QUANTUM $\mathfrak{gl}_n$, q-SCHUR ALGEBRAS AND THEIR INFINITE/INFINITESIMAL COUNTERPARTS

JIE DU AND QIANG FU

Abstract. We present a survey of recent developments of the Beilinson–Lusztig–MacPherson approach in the study of quantum $\mathfrak{gl}_n$, infinitesimal quantum $\mathfrak{gl}_n$, quantum $\mathfrak{gl}_\infty$ and their associated q-Schur algebras, little q-Schur algebras and infinite q-Schur algebras. We also use the relationship between quantum $\mathfrak{gl}_\infty$ and infinite q-Schur algebras to discuss their representations.

1. Introduction

Almost at the same time when C.M. Ringel discovered the Hall algebra realization [24] of the positive part of the quantum enveloping algebras associated with finite dimensional semisimple complex Lie algebras, A.A. Beilinson, G. Lusztig and R. MacPherson discovered a realization [1] for the entire quantum $\mathfrak{gl}_n$ via a geometric setting of quantum Schur algebras (or q-Schur algebras). This remarkable work has many applications. For example, it provides a crude model for the introduction of modified quantum groups as introduced in [21], it leads to the settlement of the integral Schur–Weyl reciprocity and, hence, the reciprocity at any root of unity [6], [11], and it has also provided a geometric approach to study quantum affine $\mathfrak{gl}_n$ [14], [22]. The BLM work has also been used to investigate the presentations of q-Schur algebras [9], infinitesimal quantum $\mathfrak{gl}_n$ and their associated little q-Schur algebras [8], and quantum $\mathfrak{gl}_\infty$, infinite q-Schur algebras and their representations [7]. This paper presents a brief account of these developments.

We organize the paper as follows. §2 collects the definitions of quantum $\mathfrak{gl}_\eta$ for any consecutive (finite or infinite) segment η of $\mathbb{Z}$, the q-Schur algebras, and the infinitesimal quantum $\mathfrak{gl}_\eta$. In §3, we generalize the geometric setting in [1] for q-Schur algebras to introduce the algebras $\mathcal{K}(\eta, r)$, for any η, and discuss the stabilization property for $\mathcal{K}(\eta, r)$. This property allows us to define a new algebra $\boldsymbol{\mathcal{K}}(\eta)$ over $\mathbb{Q}(v)$ of which a certain completion $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ of $\boldsymbol{\mathcal{K}}(\eta)$ contains a subalgebra isomorphic to the quantum group $\mathbf{U}(\eta) = \mathbf{U}_v(\mathfrak{gl}_\eta)$. This is the BLM realization which is discussed in §4. When η is an infinite segment, the completion $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$ of $\boldsymbol{\mathcal{K}}(\eta, r)$ contains a subalgebra $\mathbf{V}(\eta, r)$ which is isomorphic to a homomorphic image $\mathbf{U}(\eta, r)$ of $\mathbf{U}(\eta)$. In §5, we use the integral version $\mathcal{K}(\eta)$ of $\boldsymbol{\mathcal{K}}(\eta)$ to obtain its specialization $\mathcal{K}(\eta)_k$ at a root of unity, and then to construct a subalgebra $\mathcal{W}$ of the completion $\widehat{\mathcal{K}}(\eta)_k$ of $\mathcal{K}(\eta)_k$. The algebra $\mathcal{W}$ is isomorphic to the infinitesimal quantum $\mathfrak{gl}_\eta$. From §6 onwards, we present various applications. In §6, q-Schur algebras are investigated via their presentations. In particular, we display several bases, and mention a nice application to Hecke algebras.

Date: October 12, 2007.

2000 *Mathematics Subject Classification.* 17B20, 17B35, 20G15.

Supported by the Australian Research Council (Grant: DP 0665124). The paper was written while the second author was visiting the University of New South Wales.

Parallel to the infinitesimal theory for quantum $\mathfrak{gl}_n$, little q-Schur algebras are discussed in §7. For the rest of the paper, we focus on the case when $\eta = \mathbb{Z}$. In §8, we discuss presentations of the algebras $\mathrm{U}(\infty, r)$. Infinite q-Schur algebras $\boldsymbol{\mathcal{S}}(\infty, r)$ are introduced and the relations between $\boldsymbol{\mathcal{S}}(\infty, r)$ and quantum $\mathfrak{gl}_\infty$ are discussed in §9. In particular, we show that $\mathbf{U}(\infty, r)$ is a proper subalgebra of $\boldsymbol{\mathcal{S}}(\infty, r)$. We end the paper by discussions on the representation theory of quantum $\mathfrak{gl}_\infty$. We discuss the highest weight representations in §10 and their polynomial type representations in §11.

Acknowledgement. The first author would like to thank the organizers for the opportunity to attend the conference and for the financial support.

2. QUANTUM $\mathfrak{gl}_\eta$, INFINITESIMAL QUANTUM $\mathfrak{gl}_\eta$, AND q-SCHUR ALGEBRAS

Let $\eta \subseteq \mathbb{Z}$ be a consecutive segment of $\mathbb{Z}$. In other words, η is a subset of the form $[m, n] := \{i \in \mathbb{Z} \mid m \leqslant i \leqslant n\}$ or one of the sets $(\infty, m], [n, \infty)$ and $\mathbb{Z}$, where $m, n \in \mathbb{Z}$, Let $\eta^{\dashv} = \eta \backslash \{\max(\eta)\}$ if η is bounded above, and $\eta^{\dashv} = \eta$ otherwise. We also denote by $M_\eta(R)$ (resp. R^η) the set of all matrices $(a_{i,j})_{i,j\in\eta}$ (resp. all sequences $(a_i)_{i\in\eta}$) with all entries in a set R of numbers, and will always abbreviate the sub-/supscript η by n if $\eta = [1, n]$, and by ∞ if $\eta = \mathbb{Z}$. Moreover, we also assume that, if η is infinite, then the elements in $M_\eta(R)$ (resp., R^η) have *finite support*, i. e., all $a_{i,j} = 0$ (resp. $a_i = 0$) but finitely many of them.

Let $\mathbf{U}(\eta) := \mathbf{U}_v(\mathfrak{gl}_\eta)$ be quantum $\mathfrak{gl}_\eta$ defined over $\mathbb{Q}(v)$. Then, $\mathbf{U}(\eta)$ is the algebra over $\mathbb{Q}(v)$ presented by generators

$$E_i,\ F_i \quad (i \in \eta^{\dashv}),\ K_j,\ K_j^{-1} \quad (j \in \eta)$$

and relations

(a) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1;$

(b) $K_iE_j = v^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i;$

(c) $K_iF_j = v^{\delta_{i,j+1}-\delta_{i,j}}F_jK_i;$

(d) $E_iE_j = E_jE_i,\ F_iF_j = F_jF_i$ *when* $|i-j| > 1;$

(e) $E_iF_j - F_jE_i = \delta_{i,j}\frac{\widetilde{K}_i-\widetilde{K}_i^{-1}}{v-v^{-1}},$ *where* $\widetilde{K}_i = K_iK_{i+1}^{-1};$

(f) $E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0$ *when* $|i-j| = 1;$

(g) $F_i^2F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0$ *when* $|i-j| = 1.$

When η is an infinite segment, $\mathbf{U}(\eta)$ can be regarded as the quantum enveloping algebra associated with an $\infty \times \infty$ Cartan matrix.

We also set $\mathbf{U}(\eta) = \begin{cases} \mathbf{U}(n), & \text{if } \eta = [1, n]; \\ \mathbf{U}(\infty), & \text{if } \eta = \mathbb{Z}. \end{cases}$ Clearly, we have natural algebra embedding $\mathbf{U}([-n, n]) \subseteq \mathbf{U}([-n-1, n+1])$ for all $n \geqslant 0$. Hence, obtain an algebra isomorphism

$$\mathbf{U}(\infty) \cong \varinjlim_{n} \mathbf{U}([-n, n]) \tag{2.0.1}$$

We will see in §4 that $\mathbf{U}(\eta)$ can be reconstructed as a vector space together with a given basis and certain explicit multiplication formulas between basis elements and generators.

