see also [18]). In fact, Hubery further observed that the Frobenius map on  $KQ$ , where  $K$  is a finite extension of  $\mathbb{F}_q$ ,

<sup>1</sup>We used the term  *$\mathbb{F}_q$ -modulated quivers* [1] for  $\mathbb{F}_q$ -species in [6, 7].

considered in the going-up approach (§1) decomposes into the product of a quiver automorphism and a field automorphism (see the discussion prior to [20, Prop. 22]). This decomposition is similar to Steinberg's decomposition for an automorphism of a finite Chevalley group (see [2, 12.5.1]).

Our next result deals with a more general case and shows that every Frobenius morphism on a basic  $k$ -algebra is induced from the path algebra of a quiver with automorphism.

**Theorem 2.3** ([7, 9.5]). *If  $A$  is a finite dimensional basic  $k$ -algebra with a Frobenius morphism  $F$ , then there are a quiver  $Q$  with automorphism  $\sigma$  and an algebra epimorphism  $\varphi : kQ \rightarrow A$  such that the following diagram is commutative*

$$\begin{array}{ccc}
 kQ & \xrightarrow{\varphi} & A \\
 F_{Q,\sigma} \downarrow & & \downarrow F \\
 kQ & \xrightarrow{\varphi} & A
 \end{array}$$

*In particular, every finite dimensional basic  $\mathbb{F}_q$ -algebra  $B$  is isomorphic to  $(kQ/I)^{F_{Q,\sigma}}$  for some quiver  $Q$  with automorphism  $\sigma$  and some  $F_{Q,\sigma}$ -stable admissible ideal  $I$  of  $kQ$ .*

### 3. FOLDING REPRESENTATIONS WITH FROBENIUS MORPHISMS

We now look at several important topics in the representation theory of finite dimensional algebras.

- Morita equivalence

Up to Morita equivalence, the study of representations of algebras may be reduced to that of basic algebras. By Theorem 2.3, a Frobenius morphism on a basic algebra arises from a quiver with automorphism. Thus, it would be interesting to know if a Morita equivalence is invariant under a Frobenius morphism. The following result is given in [7, 9.5(2)].

**Theorem 3.1.** *If  $A$  is a finite dimensional  $k$ -algebra with a Frobenius morphism  $F$ . Then there exists a basic algebra  $A'$  with a Frobenius morphism  $F'$  such that both pairs  $(A, A')$  and  $(A^F, A'^{F'})$  are Morita equivalent.*

- Representation type

A (finite dimensional) algebra  $A$  is said to be of finite representation type if, up to isomorphism, it has only finitely many indecomposable representations. Other representation types including tame type and wild type can be also defined. The following result has been proved in [6, 5.1]; compare [22, Lemma 3.4], [24, 5.3] and [20, Prop. 17].

**Theorem 3.2.** *Let  $M$  be an indecomposable  $A$ -module with  $F$ -period  $r$ . Then there exists a Frobenius map  $F_M$  on  $M$  such that the pair  $(\tilde{M}, \tilde{F}_M)$  defined in 1.4 is indecomposable in  $A\text{-mod}^F$  and*

$$\text{End}_{A^F}(\tilde{M}^{\tilde{F}})/\text{Rad}(\text{End}_{A^F}(\tilde{M}^{\tilde{F}})) \cong \mathbb{F}_{q^r} \quad (\tilde{F} = \tilde{F}_M).$$

*Moreover, every indecomposable  $A^F$ -module is isomorphic to a module of the form  $\tilde{M}^{\tilde{F}}$  for some indecomposable  $A$ -module  $M$  and Frobenius map  $F_M$ .*

**Corollary 3.3.** *Let  $A$  be a  $k$ -algebra with a Frobenius morphism  $F$ . Then  $A$  is of finite representation type if and only if so is  $A^F$ .*

It is natural to expect that this result continues to hold for tame and wild types.

- Finite dimensional hereditary algebras

A quiver automorphism  $\sigma$  is called *admissible* if there are no arrows connecting vertices in the same  $\sigma$ -orbit. Call a quiver with an admissible automorphism an *ad-quiver* (see [6, 3.5]). The following characterization of finite dimensional hereditary algebras is an easy consequence of Theorem 2.1.

**Theorem 3.4.** *An algebra  $B$  is a finite dimensional hereditary basic algebra over  $\mathbb{F}_q$  if and only if  $B$  is isomorphic to  $(kQ)^{F_{Q,\sigma}}$  for some ad-quiver  $(Q, \sigma)$  without oriented cycles.*

- The Auslander-Reiten quiver (or AR-quiver) of a  $k$ -algebra

Since the algebra  $A$  is defined over the algebraically close field  $k = \overline{\mathbb{F}_q}$ , we may regard the AR-quiver  $\mathcal{Q} = \mathcal{Q}_A$  of  $A$  as an ordinary quiver. We first observe that  $\mathcal{Q}$  admits an admissible automorphism  $\mathfrak{s}$ . For each vertex  $[M] \in \mathcal{Q}$ ,  $\mathfrak{s}([M])$  is defined to be  $[M^{[1]}]$ . If  $M$  and  $N$  are indecomposable  $A$ -modules, then there are  $n_{st}$  arrows  $\gamma_{s,t}^{(m)}$  from  $[M^{[s]}]$  to  $[N^{[t]}]$  in  $\mathcal{Q}$ , where  $0 \leq s \leq p(M) - 1$ ,  $0 \leq t \leq p(N) - 1$ ,  $n_{st} = \dim_k \text{Irr}_A(M^{[s]}, N^{[t]})$  and  $1 \leq m \leq n_{st}$ . Note that  $n_{st} = n_{s+1, t+1}$  for all  $s, t$ , where subscripts are considered as integers modulo the  $F$ -periods  $p(M)$  and  $p(N)$ , respectively. We now define

$$\mathfrak{s}(\gamma_{s,t}^{(m)}) = \gamma_{s+1, t+1}^{(m)} \quad \text{for all } 0 \leq s \leq p(M) - 1 \text{ and } 0 \leq t \leq p(N) - 1.$$

Clearly,  $\mathfrak{s}$  is an admissible quiver automorphism and  $(\mathcal{Q}, \mathfrak{s})$  is an ad-quiver.

Associated to  $(\mathcal{Q}, \mathfrak{s})$ , we may define an  $\mathbb{F}_q$ -species  $\mathfrak{M}_{\mathcal{Q}, \mathfrak{s}}$  as in Section 3: let  $\mathcal{A} = k\mathcal{Q}$  denote the path algebra of  $\mathcal{Q}$  and  $F = F_{\mathcal{Q}, \mathfrak{s}}$  be the Frobenius morphism on  $\mathcal{A}$  induced by the automorphism  $\mathfrak{s}$ . For each vertex  $\mathbf{i}(M)$  (i.e., the  $\mathfrak{s}$ -orbit of  $[M]$ ) and each arrow  $\rho$  (i.e., an  $\mathfrak{s}$ -orbit of arrows in  $\mathcal{Q}$ ) in  $\Gamma(\mathcal{Q}, \mathfrak{s})$ , we define subspaces

$$\mathcal{A}_{\mathbf{i}(M)} = \bigoplus_{s=0}^{p(M)-1} ke_{[M^{[s]}}] \quad \text{and} \quad \mathcal{A}_{\rho} = \bigoplus_{\rho \in \rho} k\rho,$$

of  $\mathcal{A}$ , which are obviously  $F$ -stable. By definition, the  $\mathbb{F}_q$ -modulation  $\mathbb{M}(\mathcal{Q}, \mathfrak{s})$  is given by  $(\mathcal{A}_{\mathbf{i}(M)})^F$  and  $(\mathcal{A}_{\rho})^F$  for all vertices  $\mathbf{i}(M)$  and arrows  $\rho$  in  $\Gamma(\mathcal{Q}, \mathfrak{s})$ .

Recall from [6, 6.2] the definition of isomorphisms for  $\mathbb{F}_q$ -species. Two species are isomorphic if their associated “simple” ones (obtained by summing up the valuations and bimodules over parallel edges) are isomorphic. We now can state the following result.

**Theorem 3.5.** *The  $\mathbb{F}_q$ -species  $\mathfrak{M}_{\mathcal{Q}, \mathfrak{s}}$  associated to the AR-quiver  $(\mathcal{Q}, \mathfrak{s})$  of  $A$  defined above is isomorphic to the AR-quiver  $\mathcal{Q}_{A^F}$  of  $A^F$ . Moreover, the Auslander-Reiten translation of  $A$  naturally induces that of the fixed-point algebra  $A^F$ .*

- Other topics

Properties like global dimension, self-injectivity and preprojectivity of a  $k$ -algebra  $A$  are also  $F$ -invariant. In other words,  $A$  has such a property if and only if so does  $A^F$ . However, not every property which  $A$  possesses is  $F$ -invariant. For example, the number of irreducible modules is not  $F$ -invariant. It seems true that quasi-heredity is not  $F$ -invariant.

#### 4. APPLICATIONS TO LIE THEORY

The characterization of hereditary algebras (Theorem 3.4) in terms of quivers with automorphisms establishes direct links between representations of hereditary algebras and Kac-Moody Lie algebras. We now look at them at several levels.

- Symmetrizable generalized Cartan matrices and their root systems

Let  $Q$  be a finite quiver without oriented cycles. Then  $Q$  defines a symmetric generalized Cartan matrix  $C_Q = (a_{ij})_{i,j \in Q_0}$  by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -|\{\text{arrows between } i \text{ and } j\}| & \text{if } i \neq j. \end{cases}$$

If  $Q$  is equipped with an admissible automorphism  $\sigma$ , then the associated valued quiver  $\Gamma$  (§3) defines a symmetrizable generalized Cartan matrix  $C_\Gamma = (b_{\mathbf{i}\mathbf{j}})_{\mathbf{i}, \mathbf{j} \in I}$  by

$$b_{\mathbf{i}\mathbf{j}} = \begin{cases} 2 & \text{if } \mathbf{i} = \mathbf{j} \\ -\sum_{\rho} \varepsilon_{\rho} / \varepsilon_{\mathbf{i}} & \text{if } \mathbf{i} \neq \mathbf{j} \end{cases}$$

where the sum is taken over all arrows  $\rho$  between  $\mathbf{i}$  and  $\mathbf{j}$ . Since the definition is regardless of the orientation, it is easy to see that all symmetrizable generalized Cartan matrices can be obtained in this way.