Let $\mathscr{Z} = \mathbb{Z}[v, v^{-1}]$ and let $U(\eta)^0$ (resp., $U(\eta)^+$, $U(\eta)^-$) be the $\mathscr{Z}$-subalgebra of $\mathbf{U}(\eta)$ generated by all K_j, K_j^{-1}, $\begin{bmatrix} K_j; c \\ t \end{bmatrix}$ (resp., $E_i^{(m)}$, $F_i^{(m)}$), where

$$E_i^{(m)} = \frac{E_i^m}{[m]^!},\ F_i^{(m)} = \frac{F_i^m}{[m]^!},\ \text{and } \begin{bmatrix} K_j; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_j v^{c-s+1} - K_j^{-1} v^{-c+s-1}}{v^s - v^{-s}}$$

with $[m]^! = [1][2]\cdots[m]$ and $[t] = \frac{v^t - v^{-t}}{v - v^{-1}}$. Also, let $U(\eta) = U(\eta)^+ U(\eta)^0 U(\eta)^-$. By [25, §3] and [18, 2.3(g9)(g10)], all $\begin{bmatrix} \tilde{K}_j; c \\ t \end{bmatrix} \in U(\eta)$. Thus, $U(\eta)$ is a $\mathscr{Z}$-subalgebra of $\mathbf{U}(\eta)$ and there is a triangular decomposition

$$U(\eta) = U(\eta)^+ \otimes U(\eta)^0 \otimes U(\eta)^-.$$

Let Ω_η be a free $\mathscr{Z}$-module with basis $\{\omega_i\}_{i\in\eta}$. Let $\mathbf{\Omega}_\eta = \Omega_\eta \otimes \mathbb{Q}(v)$. Then $\mathbf{U}(\eta)$ acts on $\mathbf{\Omega}_\eta$ naturally defined by

$$K_a \omega_b = v^{\delta_{a,b}} \omega_b \,(a, b \in \eta), \quad E_a \omega_b = \delta_{a+1,b} \omega_a, \quad F_a \omega_b = \delta_{a,b} \omega_{a+1} \,(a \in \eta^{\dashv}, b \in \eta).$$

This action extends to the tensor space $\mathbf{\Omega}_\eta^{\otimes r}$ ($r \geqslant 1$) via the coalgebra structure Δ on $\mathbf{U}(\eta)$ defined by

$$\Delta(E_i) = E_i \otimes \widetilde{K}_i + 1 \otimes E_i,\ \Delta(F_i) = F_i \otimes 1 + \widetilde{K}_i^{-1} \otimes F_i,\ \Delta(K_j) = K_j \otimes K_j.$$

Thus, we have a $\mathbb{Q}(v)$-algebra homomorphism

$$\zeta_r : \mathbf{U}(\eta) \to \mathrm{End}(\mathbf{\Omega}_\eta^{\otimes r}). \tag{2.0.2}$$

Restriction induces a $\mathscr{Z}$-algebra homomorphism

$$\zeta_r|_{U(\eta)} : U(\eta) \to \mathrm{End}(\Omega_\eta^{\otimes r}). \tag{2.0.3}$$

Let

$$\mathbf{U}(\eta, r) = \mathrm{im}(\zeta_r) \quad \text{and} \quad U(\eta, r) = \mathrm{im}(\zeta_r|_{U(\eta)}). \tag{2.0.4}$$

If η is *finite* both $\mathbf{U}(\eta, r)$ and $U(\eta, r)$ are called *q-Schur algebras at* η; see [16]. Note that q-Schur algebras can be described as the endomorphism of the tensor space regarded as a module of the Hecke algebra associated with the symmetric groups on r letters (see §9 for more details).

Let $U(\eta)_k = U(\eta) \otimes_{\mathscr{Z}} k$, where k is a field containing an l'th primitive root ε of 1 ($l' \geqslant 3$), and is regarded as an $\mathscr{Z}$-module via the specialization of v to ε. We will denote the images of $E_i^{(m)} \otimes 1$ etc. in $U(\eta)_k$ by the same letters. Clearly, (2.0.1) induces a k-algebra isomorphism

$$U(\infty)_k = \varinjlim_n U([-n, n])_k \tag{2.0.5}$$

We now follow [18] to introduce infinitesimal quantum $\mathfrak{gl}_n$. Let $\tilde{u}_k(\eta)$ be the k-subalgebra of $U(\eta)_k$ generated by the elements E_i, F_i, $K_j^{\pm 1}$ for all i, j, and define $\tilde{u}_k(\eta)^+$, $\tilde{u}_k(\eta)^0$ and $\tilde{u}_k(\eta)^-$. Similarly, we have an inherited triangular decomposition

$$\tilde{u}_k(\eta) = \tilde{u}_k(\eta)^+ \tilde{u}_k(\eta)^0 \tilde{u}_k(\eta)^- \cong \tilde{u}_k(\eta)^+ \otimes \tilde{u}_k(\eta)^0 \otimes \tilde{u}_k(\eta)^-.$$

Moreover, we have

$$\tilde{u}_k(\infty) = \varinjlim_{n} \tilde{u}_k([-n,n]) \tag{2.0.6}$$

We may further introduce the so-called *infinitesimal quantum group* $u_k(\eta)$ when l' is odd. Let

$$l = \begin{cases} l', & \text{if } l' \text{ is odd} \\ l'/2, & \text{if } l' \text{ is even} \end{cases}. \tag{2.0.7}$$

Thus, $\varepsilon^l - \varepsilon^{-l} = 0$ and further $E_i^l = 0 = F_i^l$ and $K_j^{2l} = 1$ in $U(\eta)_k$. *If l' is odd*, then the elements $K_i^l - 1$ ($i \in \eta$) are central in $\tilde{u}_k(\eta)$ (and in $U(\eta)_k$). These elements generate an ideal $\langle K_i^l - 1 \mid i \in \eta \rangle$ of $\tilde{u}_k(\eta)$. Define

$$u_k(\eta) = \tilde{u}_k(\eta)/\langle K_i^l - 1 \mid i \in \eta \rangle.$$

Call $u_k(\eta)$ the *infinitesimal quantum* $\mathfrak{gl}_\eta$.

An alternative way to see the algebra $\tilde{u}_k(\eta)$ and $u_k(\eta)$ is using the DeConcini–Kac quantum group (see [2]). Let $\mathcal{U}_k(\eta)$ be the algebra over k with generators E_i, F_i and $K_j^{\pm 1}$ ($i \in \eta^{\dashv}, j \in \eta$) and the relations (a)-(g) where υ is replaced by ε (noting our assumption on l' in the definition of $u_k(\eta)$). This algebra is a $\mathfrak{gl}_\eta$ version of a DeConcini–Kac quantum group. Clearly, there is an algebra homomorphism $\delta : \mathcal{U}_k(\eta) \to U(\eta)_k$ mapping the generators of $\mathcal{U}_k(\eta)$ to their counterparts in $U(\eta)_k$. The image of δ is the algebra $\tilde{u}_k(\eta)$.

The following result is given in [8, 2.5] and [13, 2.1] (the proof for an infinite η is entirely similar).

Proposition 2.1. (1) *Let I be the (two-sided) ideal of $\mathcal{U}_k(\eta)$ generated by E_i^l, F_i^l, $K_j^{2l}-1$, $i \in \eta^{\dashv}$, $j \in \eta$. Then there is an algebra isomorphism*

$$\mathcal{U}_k(\eta)/I \xrightarrow{\sim} \tilde{u}_k(\eta).$$

In other words, $\tilde{u}_k(\eta)$ is the k-algebra defined by generators

$$E_i,\ F_i,\ K_j\ (i \in \eta^{\dashv},\ j \in \eta)$$

and relations (a)–(g) with υ replaced by ε, together with the relations:
($\tilde{h}$) *$E_i^l = 0$, $F_i^l = 0$ and $K_j^{2l} = 1$.*

(2) *If l' be odd, then $u_k(\eta)$ is the k-algebra defined by generators*

$$E_i,\ F_i,\ K_j\ (i \in \eta^{\dashv},\ j \in \eta)$$

and relations (a)–(g) with υ replaced by ε, together with the relations:
(*h*) *$E_i^l = 0$, $F_i^l = 0$ and $K_j^l = 1$.*

In §5, we will discuss the realizations of $\tilde{u}_k(\eta)$ and $u_k(\eta)$.