Let  $\Delta(Q) \subset \mathbb{N}Q_0$  (resp.  $\Delta(\Gamma) \subset \mathbb{N}\Gamma_0$ ) be the root system associated with the quiver  $Q$  (resp. the valued quiver  $\Gamma$ ), or equivalently, the root system of the Kac-Moody algebra associated with the Cartan matrix  $C_Q$  (resp.  $C_\Gamma$ ) (see [22] or [23] for its definition).

The quiver automorphism  $\sigma$  extends linearly to a group automorphism  $\sigma$  on  $\mathbb{Z}Q_0$  defined by

$$\sigma\left(\sum_{i \in Q_0} a_i i\right) = \sum_{i \in Q_0} a_i \sigma(i).$$

Let  $(\mathbb{Z}Q_0)^\sigma$  denote the subset of  $\sigma$ -fixed points in  $\mathbb{Z}Q_0$ . This set can be identified with the group  $\mathbb{Z}I$  via the canonical bijection

$$\hat{\sigma} : (\mathbb{Z}Q_0)^\sigma \longrightarrow \mathbb{Z}I; \quad \sum_{i \in Q_0} b_i i \longmapsto \sum_{i \in I} a_i i,$$

where  $a_i := b_i = b_j$  for all  $i, j \in \mathbf{i}$ .

For  $\beta \in \Delta(Q)$ , let  $t \geq 1$  be the minimal integer satisfying  $\sigma^t(\beta) = \beta$ . We call  $t$  the  $\sigma$ -period of  $\beta$ , denoted by  $p(\beta) = p_\sigma(\beta)$ . We have the folding relation between the root systems of  $Q$  and  $\Gamma$  (see [30, Prop. 2] and [20, Prop. 4]).

**Proposition 4.1.** *Let  $\beta \in \Delta(Q)$  and set*

$$\tilde{\beta} := \beta + \sigma(\beta) + \cdots + \sigma^{t-1}(\beta) \in (\mathbb{Z}Q_0)^\sigma,$$

where  $t = p_\sigma(\beta)$ . Then  $\beta \mapsto \hat{\sigma}(\tilde{\beta})$  defines a surjective map  $\Delta(Q) \rightarrow \Delta(\Gamma)$ . Moreover, if  $\hat{\sigma}(\tilde{\beta})$  is real, then  $\beta$  is real and is unique up to  $\sigma$ -orbit.

Thus, to an ad-quiver  $(Q, \sigma)$  (without oriented cycles), we may associate a finite dimensional hereditary  $\mathbb{F}_q$ -algebra (see 3.4) and a Kac-Moody algebra with the generalized Cartan matrix  $C_\Gamma$ . The connection between the two algebraic structures has been investigated since 1970s. Gabriel's theorem [13], Kac's theorem [22] and Ringel-Hall algebra approach to quantum groups [28] are the milestone contributions to this investigation.

- Counting  $F$ -stable representations

Let  $Q$  be a finite quiver with an admissible automorphism  $\sigma$ , and let  $\mathfrak{M}_{Q, \sigma}$  be the associated  $\mathbb{F}_q$ -species with underlying valued quiver  $\Gamma = \Gamma(Q, \sigma)$ . Put  $I = \Gamma_0$ .

Given a matrix  $x = (x_{ij}) \in k^{m \times n}$  and an integer  $r \geq 0$ , we define

$$x^{[r]} = (x_{ij}^{q^r}) \in k^{m \times n}.$$

For each  $\beta = \sum_{i \in Q_0} b_i i \in (\mathbb{N}Q_0)^\sigma$ , let  $V_i = k^{b_i}$  for each  $i \in Q_0$ . We consider the affine variety

$$R(\beta) = R(Q, \beta) = \prod_{\rho \in Q_1} \text{Hom}_k(k^{b_{\rho'}}, k^{b_{\rho''}}) \cong \prod_{\rho \in Q_1} k^{b_{\rho''} \times b_{\rho'}}.$$

Then a point  $x = (x_\rho)_\rho$  of  $R(\beta)$  determines a representation  $V(x) = (V_i, x_\rho)$  of  $Q$ .

The algebraic group

$$G(\beta) = \prod_{i \in Q_0} GL_{b_i}(k) \subset GL(V)$$

acts on  $R(\beta)$  by conjugation

$$(g_i)_i \cdot (x_\rho)_\rho = (g_{\rho''} x_\rho g_{\rho'}^{-1})_\rho,$$

and the  $G(\beta)$ -orbits  $\mathcal{O}_x$  in  $R(\beta)$  correspond bijectively to the isoclasses  $[V(x)]$  of representations of  $Q$  with dimension vector  $\beta$ .

Further, we define a Frobenius map  $F$  on the  $Q_0$ -graded vector space  $V = \bigoplus_{i \in Q_0} V_i$  such that, for  $v \in V_i$ ,  $v^{[1]} := F(v) \in V_{\sigma(i)}$  for all  $i \in Q_0$ . Then the Frobenius map  $F$  on  $V$  induces a Frobenius map on the variety  $R(\beta)$  such that, for  $x = (x_\rho) \in R(\beta)$ ,  $F(x) = (y_\rho)$  is defined by

$$y_\rho(F(v)) = F(x_{\sigma^{-1}(\rho)}(v)) \quad \text{for all } \rho \in Q_1, v \in V_{\sigma^{-1}(\rho')},$$

and a Frobenius map on the group  $G(\beta)$  given by

$$F(gv) = F(g)(F(v)) \quad \text{for all } g \in G(\beta), v \in V.$$

The action of  $G(\beta)$  on  $R(\beta)$  restricts to an action of  $G(\beta)^F$  on  $R(\beta)^F$ . Then, the  $G(\beta)^F$ -orbits in  $R(\beta)^F$  correspond bijectively to the isoclasses of  $F$ -stable representations of  $Q$  with dimension vector  $\beta$ , or equivalently, to the isoclasses of  $T(\mathfrak{M}_{Q,\sigma})$ -modules with dimension vector  $\hat{\sigma}(\beta) =: \alpha$ .