3. The algebra $\mathcal{K}(\eta,r)$ and the stabilization property

For a matrix $A=(a_{ij})\in M_\eta(\mathbb{N})$ (resp. $\mathbf{j}=(j_i)\in\mathbb{N}^\eta$), let $|A|=\sum_{i,j}a_{ij}$ (resp., $|\mathbf{j}|=\sum_i j_i$). Let $\Xi(\eta)=M_\eta(\mathbb{N})$, $\Xi(\eta,r)=\{A\in\Xi(\eta)\mid r=|A|\}$ and $\widetilde{\Xi}(\eta)=\{(a_{ij})\in M_\eta(\mathbb{Z}):a_{ij}\geqslant 0\ \forall i\neq j\}$.

Let V be a vector space of dimension r over a field k. An η-step flag is a collection $\mathfrak{f}=(V_i)_{i\in\eta}$ of subspaces of V such that $V=\cup_{i\in\eta}V_i$, $V_i\subseteq V_{i+1}$ for all $i\in\eta^{\dashv}$ and $V_i=0$ for $i\ll 0$ if η has no lower bound. Let $\mathcal{F}$ be the set of η-step flags. The group $G=G(k):=GL(V)$ acts naturally on $\mathcal{F}$ with orbits being the fibres of the map $\mathcal{F}\to\Lambda(\eta,r):=\{\lambda\in\mathbb{N}^\eta\mid r=|\lambda|\}$ given by

$$(V_i)_{i\in\eta}\mapsto(\dim V_i/V_{i-1})_{i\in\eta},$$

where $V_{\min(\eta)-1}=0$ if η is bounded below. If $\mathcal{F}_\lambda$ denotes the inverse image of $\lambda\in\Lambda(\eta,r)$, then $\mathcal{F}=\cup_{\lambda\in\Lambda(\eta,r)}\mathcal{F}_\lambda$, a disjoint union of G-orbits. If $\mathfrak{f}\in\mathcal{F}_\lambda$ and P_λ is the stabilizer of $\mathfrak{f}$ in G, then $\mathcal{F}_\lambda\cong G/P_\lambda$.

Let $\mathcal{V}=\mathcal{V}(k)=\mathcal{F}\times\mathcal{F}$ and let G act on $\mathcal{V}$ diagonally. For $(\mathfrak{f},\mathfrak{f}')\in\mathcal{V}$ where $\mathfrak{f}=(V_i)_{i\in\eta}$ and $\mathfrak{f}'=(V_i')_{i\in\eta}$, the subspaces

$$X_{i,j}=X_{i,j}(\mathfrak{f},\mathfrak{f}')=V_{i-1}+(V_i\cap V_j')$$

(where $i,j\in\eta$) form an $(\eta\times\eta)$-flag:

$$\cdots\subseteq X_{i,j}\subseteq X_{i,j+1}\cdots\subseteq X_{i+1,j}\subseteq X_{i+1,j+1}\cdots\subseteq V.$$

Let $a_{i,j}=\dim X_{i,j}/X_{i,j-1}$. Setting $\Psi(\mathfrak{f},\mathfrak{f}')=(a_{i,j})$ defines a map $\Psi:\mathcal{V}\to\Xi(\eta)$. Then $\Xi(\eta,r)=\operatorname{im}\Psi$. The G-orbits $\mathcal{O}_A$, $A\in\Xi(\eta,r)$, in $\mathcal{V}$ are the fibres of Ψ.

When $k=\mathbb{F}_q$ is a finite field of q elements, then the action of $G(q):=G(k)$ on $\mathcal{V}(q):=\mathcal{V}(k)$ induces a permutation module $\mathbb{Z}\mathcal{V}(q)$. Let $\mathcal{E}_{\eta,r}(q)$ be the $\mathbb{Z}$-algebra with basis $\{e_A\}_{A\in\Xi(\eta,r)}$ and multiplication: for $A,A'\in\Xi(\eta,r)$,

$$e_A\cdot e_{A'}=\sum_{A''\in\Xi(\eta,r)}c_{A,A',A''}e_{A''}$$

where, for fixed $(\mathfrak{f}_1,\mathfrak{f}_2)\in\mathcal{O}_{A''}$,

$$c_{A,A',A''}=\{\mathfrak{f}\in\mathcal{V}(q)\mid(\mathfrak{f}_1,\mathfrak{f})\in\mathcal{O}_A,(\mathfrak{f},\mathfrak{f}_2)\in\mathcal{O}_{A'}\}.$$

Note that, if η is finite, then

$$\mathcal{E}_{\eta,r}(q)=\operatorname{End}_{\mathbb{Z}G(q)}(\mathbb{Z}\mathcal{V}(q))^{\rm op}.$$

Thus, $\mathcal{E}_{\eta,r}(q)$ has an identity. This is not the case if η is infinite.

It is well known that there are polynomials $f_{A,A',A''}$ such that $f_{A,A',A''}(q)=c_{A,A',A''}$ for all $A,A',A''\in\Xi(\eta,r)$. Thus, we define a $\mathcal{Z}$-algebra $\mathcal{K}(\eta,r)$ with basis $\{\phi_B\}_{B\in\Xi(\eta,r)}$ and multiplication

$$\phi_A\cdot\phi_{A'}=\sum_{A''}f_{A,A',A''}(v^2)\phi_{A''}.$$

For any $n\geqslant 1$, we may embed $\Xi([-n,n],r)$ into $\Xi([-n-1,n+1],r)$ by adding zeros in rows and columns labelled with $-n-1$ or $n+1$. This induces an algebra embedding $\mathcal{K}([-n,n],r)$ into $\mathcal{K}([-n-1,n+1],r)$ as a centralizer subalgebra $e\mathcal{K}([-n-1,n+1],r)e$ for an idempotent e. Thus, we obtain a direct system $\{\mathcal{K}([-n,n],r)\}_{n\geqslant 1}$.

Lemma 3.1. (1) *Let* $\mathbb{Z}[\sqrt{q},\sqrt{q}^{-1}]$ *be the subring of* $\mathbb{C}$ *generated by* $\mathbb{Z}$ *and* $\sqrt{q},\sqrt{q}^{-1}$*, and let* $\mathcal{K}(\eta,r)|_{\sqrt{q}}$ *be the specialization of* $\mathcal{K}(\eta,r)$ *at* $v=\sqrt{q}$*. Then we have an algebra isomorphism*

$$\mathcal{K}(\eta,r)|_{\sqrt{q}}\cong\mathcal{E}_{\eta,r}(q)\otimes_{\mathbb{Z}}\mathbb{Z}[\sqrt{q},\sqrt{q}^{-1}].$$

(2) *We have an algebra isomorphism:*

$$\mathcal{K}(\infty,r)=\varinjlim_{n}\mathcal{K}([-n,n],r).$$

For any $A=(a_{i,j})\in\widetilde{\Xi}(\eta)$, let $\mathrm{row}\,(A)=\big(\sum_{j\in\eta}a_{i,j}\big)_{i\in\eta}$ and $\mathrm{col}\,(A)=\big(\sum_{i\in\eta}a_{i,j}\big)_{j\in\eta}$ be the sequences of row and column sums of A. Note that $\mathrm{row}\,(A)$ and $\mathrm{col}\,(A)$ are well-defined for any $A\in M_\infty(\mathbb{N})$ since A has finite support.

Let $\mathcal{Z}_1$ be the subring of $\mathbb{Q}(v)[v']$ generated by v^j $(j\in\mathbb{Z})$ and the elements

$$\prod_{i\in[1,t]}\frac{v^{-2(a-i)}v'^2-1}{v^{-2i}-1}$$

for all $a\in\mathbb{Z}$ and $t\geqslant 1$. For $n\geqslant n_0$ and $A\in\Xi([-n_0,n_0])$, define $A^{[-n,n]}\in\Xi([-n,n])$ by adding zeros at the (i,j)-position for all $i<-n_0$ or $i>n_0$ or $j<-n_0$ or $j>n_0$.

The following stabilization property is a slight modification of the result [1, 4.2] for q-Schur algebras; see [7, §3] for more details.

Theorem 3.2. *For* $B\in\Xi(\eta,r)$ *let* $d(B)=\sum_{i\geqslant k,j<l}b_{ij}b_{kl}$*. The basis* $\{[B]:=v^{-d_B}\phi_B\}_{B\in\Xi(\eta,r)}$ *for* $\mathcal{K}(\eta,r)$ *satisfies the following property: if* $A,B\in\widetilde{\Xi}([-n_0,n_0])$ *for some* $n_0\geqslant 1$ *with* $\mathrm{col}\,(A)=\mathrm{row}\,(B)$*, then there exist* $g_{A,B,C}(v,v')\in\mathcal{Z}_1$ *with* $C\in\widetilde{\Xi}([-n_0,n_0])$ *such that*

$$[{}_pA^{[-n,n]}]\cdot[{}_pB^{[-n,n]}]=\sum_{C\in\widetilde{\Xi}([-n_0,n_0])}g_{A,B,C}(v,v^{-p})[{}_pC^{[-n,n]}]$$

for all large p *and* $n\geqslant n_0$*, where* ${}_pC^{[-n,n]}=pI_{[-n,n]}+C^{[-n,n]}$ *and* $I_{[-n,n]}$ *is the identity matrix in* $\Xi[-n,n]$*.*

4. The BLM construction of $\mathbf{U}(\eta)$

Beilinson–Lusztig–MacPherson [1] used the stabilization property of the q-Schur algebras to define an algebra over $\mathcal{Z}_1$ with a basis indexed by $\widetilde{\Xi}(n)$, and then, by specializing v' to 1, to obtain an algebra $\mathcal{K}$ free over $\mathcal{Z}$ with a basis $\{[A]\}_{A\in\widetilde{\Xi}(n)}$. The quantum group $\mathbf{U}(n)$ is realized as a subalgebra of a certain completion of $\mathbf{K}:=\mathcal{K}\otimes\mathbb{Q}(v)$. In this section, we state the result with respect to a consecutive segment η of $\mathbb{Z}$. We refer [7] for the treatment of the case when $\eta=\mathbb{Z}$.

Let $\mathcal{K}_\eta(v,v')$ be the free $\mathcal{Z}_1$-module with basis $\{A\mid A\in\widetilde{\Xi}(\eta)\}$. Define a multiplication $\cdot$ on $\mathcal{K}_\eta(v,v')$ by linearly extending the products on basis elements:

$$A\cdot A':=\begin{cases}\sum\limits_{A''\in\Xi(\eta)}g_{A,A',A''}A'', & \text{if } \mathrm{col}\,(A)=\mathrm{row}\,(A')\\ 0, & \text{otherwise}\end{cases}$$

for all $A,A'\in\widetilde{\Xi}(\eta)$. $\mathcal{K}_\eta(v,v')$ is an associative algebra without 1.

Let $\mathcal{K}(\eta) = \mathcal{K}_\eta(v, v') \otimes_{\mathcal{Z}_1} \mathcal{Z}$ and $\boldsymbol{\mathcal{K}}(\eta) = \mathcal{K}_\eta(v, v') \otimes_{\mathcal{Z}_1} \mathbb{Q}(v)$ where $v' \mapsto 1$. Then $\mathcal{K}(\eta)$ and $\boldsymbol{\mathcal{K}}(\eta)$ are associative algebras with basis $\{[A] := A \otimes 1 \mid A \in \widetilde{\Xi}(\eta)\}$. We also have

$$\mathcal{K}(\infty) = \varinjlim_n \mathcal{K}([-n, n]), \qquad \boldsymbol{\mathcal{K}}(\infty) = \varinjlim_n \boldsymbol{\mathcal{K}}([-n, n]). \tag{4.0.1}$$

Let $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ be the vector space of all formal (possibly infinite) $\mathbb{Q}(v)$-linear combinations $\sum_{A \in \widetilde{\Xi}(\eta)} \beta_A [A]$ which have the following properties: for any $\mathbf{x} \in \mathbb{Z}^\eta$,

$$\text{the sets } \begin{matrix} \{A \in \widetilde{\Xi}(\eta) \mid \beta_A \neq 0, \ \mathrm{row}\,(A) = \mathbf{x}\} \\ \{A \in \widetilde{\Xi}(\eta) \mid \beta_A \neq 0, \ \mathrm{col}\,(A) = \mathbf{x}\} \end{matrix} \text{ are finite.} \tag{4.0.2}$$

In other words, for any $\lambda, \mu \in \mathbb{Z}^\eta$, the sums $\sum_{A \in \widetilde{\Xi}(\eta)} \beta_A [\mathrm{diag}(\lambda)] \cdot [A]$ and $\sum_{A \in \widetilde{\Xi}(\eta)} \beta_A [A] \cdot [\mathrm{diag}(\mu)]$ are finite. We can define the product of two elements $\sum_{A \in \widetilde{\Xi}(\eta)} \beta_A [A]$, $\sum_{B \in \widetilde{\Xi}(\eta)} \gamma_B [B]$ in $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ to be $\sum_{A,B} \beta_A \gamma_B [A] \cdot [B]$. This defines an associative algebra structure on $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$. This algebra has an identity element $\sum_{\lambda \in \mathbb{Z}^\eta} [\mathrm{diag}(\lambda)]$, the sum of all $[D]$ with D a diagonal matrix in $\widetilde{\Xi}(\eta)$. $\boldsymbol{\mathcal{K}}(\eta)$ is naturally a subalgebra of $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ (without 1).

Note that one may define $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$ similarly. However, if $|\eta| < \infty$, then $\boldsymbol{\mathcal{K}}(\eta, r)$ is finite dimensional and $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r) = \boldsymbol{\mathcal{K}}(\eta, r)$. If $\eta = \mathbb{Z}$, then $\boldsymbol{\mathcal{K}}(\infty, r)$ is infinite dimensional. We will see in §9 that the algebra $\mathbf{U}(\infty, r)$ defined in (2.0.4) is embedded in the completion algebra $\widehat{\boldsymbol{\mathcal{K}}}(\infty, r)$.

Let $\Xi^\pm(\eta)$ be the set of all $A \in \Xi(\eta)$ whose diagonal entries are zero, and let $\Xi^+(\eta)$ (resp., $\Xi^-(\eta)$, $\widetilde{\Xi}(\eta)^0$) denote the subset of $\widetilde{\Xi}(\eta)$ consisting of those matrices $(a_{i,j})$ with $a_{i,j} = 0$ for all $i \geqslant j$ (resp. $i \leqslant j$, $i \neq j$). For any $A \in \Xi(\eta)$, there exist unique $A^+ \in \Xi^+(\eta)$, $A^- \in \Xi^-(\eta)$ and $A^0 \in \Xi(\eta)^0$ such that $A = A^+ + A^0 + A^-$.

Given $r > 0$, $A \in \Xi^\pm(\eta)$ and $\mathbf{j} = (j_i)_{i \in \eta} \in \mathbb{Z}^\eta$, we define

$$A(\mathbf{j}) = A(\mathbf{j})_\eta = \sum_{D \in \widetilde{\Xi}(\eta)^0} v^{\sum_i d_i j_i} [A + D] \in \widehat{\boldsymbol{\mathcal{K}}}(\eta),$$

$$A(\mathbf{j}, r) = A(\mathbf{j}, r)_\eta = \sum_{\substack{D \in \Xi(\eta)^0 \\ |A+D| = r}} v^{\sum_i d_i j_i} [A + D] \in \widehat{\boldsymbol{\mathcal{K}}}(\eta, r),$$

where $\Xi(\eta)^0$ denotes the subset of diagonal matrices in $\Xi(\eta)$ and d_i are diagonal entries of D.

Let $\mathbf{V}(\eta)$ (resp., $\mathbf{V}(\eta, r)$) be the subspace of $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ (resp., $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$) spanned by

$$\mathfrak{B}(\eta) = \{A(\mathbf{j}) \mid A \in \Xi^\pm(\eta), \ \mathbf{j} \in \mathbb{Z}^\eta\}$$

(resp., $\mathfrak{B}(\eta, r) = \{A(\mathbf{j}, r) \mid A \in \Xi^\pm(\eta, r), \ \mathbf{j} \in \mathbb{Z}^\eta\}$). Note that $\mathbf{V}(\eta, r) = \mathbf{U}(\eta, r) = \boldsymbol{\mathcal{K}}(\eta, r)$ if $|\eta| < \infty$.

The following results are proved by using the multiplication formulas given in [1, Lemma 5.3]. The details in the infinite case can be found in [7, 4.3]. For $1 \leqslant i, j \leqslant n$, let $E_{i,j} \in \Xi(\eta)$ be the matrix $(a_{k,l})$ with $a_{k,l} = \delta_{i,k} \delta_{j,l}$.