Now let

$$\begin{aligned} M_{Q,\sigma}(\alpha, q) &= \# \text{ of isoclasses of } T(\mathfrak{M}_{Q,\sigma})\text{-modules of dimension vector } \alpha, \\ I_{Q,\sigma}(\alpha, q) &= \# \text{ of isoclasses of indecomposable } T(\mathfrak{M}_{Q,\sigma})\text{-modules} \\ &\quad \text{of dimension vector } \alpha. \end{aligned}$$

The following result is proved in [6, 9.1-2]. See [18] for some *natural*  $\mathbb{F}_q$ -species in which the  $\mathbb{F}_{q^r}$ - $\mathbb{F}_{q^s}$ -bimodule structure on  $\mathbb{F}_{q^m}$  in the modulation is given naturally as subfields.

**Theorem 4.2.** *Both  $M_{Q,\sigma}(\alpha, q)$  and  $I_{Q,\sigma}(\alpha, q)$  are polynomials in  $q$  with rational coefficients and are independent of the  $\sigma$ -admissible orientation of  $Q$ .*

The proof uses some standard counting formulas for  $GL(n, q)$  given in [26, p.272] together with Burnside's counting formula:

$$M_{Q,\sigma}(\alpha, q) = \frac{1}{|G|} \sum_{g \in G} |X^g| = \sum_{g \in \text{ccl}(G)} \frac{|X^g|}{|G_g|},$$

where  $G = G(\beta)^F$ ,  $X = R(\beta)^F$ , and  $\text{ccl}(G)$  is a set of representatives of conjugacy classes of  $G$ .

**Remark 4.3.** In [20], the so-called isomorphically invariant representations (ii-representations for short) of  $Q$  have been studied. This generalizes some results in [30]. Given a representation  $V = (V_i, V_\alpha)$  of  $Q$  over  $k$ , we define a new representation  ${}^\sigma V = (W_i, W_\alpha)$  of  $Q$  by  $W_i = V_{\sigma^{-1}(i)}$  and  $W_\alpha = V_{\sigma^{-1}(\alpha)}$  for all  $i \in Q_0$  and  $\alpha \in Q_1$ . The representation  $V$  is called *isomorphically invariant* if  $V \cong {}^\sigma V$ . It is said to be *ii-indecomposable* if it is not a direct sum of two non-zero ii-representations. In case  $k$  is an algebraically closed field of characteristic not dividing the order of  $\sigma$ , the dimension vectors of ii-indecomposable representations of  $Q$  over  $k$  have been described (see [20, Thm 1]). Further, the polynomials for counting the number of isoclasses of ii-indecomposable and absolutely ii-indecomposable representations of  $Q$  over finite fields have been investigated. In the affine case these polynomials are explicitly calculated. We refer to [19, Chapter 4] for details.

- A generalization of Kac's theorem

Let  $Q$  be a finite quiver without oriented cycles and  $\sigma$  an admissible automorphism of  $Q$ . Let  $\mathfrak{M}_{Q,\sigma}$  be the associated  $\mathbb{F}_q$ -species with the underlying valued quiver  $\Gamma$  and vertex set  $I = \Gamma_0$ .

The following theorem is known as Kac's theorem when  $\sigma = 1$  and is proved in [6, 10.3].

**Theorem 4.4.** (1) *The polynomial  $I_{Q,\sigma}(\alpha, q)$  is non-zero  $\iff \alpha \in \Delta(\Gamma)^+$ .*  
 (2)  *$\alpha \in \Delta(\Gamma)^+$  is real  $\implies I_{Q,\sigma}(\alpha, q) = 1$ .*

Note that our proof requires Kac's theorem together with a result of Hubery [20, Thm1]. It should be interesting to find an independent proof of the theorem in the species case. See the conjecture [6, 10.5].

**Remarks 4.5.** (a) This theorem is a generalization of Kac's theorem for quivers over a finite field to a result for all  $\mathbb{F}_q$ -species. This result has been proved by Hua and Hubery for the natural  $\mathbb{F}_q$ -species.

(b) By using Ringel's Hall algebra approach, Deng and Xiao proved in [8] that, for any prime power  $q$  and dimension vector  $\alpha$ , the number  $I_{Q,\sigma}(\alpha, q) \neq 0 \iff \alpha \in \Delta(\Gamma)^+$ .

- Kac conjectures

For each  $\alpha \in \Delta(\Gamma)^+$  and each  $r \geq 1$ , let

$$I_{Q,\sigma}(\alpha, q; r) = \# \text{ of isoclasses of indecomposable } T(\mathfrak{M}_{Q,\sigma})\text{-modules} \\ \text{of dimension vector } \alpha \text{ arising from indecomposable} \\ kQ\text{-modules of } F\text{-period } r.$$

Clearly,  $I_{Q,\sigma}(\alpha, q) = \sum_{r \geq 1} I_{Q,\sigma}(\alpha, q; r)$ , and  $I_{Q,\sigma}(\alpha, q; 1)$  is the number of isoclasses of *absolutely* indecomposable representations of  $Q$  over  $\mathbb{F}_q$ .