Theorem 4.1. *Let $\eta \subseteq \mathbb{Z}$ be a consecutive segment of $\mathbb{Z}$.*

(1) $\mathbf{V}(\eta)$ *is a subalgebra of* $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ *with* $\mathbb{Q}(v)$*-basis* $\mathfrak{B}(\eta)$*. It is generated by* $E_{h,h+1}(\mathbf{0})$, $E_{h+1,h}(\mathbf{0})$ *and* $0(\mathbf{j})$ *for all* $h \in \eta^{\dashv}$ *and* $\mathbf{j} \in \mathbb{Z}^\eta$.

(2) *There is an algebra monomorphism* $\iota : \mathbf{U}(\eta) \to \widehat{\boldsymbol{\mathcal{K}}}(\eta)$ *satisfying*

$$E_h \mapsto E_{h,h+1}(\mathbf{0}),\ K_1^{j_1}K_2^{j_2}\cdots K_n^{j_n} \mapsto 0(\mathbf{j}),\ F_h \mapsto E_{h+1,h}(\mathbf{0}).$$

Hence, $\mathrm{im}(\iota) = \mathbf{V}(\eta)$.

(3) *For any positive integer* r*, there is an algebra homomorphism* $\xi_r : \mathbf{U}(\eta) \to \widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$ *satisfying*

$$E_h \mapsto E_{h,h+1}(\mathbf{0}, r),\ K_1^{j_1}K_2^{j_2}\cdots K_n^{j_n} \mapsto 0(\mathbf{j}, r),\ F_h \mapsto E_{h+1,h}(\mathbf{0}, r)$$

such that $\mathbf{V}(\eta, r) = \mathrm{im}(\xi_r)$ *is a subalgebra of* $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$. *In particular,* $\mathbf{V}(\eta, r)$ *is generated by the elements* $E_{h,h+1}(\mathbf{0}, r),\ E_{h+1,h}(\mathbf{0}, r),$ *and* $0(\mathbf{j}, r)$ *for all* $h \in \eta^{\dashv}$ *and* $\mathbf{j} \in \mathbb{N}^{\eta}$.

(4) *By identifying* $\mathbf{U}(\eta)$ *with* $\mathbf{V}(\eta)$ *via* ι*, the algebra homomorphism* ξ_r *in (3) satisfies* $\xi_r(A(\mathbf{j})) = A(\mathbf{j}, r)$*, for any* $A \in \Xi^{\pm}(\eta)$*,* $\mathbf{j} \in \mathbb{Z}^{\eta}$.

Remarks 4.2. (1) From Theorem 4.1(2), $\mathbf{U}(\eta) \cong \mathbf{V}(\eta)$ gives a realization of quantum $\mathfrak{gl}_{\eta}$. This realization provides explicit multiplication formulas between the generators and basis elements $A(\mathbf{j})$; see [1, 5.3] and [7, 4.2].

(2) It is known $\mathbf{U}(\eta, r) \cong \mathbf{V}(\eta, r)$ if η is finite. We shall see in §9 that $\mathbf{U}(\infty, r) \cong \mathbf{V}(\infty, r)$. In particular, $\mathbf{V}(\infty, r)$ is a realization of $\mathbf{U}(\infty, r)$.

5. Constructions of $u_k(\eta)$

In this section, we discuss the realizations of $\tilde{u}_k(\eta)$ and $u_k(\eta)$. As set in §2, k is a field and $\varepsilon \in k$ is an l'th primitive root of 1.

Let $\widetilde{\Xi}_l(\eta)$ be the set of all $A = (a_{i,j}) \in \widetilde{\Xi}(\eta)$ such that $a_{i,j} < l$ for all $i \neq j$. Let $\widetilde{\Xi}_l^{\pm}(\eta)$ be the set of all $A \in \widetilde{\Xi}_l(\eta)$ whose diagonal entries are zero. Let $\Xi_{l,l'}(\eta)$ be the set of all $n \times n$ matrices $A = (a_{i,j})$ with $a_{i,j} \in \mathbb{N},\ a_{i,j} < l$ for all $i \neq j$ and $a_{i,i} \in \mathbb{Z}_{l'} := \mathbb{Z}/l'\mathbb{Z}$ for all i. We have an obvious map $pr : \widetilde{\Xi}_l(\eta) \to \Xi_{l,l'}(\eta)$ defined by reducing the diagonal entries modulo l'.

Let $\mathcal{K}(\eta)_k = \mathcal{K}(\eta) \otimes_{\mathcal{Z}} k$. Mimicking the construction of $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$, we define $\widehat{\mathcal{K}}(\eta)_k$ to be the k-vector space of all formal (possibly infinite) k-linear combinations $\sum_{A \in \widetilde{\Xi}(\eta)} \beta_A[A]$ satisfying the property (4.0.2) with a similar multiplication. This is an associative algebra with an identity: the sum of all $[D]$ with D a diagonal matrix in $\widetilde{\Xi}(\eta)$. The elements $A(\mathbf{j})$ defined earlier becomes

$$A(\mathbf{j}) = \sum_{\mathbf{z} \in \mathbb{Z}^n} \varepsilon^{\mathbf{j}\cdot\mathbf{z}}[A + \mathrm{diag}(\mathbf{z})] \in \widehat{\mathcal{K}}(\eta)_k,$$

where $\mathbf{j} \cdot \mathbf{z} = \Sigma_i j_i z_i$. Clearly, $A(\mathbf{j}) = A(\mathbf{j}')$ whenever $\bar{\mathbf{j}} = \bar{\mathbf{j}}'$. Here $^{-} : \mathbb{Z}^{\eta} \to (\mathbb{Z}_{l'})^{\eta}$ is the map defined by $\overline{(j_1, j_2, \cdots, j_n)} = (\bar{j}_1, \bar{j}_2, \cdots, \bar{j}_n)$. Thus, we shall write $A(\bar{\mathbf{j}}) := A(\mathbf{j})$. Similarly, we shall use $A(\bar{\mathbf{j}}, r) := A(\mathbf{j}, r)$ to denote the element defined earlier with υ replaced by ε for the q-Schur algebra $U(n, r)_k = U(n, r) \otimes_{\mathcal{Z}} k$ over k.

Let $\mathcal{W}$ be the subspace of $\widehat{\mathcal{K}}(\eta)_k$ spanned by

$$\mathfrak{B}_k = \{A(\bar{\mathbf{j}}) \mid A \in \widetilde{\Xi}_l^{\pm}(\eta), \bar{\mathbf{j}} \in (\mathbb{Z}_{l'})^n\}.$$

We have clearly $\dim \mathcal{W} \leqslant (l')^n l^{n^2-n}$ if $|\eta| = n$.

We remark that all the results given below are stated for an arbitrary η. Their proofs for an infinite η is entirely similar to the proofs given in [8] (when l' is odd) and in [13] (when l' is even).

The following result is [8, 4.4] and [13, 4.1].

Lemma 5.1. (1) *$\mathcal{W}$ is a subalgebra of $\widehat{\mathcal{K}}(\eta)_k$.*

(2) *The elements $E_{h,h+1}(\bar{\mathbf{0}})$, $E_{h+1,h}(\bar{\mathbf{0}})$, $0(\bar{\mathbf{e}}_i)$ (for $h \in \eta^{\dashv}$ and $i \in \eta$) generate $\mathcal{W}$ as an algebra.*

Given $A \in \widetilde{\Xi}_l^{\pm}(\eta)$ and $\mathbf{j} = (j_i)_{i\in\eta} \in \mathbb{Z}^\eta$, we rewrite

$$\begin{aligned} A(\bar{\mathbf{j}}) &:= \sum_{\mathbf{z}\in\mathbb{Z}^\eta} \varepsilon^{\mathbf{j}\cdot\mathbf{z}}[A+\mathrm{diag}(\mathbf{z})] \in \widehat{\mathcal{K}}(\eta)_k \\ &= \sum_{\bar{\mathbf{z}}\in(\mathbb{Z}_{l'})^\eta} \varepsilon^{\mathbf{j}\cdot\bar{\mathbf{z}}} \sum_{\substack{\mathbf{x}\in\mathbb{Z}^\eta\\ \bar{\mathbf{x}}=\bar{\mathbf{z}}}} [A+\mathrm{diag}(\mathbf{x})] \\ &= \sum_{\bar{\mathbf{z}}\in(\mathbb{Z}_{l'})^\eta} \varepsilon^{\mathbf{j}\cdot\bar{\mathbf{z}}} [\![A+\mathrm{diag}(\bar{\mathbf{z}})]\!] \end{aligned}$$

where $\varepsilon^{\mathbf{j}\cdot\bar{\mathbf{z}}} = \varepsilon^{\mathbf{j}\cdot\mathbf{z}}$ and

$$[\![A+\mathrm{diag}(\bar{\mathbf{z}})]\!] = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^\eta\\ \bar{\mathbf{x}}=\bar{\mathbf{z}}}} [A+\mathrm{diag}(\mathbf{x})] \in \widehat{\mathcal{K}}(\eta)_k.$$

Note that $A+\mathrm{diag}(\bar{\mathbf{z}}) \in \Xi_{l,l'}(\eta)$ and the elements $[\![A]\!]$, $A \in \Xi_{l,l'}(\eta)$, are linearly independent. Thus, we have the following; see [8], [13].