For  $\sigma = 1$ , we put  $I_Q(\alpha, q) = I_{Q,1}(\alpha, q)$  and  $I_Q(\alpha, q; r) = I_{Q,1}(\alpha, q, r)$ . Thus, since the constant term of  $I_Q(\alpha, q; r)$ ,  $r \geq 2$ , is zero (see [17, 2.3]), it follows that the constant terms of  $I_Q(\alpha, q)$  and  $I_Q(\alpha, q; 1)$  coincide.

Kac proves that  $I_Q(\alpha, q; 1) \in \mathbb{Z}[q]$  and makes the following two conjectures:

**Kac Conjecture for Positivity:**  $I_Q(\alpha, q; 1) \in \mathbb{N}[q]$ .

**Kac Conjecture for Multiplicity:**  $I_Q(\alpha, 0; 1) = \text{mult}_{\mathfrak{g}}(\alpha)$ .

Here  $\mathfrak{g}$  is the Kac-Moody algebra associated to  $C_Q$  and  $\text{mult}_{\mathfrak{g}}(\alpha) = \dim \mathfrak{g}_{\alpha}$ .

These two conjectures have been proved by Crawley-Boevey and Van den Bergh [4] for indivisible roots  $\alpha$ , but are still open in general.

It would be interesting to study the polynomials  $I_{Q,\sigma}(\alpha, q; r)$  for an arbitrary  $\sigma$ .

- Some other applications

Applications of this theory to Hall algebras, Lie algebras and quantum groups can be found in [3, 31, 32]. More precisely, in [3], by studying the Frobenius morphism of the tubular algebra of type  $\mathbb{T}(3, 3, 3)$  and the root category of its fixed point algebra, the author obtains a realization of the elliptic Lie algebra of type  $F_4^{(2,2)}$ . In [31], based on the folding relation between quivers and valued quivers in §3, the author studies representations of (arbitrary)  $\mathbb{F}_q$ -species with oriented cycles and their associated Ringel-Hall algebras. Finally, in [32], by describing  $F$ -stable representations of affine quivers with automorphisms, the PBW type bases of composition algebras of affine valued quivers (thus of the corresponding quantum groups) have been constructed. Moreover, the minimal generating system for the Ringel-Hall algebra of an affine valued quiver is obtained. The author also presents a  $q$ -analogue of the Weyl-Kac denominator identity for affine valued quivers of type  $\tilde{B}_n$ ,  $\tilde{DD}_n$ ,  $\tilde{F}_{41}$ , and  $\tilde{G}_{21}$ .

## 5. FOLDING DERIVED/ROOT CATEGORIES WITH FROBENIUS FUNCTORS

The Frobenius functor  $(\ )_{\mathbf{mod}}^{[1]} : A\text{-}\mathbf{mod} \rightarrow A\text{-}\mathbf{mod}$  and the category equivalence  $A^F\text{-}\mathbf{mod} \rightarrow A\text{-}\mathbf{mod}^F$  can be naturally lifted to the derived and root category levels.

- The derived category level

Let  $\mathcal{C}(A) := \mathcal{C}(\mathbf{mod}\text{-}A)$  denote the category of (cochain) complexes of  $A$ -modules

$$M = (M^i, d^i) = \dots \longrightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \dots$$

where  $d^2 = 0$ . Let  $\mathcal{K}(A) := \mathcal{K}(\mathbf{mod}\text{-}A)$  be the *homotopy category* defined by

$$(5.0.1) \quad \begin{cases} \text{Ob}(\mathcal{K}(A)) = \text{Ob}(\mathcal{C}(A)), \text{ and for } M, N \in \text{Ob}(\mathcal{K}(A)), \\ \text{Hom}_{\mathcal{K}(A)}(M, N) = \text{Hom}_{\mathcal{C}(A)}(M, N)/\text{Ht}(M, N), \end{cases}$$

where  $\text{Ht}(M, N)$  denotes the subspace of  $\text{Hom}_{\mathcal{C}(A)}(M, N)$  consisting of morphisms homotopic to zero. Further we let  $\mathcal{D}(A)$  be the *derived category* of  $A$ , i.e.,  $\mathcal{D}(A)$  is the localization of  $\mathcal{K}(A)$  at the class  $\mathcal{S}$  of all quasi-isomorphisms. (A morphism  $f : M \rightarrow N$  in  $\mathcal{K}(A)$  is called a *quasi-isomorphism* if  $H^i(f)$  is an isomorphism for each  $i \in \mathbb{Z}$ , where  $H^i : \mathcal{K}(A) \rightarrow \mathbf{mod}\text{-}A$  is the  $i$ -th cohomological functor.) We shall denote the full subcategory of  $\mathcal{C}(A)$ , where  $\mathcal{C} = \mathcal{C}, \mathcal{K}$  or  $\mathcal{D}$ , consisting of bounded complexes (resp. complexes bounded below, complexes bounded above) by  $\mathcal{C}^b(A)$  (resp.  $\mathcal{C}^+(A), \mathcal{C}^-(A)$ , etc.).

Applying the Frobenius functor to each  $M^i$  in (5.0.1), we obtain a new complex

$$M^{[1]} = \dots \longrightarrow (M^{i-1})^{[1]} \xrightarrow{(d^{i-1})^{[1]}} (M^i)^{[1]} \xrightarrow{(d^i)^{[1]}} (M^{i+1})^{[1]} \xrightarrow{(d^{i+1})^{[1]}} \dots$$

This will be called the *Frobenius twist* of  $M$ . Thus, the Frobenius functor on  $\mathbf{mod}\text{-}A$  defined in (1.4.1) induces a functor

$$(\ )^{[1]} = (\ )_{\mathcal{C}(A)}^{[1]} : \mathcal{C}(A) \rightarrow \mathcal{C}(A),$$

which we still call the *Frobenius (twist) functor* (on complexes). Since a morphism  $f : M \rightarrow N$  is homotopic to zero (resp. a quasi-isomorphism) if and only if so is  $f^{[1]}$ , the Frobenius functor  $(\ )^{[1]}$  on  $\mathcal{C}(A)$  induces functors

$$(\ )^{[1]} = (\ )_{\mathcal{C}(A)}^{[1]} : \mathcal{C}(A) \rightarrow \mathcal{C}(A) \quad (\mathcal{C} = \mathcal{K}, \mathcal{D}),$$

which clearly preserve distinguished triangles. Thus, the Frobenius functors on  $\mathcal{K}(A)$  and  $\mathcal{D}(A)$  are equivalences of triangulated categories.

For  $\mathcal{C} \in \{\mathcal{C}, \mathcal{K}, \mathcal{D}\}$ , let  $\mathcal{C}^b(A)^F$  be the category whose objects consist of bounded complexes  $M$  satisfying  $M^{[1]} \cong_{\phi_M^{\mathcal{C}}} M$  and whose morphisms are compatible with these isomorphisms:

$$\text{Hom}_{\mathcal{C}^b(A)^F}(M, N) = \{f \in \text{Hom}_{\mathcal{C}(A)}(M, N) \mid \phi_N^{\mathcal{C}} \circ f^{[1]} = f \circ \phi_M^{\mathcal{C}}\}.$$

Clearly, a different selection of the isomorphisms results in an equivalent category. The following results are proved in [7, 4.1-2, 5.2-4] (cf. 1.5).

**Theorem 5.1.** *The embedding  $\mathcal{C}^b(A^F) \rightarrow \mathcal{C}^b(A)$  sending  $X$  to  $X_k = X \otimes k$  induces a category equivalence  $\mathcal{C}^b(A^F) \cong \mathcal{C}^b(A)^F$  and faithful functors  $\Phi^{\mathcal{C}} : \mathcal{C}^b(A^F) \rightarrow$*

$\mathcal{C}^b(A)$  for  $\mathcal{C} = \mathcal{H}, \mathcal{D}$ . Moreover, these embeddings result in triangulated category equivalences

$$\mathcal{H}^b(A^F) \cong \mathcal{H}^b(A)^F \quad \text{and} \quad \mathcal{D}^b(A^F) \cong \mathcal{D}^b(A)^F.$$

- The root category level

Following [15, 5.1], the *root category*  $\mathcal{R}(A)$  of  $A$  is the quotient (or orbit) category  $\mathcal{D}^b(A)/(\mathcal{T}^2)$  of  $\mathcal{D}^b(A)$  by the automorphism  $\mathcal{T}^2$ , where  $\mathcal{T}$  is the shift functor on  $\mathcal{D}^b(A)$  induced from the one on  $\mathcal{C}(A)$ .<sup>2</sup> Thus, by definition, the objects in  $\mathcal{R}(A)$  are  $\mathcal{T}^2$ -orbits of objects in  $\mathcal{D}^b(A)$ , i. e.,  $\mathcal{O}_{\mathbf{M}} = \{\mathcal{T}^{2i}\mathbf{M} \mid i \in \mathbb{Z}\}$ ,  $\mathbf{M} \in \mathcal{D}^b(A)$ . A morphism  $f = (f_{ji}) : \mathcal{O}_{\mathbf{M}} \rightarrow \mathcal{O}_{\mathbf{N}}$  is given by morphisms  $f_{ji} : \mathcal{T}^{2i}\mathbf{M} \rightarrow \mathcal{T}^{2j}\mathbf{N}$  in  $\mathcal{D}^b(A)$  satisfying

- (1)  $\mathcal{T}(f_{ji}) = f_{j+1, i+1}$  for all  $i, j \in \mathbb{Z}$ ,
- (2) For each fixed  $i \in \mathbb{Z}$ , all but finitely many  $f_{ji}$  are zero.

The composition of the morphisms  $f : \mathcal{O}_{\mathbf{M}} \rightarrow \mathcal{O}_{\mathbf{N}}$  and  $g : \mathcal{O}_{\mathbf{N}} \rightarrow \mathcal{O}_{\mathbf{L}}$  is the morphism  $h = (h_{ji}) = gh$  with  $h_{ji} = \sum_{s \in \mathbb{Z}} g_{js} f_{si}$ .