Lemma 5.2. *Each of the following sets forms a k-basis for $\mathcal{W}$:*

(1) $\{A(\bar{\mathbf{j}}) \mid A \in \widetilde{\Xi}_l^{\pm}(\eta),\ \bar{\mathbf{j}} \in (\mathbb{Z}_{l'})^\eta\}$;

(2) $\{[\![A]\!] \mid A \in \Xi_{l,l'}(\eta)\}$.

In particular, if $n = |\eta|$ is finite, then $\dim \mathcal{W} = (l')^n l^{n^2-n}$.

Following [1, 6.3], let $\mathrm{K}(\eta)$ be the k-subspace of $\mathcal{K}(\eta)_k$ spanned by the elements $[A]$ with $A \in \widetilde{\Xi}_l(\eta)$. It is proved that $\mathrm{K}(\eta)$ is a subalgebra of $\mathcal{K}(\eta)_k$.

Let $\mathrm{K}'(\eta)$ be the free k-module with basis elements $[A]$ in bijection with the elements $A \in \Xi_{l,l'}(\eta)$. There is an algebra structure on $\mathrm{K}'(\eta)$ given, for $A, A' \in \Xi_{l,l'}(\eta)$, by

$$[A]\cdot[A'] = \begin{cases} 0, & \text{if } \mathrm{col}\,(A) \neq \mathrm{row}\,(A') \text{ in } \mathbb{Z}_{l'}, \\ \sum \rho_{\widetilde{A}''}[pr(\widetilde{A}'')], & \text{otherwise,} \end{cases}$$

where $\rho_{\widetilde{A}''}$ and $\widetilde{A}''$ are determined by a product in $\mathrm{K}(\eta)$: $[\widetilde{A}]\cdot[\widetilde{A}'] = \sum \rho_{\widetilde{A}''}[\widetilde{A}'']$ for any $\widetilde{A}$, $\widetilde{A}' \in \widetilde{\Xi}_l(\eta)$ satisfying $\mathrm{col}\,(\widetilde{A}) = \mathrm{row}\,(\widetilde{A}')$ (in $\mathbb{Z}$), $pr(\widetilde{A}) = A$ and $pr(\widetilde{A}') = A'$. Unlike $\mathcal{K}(\eta)$ or $\mathrm{K}(\eta)$, the algebra $\mathrm{K}'(\eta)$ has unit element: the sum $\sum_{\lambda\in\mathbb{Z}_{l'}^\eta}[\mathrm{diag}(\lambda)]$.

We now have the following results.

Theorem 5.3 ([8], [13])**.** (1) *There is an algebra isomorphism $\psi : \mathcal{W} \xrightarrow{\sim} \mathrm{K}'(\eta)$ satisfying $[\![A]\!] \mapsto [A]$ for $A \in \Xi_{l,l'}(\eta)$.*

(2) *There is an algebra epimorphism* $\varphi : \tilde{u}_k(\eta) \twoheadrightarrow \mathcal{W}$ *satisfying* $E_h \mapsto E_{h,h+1}(\bar{\mathbf{0}})$, $F_h \mapsto E_{h+1,h}(\bar{\mathbf{0}})$, $K_j \mapsto 0(\bar{\boldsymbol{e}}_j)$.

(3) *If* l' *is even,* φ *is an isomorphism and, if* l' *is odd, then* φ *induces isomorphism* $\bar{\varphi} : u_k(\eta) \xrightarrow{\sim} \mathcal{W}$.

Corollary 5.4 ([1])**.** *If* $l' = l$ *is odd, then we have an algebra isomorphism* $u_k(\eta) \cong \mathrm{K}'(\eta)$.

6. Bases for q-Schur algebras

In this section, we first describe the Drinfeld–Jimbo type presentations for q-Schur algebras at $\eta = [1, n]$, and then we introduce several resulting bases.

Theorem 6.1. *The* q*-Schur algebra* $\mathbf{U}(n,r)$ *over* $\mathbb{Q}(\upsilon)$ *has the following presentations with*

(1) *(*[4]*) generators*

$$\mathtt{e}_i,\ \mathtt{f}_i\,(1 \leqslant i \leqslant n-1),\ \mathtt{k}_j\,(1 \leqslant j \leqslant n)$$

and relations:

(S1) $\mathtt{k}_i\mathtt{k}_j = \mathtt{k}_j\mathtt{k}_i$;

(S2) $\mathtt{k}_i\mathtt{e}_j = \upsilon^{\delta_{i,j}-\delta_{i,j+1}}\mathtt{e}_j\mathtt{k}_i,\ \mathtt{k}_i\mathtt{f}_j = \upsilon^{\delta_{i,j+1}-\delta_{i,j}}\mathtt{f}_j\mathtt{k}_i$;

(S3) $\mathtt{e}_i\mathtt{e}_j = \mathtt{e}_j\mathtt{e}_i,\ \mathtt{f}_i\mathtt{f}_j = \mathtt{f}_j\mathtt{f}_i$ *when* $|i-j| > 1$;

(S4) $\mathtt{e}_i^2\mathtt{e}_j - (\upsilon+\upsilon^{-1})\mathtt{e}_i\mathtt{e}_j\mathtt{e}_i + \mathtt{e}_j\mathtt{e}_i^2 = 0$ *when* $|i-j| = 1$;

(S5) $\mathtt{f}_i^2\mathtt{f}_j - (\upsilon+\upsilon^{-1})\mathtt{f}_i\mathtt{f}_j\mathtt{f}_i + \mathtt{f}_j\mathtt{f}_i^2 = 0$ *when* $|i-j| = 1$;

(S6) $\mathtt{e}_i\mathtt{f}_j - \mathtt{f}_j\mathtt{e}_i = \delta_{ij}\frac{\widetilde{\mathtt{k}}_i - \widetilde{\mathtt{k}}_i^{-1}}{\upsilon-\upsilon^{-1}}$, *where* $\widetilde{\mathtt{k}}_i = \mathtt{k}_i\mathtt{k}_{i+1}^{-1}$;

(S7) $\mathtt{k}_1\cdots\mathtt{k}_n = \upsilon^r$ *and, for all* $1 \leqslant i \leqslant n$, $(\mathtt{k}_i - 1)(\mathtt{k}_i - \upsilon)\cdots(\mathtt{k}_i - \upsilon^r) = 0$.

(2) *(*[9]*) generators*

$$\mathtt{e}_i,\ \mathtt{f}_i,\ \mathtt{k}_j\,(1 \leqslant i \leqslant n-1)$$

and relations (S1)–(S6) *together with*

(S0) $[\mathtt{k}_1; t_1]![\mathtt{k}_2; t_2]!\cdots[\mathtt{k}_{n-1}; t_{n-1}]! = 0$ *for* $t_i \in \mathbb{N}$ *such that* $t_1 + \cdots + t_{n-1} = r+1$, *where* $[X;t]^! = (X-1)(X-\upsilon)\cdots(X-\upsilon^{t-1})$, *and* $\mathtt{k}_n = \upsilon^r\mathtt{k}_1^{-1}\cdots\mathtt{k}_{n-1}^{-1}$ *in (S6).*

Proof. Let $\widetilde{\mathbf{U}}(n,r)$ be the algebra with either of the described presentations. Then there exist an algebra epimorphism

$$\tilde{\zeta}_r : \mathbf{U}(n) \twoheadrightarrow \widetilde{\mathbf{U}}(n,r).$$

The algebra epimorphism $\zeta_r : \mathbf{U}(n) \twoheadrightarrow \mathbf{V}(n,r) = \mathbf{U}(n,r)$ given by

$$E_h \mapsto E_{h,h+1}(\mathbf{0}, r), K_i \mapsto 0(\boldsymbol{e}_i, r), F_h \mapsto E_{h+1,h}(\mathbf{0}, r),$$

where $\boldsymbol{e}_i = (\ldots, 0, \underset{(i)}{1}, 0 \ldots)$, induces an epimorphism $\widetilde{\mathbf{U}}(n,r) \twoheadrightarrow \mathbf{U}(n,r)$. Thus, the result follows if we could prove that $\dim\widetilde{\mathbf{U}}(n,r) \leqslant \dim\mathbf{U}(n,r)$.