It is easy to check that the Frobenius functor  $(\ )_{\mathcal{D}^b(A)}^{[1]}$  commutes with the shift functor  $\mathcal{T}$  and induces a functor

$$(\ )_{\mathcal{R}(A)}^{[1]} : \mathcal{R}(A) \longrightarrow \mathcal{R}(A).$$

Note that  $(\mathcal{O}_{\mathbf{M}})^{[1]} = \mathcal{O}_{\mathbf{M}^{[1]}}$  for each  $\mathbf{M}$  in  $\mathcal{D}^b(A)$  and  $f^{[1]} = (f_{ji}^{[1]})$  for a morphism  $f = (f_{ji})$  in  $\mathcal{R}(A)$ .

Let  $\mathcal{R}(A)^F$  be the category consisting of

$$\left\{ \begin{array}{l} \text{Objects: } \mathcal{O}_{\mathbf{M}} \text{ such that } \mathcal{O}_{\mathbf{M}}^{[1]} \stackrel{\phi_{\mathbf{M}}^{\mathcal{R}}}{\cong} \mathcal{O}_{\mathbf{M}} \text{ in } \mathcal{R}(A), \\ \text{Morphisms: } \text{Hom}_{\mathcal{R}(A)^F}(\mathbf{M}, \mathbf{N}) = \{\xi \in \text{Hom}_{\mathcal{R}(A)}(\mathbf{M}, \mathbf{N}) \mid \phi_{\mathbf{N}}^{\mathcal{R}} \circ \xi^{[1]} = \xi \circ \phi_{\mathbf{M}}^{\mathcal{R}}\}. \end{array} \right.$$

**Theorem 5.2.** *Let  $A$  be of finite global dimension. Then  $\Phi^{\mathcal{D}} : \mathcal{D}^b(A^F) \rightarrow \mathcal{D}^b(A)$  induces a faithful functor  $\mathcal{R}(A^F) \rightarrow \mathcal{R}(A)$  and a category equivalence*

$$\mathcal{R}(A^F) \cong \mathcal{R}(A)^F.$$

Let  $A$  be a hereditary (basic)  $k$ -algebra with a Frobenius map  $F$ . Then  $A$  and  $A^F$  are related by an ad-quiver  $(Q, \sigma)$  (Thm 3.4). So  $A$  can be identified as the path algebra of  $Q$  and  $A^F$  as the path (or tensor) algebra of the associated  $\mathbb{F}_q$ -species  $(\Gamma, \mathbb{M})$  of  $Q$  via  $\sigma$ . Now, using the notation introduced in the previous section, the theorem above together with Theorem 4.4 and [15, 4.7] implies immediately the following.

**Corollary 5.3.** *Suppose that  $A$  is hereditary (and basic). If  $\mathcal{O}_{\mathbf{M}}$  is an indecomposable object in  $\mathcal{R}(A)$ , then  $\hat{\sigma}(\mathbf{dim} \mathcal{O}_{\mathbf{M}}) = \mathbf{dim} \mathcal{O}_{\mathbf{X}}$ , where  $\mathbf{X} \in \mathcal{D}^b(A^F)$  with  $\mathbf{M} \cong \mathbf{X}_k$  in  $\mathcal{D}^b(A)$ , and the root system*

$$\Delta(\Gamma) = \{\hat{\sigma}(\mathbf{dim} \mathcal{O}_{\mathbf{M}}) \mid \mathbf{M} \text{ indecomposable in } \mathcal{D}^b(A)\}.$$

<sup>2</sup>The shift functor on  $\mathcal{C}$  is defined by  $(\mathcal{T}\mathbf{M})^i = \mathbf{M}^{i+1}$ ,  $d_{\mathcal{T}\mathbf{M}}^i = -d_{\mathbf{M}}^{i+1}$  and  $\mathcal{T}(f)^i = f^{i+1}$  if  $f$  is a morphism in  $\mathcal{C}(A)$ .

We remark that it is known that the results given in [15, 4.7], [16] are over an algebraically closed field and continue to hold for a species over a finite field. However, the argument in the latter case is not given there. Using Frobenius morphisms, the argument to the theory is now complete and elegant.

- Frobenius maps on complexes

Let  $\mathbf{M} = (M^i, d^i)$  be a complex in  $\mathcal{C}(A)$  and let  $\mathcal{F} := \{F_i : M^i \rightarrow M^i \mid i \in \mathbb{Z}\}$  be a family of Frobenius maps. We shall call  $\mathcal{F}$  a *Frobenius map* on  $\mathbf{M}$ . For each  $i \in \mathbb{Z}$ , let  $(M^i)^{[F_i]}$  denote the  $F_i$ -twist of  $M^i$ . Then each  $d^i : M^i \rightarrow M^{i+1}$  gives an  $A$ -module homomorphism (see (1.2.1) for the notation)

$$d^{i[F]} = F_{i+1} \circ d_M^i \circ F_i^{-1} : (M^i)^{[F_i]} \rightarrow (M^{i+1})^{[F_{i+1}]}.$$

Thus we obtain a complex  $((M^i)^{[F_i]}, d^{i[F]})$ , which is called the  $\mathcal{F}$ -twist of  $\mathbf{M}$  and is denoted by  $\mathbf{M}^{[\mathcal{F}]}$ .

There is a complex version of 1.3.