We prove this inequality by constructing some integral monomial bases below. $\square$

We first display an integral monomial basis for $U(n)$. For $A = A^+ + A^- \in \Xi^{\pm}(n)$, where $A^+ \in \Xi^+(n)$ and $A^- \in \Xi^-(n)$, and $\mathbf{j} \in \mathbb{Z}^n$ let

$$M^{(A,\mathbf{j})} = E^{(A^+)} \cdot K(\mathbf{j}) \cdot F^{(A^-)} \in \mathbf{U}(n)$$

where $E^{(A^+)} = \prod_{1\leqslant i\leqslant h<j\leqslant n} E_h^{(a_{i,j})}$, $F^{(A^-)} = \prod_{1\leqslant j\leqslant h<i\leqslant n} F_h^{(a_{i,j})}$ and

$$K(\mathbf{j}) = K_1^{\delta_1}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\|j_1|\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\|j_n|\end{bmatrix},$$

where $\delta_i = 0$ resp. 1 if $j_i \geqslant 0$ resp. $j_i < 0$. The order in the products $E^{(A^+)}$ and $F^{(A^-)}$ are taken as follows: For the jth column (reading upwards) $a_{j-1,j},\cdots,a_{1,j}$ $(2\leqslant j\leqslant n)$ of A^+, fix the reduced expression for the longest word of the symmetric grup $\mathfrak{S}_j$ on j letters:

$$w_{0,j} := s_{j-1}(s_{j-2}s_{j-1})\cdots(s_1s_2\cdots s_{j-1})$$

and put

$$M_j = E_{j-1}^{(a_{j-1,j})}(E_{j-2}^{(a_{j-2,j})}E_{j-1}^{(a_{j-2,j})})\cdots(E_1^{(a_{1,j})}E_2^{(a_{1,j})}\cdots E_{j-1}^{(a_{1,j})}).$$

Similarly, for the jth row (reading to the right) $a_{j,1},\cdots,a_{j,j-1}$ $(2\leqslant j\leqslant n)$ of A^-, put

$$M'_j = (F_{j-1}^{(a_{j,1})}\cdots F_2^{(a_{j,1})}F_1^{(a_{j,1})})\cdot(F_{j-1}^{(a_{j,j-2})}F_{j-2}^{(a_{j,j-2})})F_{j-1}^{(a_{j,j-1})}.$$

Then $E^{(A^+)} = M_nM_{n-1}\cdots M_2$ and $F^{(A^-)} = M'_2M'_3\cdots M'_n$.

Lemma 6.2. *The set*

$$\{M^{(A,\mathbf{j})} \mid A\in\Xi^{\pm}(n), \mathbf{j}\in\mathbb{Z}^n\}$$

forms a $\mathcal{Z}$-basis for $U(n)$.

We want to construct a similar basis for $\widetilde{\mathbf{U}}(n,r)$. Let

$$\mathtt{e}^{(A^+)} = \tilde{\zeta}_r(E^{(A^+)}),\quad \mathtt{f}^{(A^-)} = \tilde{\zeta}_r(F^{(A^-)}) \text{ and } \begin{bmatrix}\mathtt{k}_i;0\\t\end{bmatrix} = \tilde{\zeta}_r\Big(\begin{bmatrix}K_i;0\\t\end{bmatrix}\Big),$$

and put, for $\lambda\in\mathbb{N}^n$,

$$\mathtt{k}_\lambda = \prod_{i=1}^n\begin{bmatrix}\mathtt{k}_i;0\\\lambda_i\end{bmatrix}. \tag{6.2.1}$$

Lemma 6.3. *We have, for any $\lambda\in\mathbb{N}^n$,*

$$\mathtt{k}_\lambda = \begin{cases}[\mathrm{diag}(\lambda)], & \text{if } |\lambda| = r,\\ 0, & \text{otherwise.}\end{cases}$$

The following result is proved in [9].

Theorem 6.4. *Let*

$$\mathcal{M} = \{\mathfrak{m}^{(A)} := \mathtt{e}^{(A^+)}\mathtt{k}_{\boldsymbol{\sigma}(A)}\mathtt{f}^{(A^-)} \mid A\in\Xi(n,r)\}$$

where $\boldsymbol{\sigma}(A) = (\sigma_1(A),\cdots,\sigma_n(A))$ with $\sigma_i(A) = a_{i,i} + \sum_{1\leqslant j<i}(a_{i,j}+a_{j,i})$. Then $\mathcal{M}$ is a spanning set and, hence, forms a $\mathbb{Q}(v)$-basis for $\widetilde{\mathbf{U}}(n,r)$. (So we may identify $\widetilde{\mathbf{U}}(n,r)$ with $\mathbf{U}(n,r)$, and ζ_r with $\tilde{\zeta}_r$.) Moreover, $\mathcal{M}$ gives a $\mathcal{Z}$-basis for the integral q-Schur algebra $U(n,r)$.

We call $\mathcal{M}$ an integral monomial basis. This basis leads to several other bases for $\mathbf{U}(n,r)$ which we now describe.

Fix the reduced expression

$$\mathbf{i} = (i_1, i_2, \cdots, i_\nu) = (n-1, \cdots, 2, 1, \cdots, n-1, n-2, n-1) \tag{6.4.1}$$

of the longest word w_0 of the symmetric group $\mathfrak{S}_n$, that is, $w_0 = s_{i_1}s_{i_2}\cdots s_{i_\nu}$. For any $\mathbf{c} = (c_1, \cdots, c_\nu) \in \mathbb{N}^\nu$, define monomials in root vectors $E_{\mathbf{i}}^{\mathbf{c}}$ and $F_{\mathbf{i}}^{\mathbf{c}}$ as in [19, 2.2] (using braid group actions). Let $\mathtt{e}_{\mathbf{i}}^{\mathbf{c}} = \zeta_r(E_{\mathbf{i}}^{\mathbf{c}})$ and $\mathtt{f}_{\mathbf{i}}^{\mathbf{c}} = \zeta_r(F_{\mathbf{i}}^{\mathbf{c}})$.

For any $A \in \Xi^\pm(n)$, let $\mathbf{c}(A^+) = (c_1, \cdots, c_\nu) \in \mathbb{N}^\nu$, where the first $n-1$ components of $\mathbf{c}(A^+)$ is the n-th column of A^+ reading upwords, and the next $n-2$ components is the $(n-1)$-th column and so on, i.e.,

$$c_1 = a_{n-1,n}, \cdots, c_{n-1} = a_{1n}, c_n = a_{n-2,n-1}, \cdots.$$

Define $\mathbf{c}(A^-)$ symmetrically.

Theorem 6.5 ([8],[13]). *We list the following $\mathbb{Q}(v)$-bases for the q-Schur algebra $\mathbf{U}(n,r)$. Let i_0 be a fixed integer with $1 \leqslant i_0 \leqslant n$.*

(1) *The monomial basis:*

$$\mathcal{N}_{i_0} = \Big\{\mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)} \;\Big|\; A \in \Xi^\pm(n), \mathbf{j} \in \mathbb{N}^n, j_{i_0} = 0, |\mathbf{j}| + |A| \leqslant r\Big\};$$

(2) *The BLM basis:*

$$\mathcal{B}_{i_0} = \big\{A(\mathbf{j}, r) \;\big|\; A \in \Xi^\pm(n),\, \mathbf{j} \in \mathbb{N}^n,\, j_{i_0} = 0, |\mathbf{j}| + |A| \leqslant r\big\};$$

(3) *The PBW basis:*

$$\mathcal{P}_{i_0} = \big\{\mathtt{e}_{\mathbf{i}}^{\mathbf{c}(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} \;\big|\; A \in \Xi^\pm(n), \mathbf{j} \in \mathbb{N}^n, j_{i_0} = 0, |\mathbf{j}| + |A| \leqslant r\big\}.$$

We end this section with an application. We identify the Hecke algebra part of the monomial basis $\mathcal{M}$. We assume now $n = r$ though all results hold for $n \geqslant r$. Let

$$\varpi = (1, 1, \cdots, 1) \in \mathbb{N}^n,$$

and let $\boldsymbol{\mathcal{H}} = \mathtt{k}_\varpi \mathbf{U}(n,n)\mathtt{k}_\varpi$ and $\mathcal{H} = \mathtt{k}_\varpi U(n,n)\mathtt{k}_\varpi$. For any $w \in \mathfrak{S}_n$, let $A_w \in \Xi(n,n)$ be the permutation matrix defined inductively by $A_w = A_y A_{s_i}$, where $w = ys_i$ with $y < w$ and $s_i = (i, i+1)$ for some i.