**Lemma 5.4.** *Let  $\mathbf{M} = (M^i, d^i)$  be a complex in  $\mathcal{C}(A)$ . Then  $\mathbf{M} \cong \mathbf{M}^{[r]}$  if and only if there exists a Frobenius map  $\mathcal{F} = \{F_i : M^i \rightarrow M^i \mid i \in \mathbb{Z}\}$  on  $\mathbf{M}$  such that  $\mathbf{M}^{[\mathcal{F}^r]} = \mathbf{M}$  as complexes of  $A$ -modules.*

If  $\mathbf{M} \cong \mathbf{M}^{[1]}$  then  $\mathbf{M}$  is called an *F-stable complex*. The method of constructing  $F$ -stable modules from  $F$ -periodic modules given at the end of §2 can be generalized to complexes. Let  $\mathbf{M}$  be an  $F$ -periodic complex in  $\mathcal{C}(A)$  with  $F$ -period  $r$ , i.e.,  $\mathbf{M} \cong \mathbf{M}^{[r]}$  in  $\mathcal{C}(A)$  with  $r$  minimal. By Lemma 5.4, there is a Frobenius map  $\mathcal{F} = \{F_i : M^i \rightarrow M^i \mid i \in \mathbb{Z}\}$  such that  $\mathbf{M} = \mathbf{M}^{[\mathcal{F}^r]}$  as complexes. For each  $i$ , let

$$\tilde{M}^i = M^i \oplus (M^i)^{[F_i]} \oplus \cdots \oplus (M^i)^{[F_i^{r-1}]}$$

and define a Frobenius map  $\tilde{F}_i : \tilde{M}^i \rightarrow \tilde{M}^i$  by

$$(5.4.1) \quad \tilde{F}_i(x_0, x_1, \dots, x_{r-1}) = (F_i(x_{r-1}), F_i(x_0), \dots, F_i(x_{r-2})).$$

Further, let  $\tilde{d}^i = \text{diag}(d^i, (d^i)^{[F_i]}, \dots, (d^i)^{[F_i^{r-1}]}) : \tilde{M}^i \rightarrow \tilde{M}^{i+1}$ . Then we obtain an  $F$ -stable complex  $\tilde{\mathbf{M}} = (\tilde{M}^i, \tilde{d}^i)$  satisfying  $\tilde{\mathbf{M}}^{[\tilde{\mathcal{F}}]} = \tilde{\mathbf{M}}$ , where  $\tilde{\mathcal{F}} = \{\tilde{F}_i \mid i \in \mathbb{Z}\}$ . This gives a complex  $\tilde{\mathbf{M}}^{\tilde{\mathcal{F}}}$  in  $\mathcal{C}(A^F)$ . Since every bounded complex has a finite  $F$ -period, this construction applies to every object in  $\mathcal{C}^b(A)$ . Moreover, every complex is isomorphic to one containing no non-zero *contractible* summands in the homotopy category  $\mathcal{K}(A)$ , the same construction applies to every object in  $\mathcal{K}^b(A)$  which contains no non-zero *contractible* summands. Thus the first part of the following theorem generalizes Theorem 3.2 (see [6, 5.1] and [7, 4.4]), while the second part is given in [7, 5.1, 5.6].

**Theorem 5.5.** *Maintain the notation above. Let  $\mathbf{M}$  be an indecomposable complex in  $\mathcal{C}^b(A)$  with  $F_{\mathcal{C}}$ -period  $r$ , where  $\mathcal{C} \in \{\mathcal{C}, \mathcal{K}, \mathcal{D}\}$ .*

(1) *If  $\mathcal{C} = \mathcal{C}$  or  $\mathcal{K}$ ,  $\tilde{\mathbf{M}}^{\tilde{\mathcal{F}}}$  is indecomposable in  $\mathcal{C}^b(A^F)$  and*

$$\text{End}_{\mathcal{C}^b(A^F)}(\tilde{\mathbf{M}}^{\tilde{\mathcal{F}}}) / \text{Rad}(\text{End}_{\mathcal{C}^b(A^F)}(\tilde{\mathbf{M}}^{\tilde{\mathcal{F}}})) \cong \text{End}_{\mathcal{C}^b(A)^F}(\tilde{\mathbf{M}}) / \text{Rad}(\text{End}_{\mathcal{C}^b(A)^F}(\tilde{\mathbf{M}})) \cong \mathbb{F}_{q^r}.$$

*Moreover, every indecomposable complex in  $\mathcal{C}^b(A^F)$  is isomorphic to a complex of the form  $\tilde{\mathbf{M}}^{\tilde{\mathcal{F}}}$  for some  $F$ -periodic indecomposable complex  $\mathbf{M}$  in  $\mathcal{C}^b(A)$ .*

(2) Suppose in addition that  $A$  has a finite global dimension. Then there exists  $X \in \mathcal{D}^b(A^F)$  such that  $\tilde{M} \cong X_k$  in  $\mathcal{D}^b(A)$  and

$$\text{End}_{\mathcal{D}^b(A^F)}(X)/\text{Rad}(\text{End}_{\mathcal{D}^b(A^F)}(X)) \cong \text{End}_{\mathcal{D}^b(A)^F}(\tilde{M})/\text{Rad}(\text{End}_{\mathcal{D}^b(A)^F}(\tilde{M})) \cong \mathbb{F}_{q^r}.$$

Hence  $X$  is indecomposable. Moreover, every indecomposable object in  $\mathcal{D}^b(A^F)$  can be obtained in this way.

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