Lemma 6.6 ([9, 9.1]). *The algebra $\mathcal{H}$ is free over $\mathcal{Z}$ with basis*

$$\mathcal{M}_\varpi = \{\mathfrak{m}^{(A_w)} \mid w \in \mathfrak{S}_n\}.$$

Let $\mathtt{C}_i = \mathfrak{m}^{(A_{s_i})} = \mathtt{e}^{(A_{s_i}^+)}\mathtt{k}_{\boldsymbol{\sigma}(A_{s_i})}\mathtt{f}^{(A_{s_i}^-)}$, and let $\mathcal{T}_i = \mathtt{C}_i - v^{-1}$.

Theorem 6.7 ([9, 9.3-4]). (1) *The elements $\mathcal{T}_i$, $1 \leqslant i \leqslant n-1$, satisfy the following relations:*

$$\begin{cases} (a) & (\mathcal{T}_i - v)(\mathcal{T}_i + v^{-1}) = 0; \\ (b) & \mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i, \quad |i-j| > 1; \\ (c) & \mathcal{T}_i\mathcal{T}_{i+1}\mathcal{T}_i = \mathcal{T}_{i+1}\mathcal{T}_i\mathcal{T}_{i+1}, 1 \leqslant i \leqslant n-2. \end{cases}$$

In particular, $\mathcal{T}_i$ is invertible and $\mathcal{T}_i^{-1} = \mathtt{C}_i - v$.

(2) *For any $w \in \mathfrak{S}_n$, there is a reduced expression $w = s_{i_1} \cdots s_{i_l}$ satisfying*

$$\mathfrak{m}^{(A_w)} = \mathtt{C}_{i_1} \cdots \mathtt{C}_{i_l}.$$

Therefore, $\mathcal{H}$ is isomorphic to the Hecke algebra over $\mathcal{Z}$ associated with $\mathfrak{S}_n$.

Remark 6.8. (1) Some commuting relations between the $\mathtt{e}_i$ and $\mathtt{f}_i$ and the quantum Serre relations give rise to the braid relations (b) and (c).

(2) Using the notation in [17], $\mathtt{C}_i$ corresponds to C'_{s_i}, and $\mathcal{T}_s = v^{-1}T_{s_i}$.

7. Little q-Schur algebras

In this section, we continue to assume $\eta = [1, n]$. By restricting the map $\zeta_r : \mathbf{U}(n) \twoheadrightarrow \mathbf{U}(n, r)$ defined in (2.0.2) to the $\mathcal{Z}$-form $U(n)$, we obtain a surjective map $\zeta_r : U(n) \twoheadrightarrow U(n, r)$ (see [6]).

As in §2, let k be a field which is a $\mathcal{Z}$-module via $v \mapsto \varepsilon$, where ε is an l'th primitive root of 1. Then base change induces a surjective homomorphism

$$\zeta_{r,k} := \zeta_r \otimes 1 : U(n)_k \twoheadrightarrow U(n, r)_k$$

and hence, a map

$$\zeta_{r,k} : \tilde{u}_k(n) \to U(n, r)_k$$

by restriction. The image $\zeta_{r,k}(\tilde{u}_k(n))$, denoted $u_k(n, r)$, is called a *little q-Schur algebra*. Note that, if l' is odd, then $\zeta_{r,k}(K_i^l - 1) = 0$ for all i. Hence, $\zeta_{r,k}$ induces a surjective map, the version of ζ_r over k,

$$\zeta_{r,k} : u_k(n) \twoheadrightarrow u_k(n, r).$$

Let $u_k(n, r)^+ = \zeta_{r,k}(\tilde{u}_k(n)^+)$, $u_k(n, r)^- = \zeta_{r,k}(\tilde{u}_k(n)^-)$, $u_k(n, r)^0 = \zeta_{r,k}(\tilde{u}_k(n)^0)$. By abuse of notation, we shall continue to denote the images of the generators E_i, F_i, K_j for $\tilde{u}_k(n)$ by the same letters $\mathtt{e}_i$, $\mathtt{f}_i$, $\mathtt{k}_j$ used for $\mathbf{U}(n, r)$.

We first have the following.

Theorem 7.1 ([8, 7.6]). *The set $\{\mathtt{e}^{(A)} \mid A \in \widetilde{\Xi}_l^+(n),\ |A| \leqslant r\}$ (resp., $\{\mathtt{f}^{(A)} \mid A \in \widetilde{\Xi}_l^-(n),\ |A| \leqslant r\}$) forms a k-basis of $u_k(n, r)^+$ (resp., $u_k(n, r)^-$).*

For any $\lambda \in \Lambda(n, r)$, let $\overline{\lambda}$ be the image in $(\mathbb{Z}_{l'})^n$ obtained by reducing every entry modulo l'. Let, for $A \in \widetilde{\Xi}_l^\pm(n)$ with $|A| \leqslant r$ and $\lambda \in \Lambda(n, r - |A|)$,

$$[\![A + \mathrm{diag}(\overline{\lambda}), r]\!] = \sum_{\substack{\mu \in \Lambda(n, r-|A|) \\ \overline{\mu} = \overline{\lambda}}} [A + \mathrm{diag}(\mu)] \in U_k(n, r).$$

We also set $\overline{\Lambda(n, r)}_{l'} = \{\overline{\lambda} \in (\mathbb{Z}_{l'})^n \mid \lambda \in \Lambda(n, r)\}$. For $\nu \in (\mathbb{Z}_{l'})^n$, define

$$\mathtt{p}_\nu = \begin{cases} \displaystyle\sum_{\mu \in \Lambda(n,r), \overline{\mu} = \nu} \mathtt{k}_\mu & \text{if } \nu \in \overline{\Lambda(n, r)}_{l'}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{N}_{l'}^n = \{\lambda \in \mathbb{N}^n \mid \lambda_i < l' \,\forall i\}$. In contrast with Theorems 6.4 and 6.5, we have the following bases for $u_k(n, r)$.

Theorem 7.2 ([8, 8.2, 8.5]). *Fix any integer i_0 with $1 \leqslant i_0 \leqslant n$. Each of the following sets forms a k-basis for $u_k(n, r)$.*

(1) $\mathcal{L}_k := \{[\![A+\mathrm{diag}(\overline{\lambda}), r]\!] \mid A \in \widetilde{\Xi}^{\pm}_l(n),\ |A| \leqslant r,\ \lambda \in \Lambda(n, r-|A|)\}$.
(2) $\mathcal{M}_k := \{\mathtt{e}^{(A^+)}\mathtt{p}_{\overline{\lambda}}\mathtt{f}^{(A^-)} \mid A \in \widetilde{\Xi}^{\pm}_l(n), \lambda \in \Lambda(n,r),\ \lambda_i \geqslant \sigma_i(A)\,\forall i\}$
(3) $\mathcal{N}_{i_0,k} = \big\{\mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)} \mid A \in \widetilde{\Xi}^{\pm}_l(n), \mathbf{j} \in \mathbb{N}^n_{l'}, j_{i_0} = 0, |\mathbf{j}|+|A| \leqslant r\big\}$;
(4) $\mathcal{B}_{i_0,k} = \big\{A(\mathbf{j}, r) \mid A \in \widetilde{\Xi}^{\pm}_l(n), \mathbf{j} \in \mathbb{N}^n_{l'}, j_{i_0} = 0, |\mathbf{j}|+|A| \leqslant r\big\}$;
(5) $\mathcal{P}_{i_0,k} = \big\{\mathtt{e}_{\mathbf{i}}^{\mathbf{c}(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} \mid A \in \widetilde{\Xi}^{\pm}_l(n), \mathbf{j} \in \mathbb{N}^n_{l'}, j_{i_0} = 0, |\mathbf{j}|+|A| \leqslant r\big\}$.

We may use the bases to derive certain dimension formulas. Let, for $n \geqslant m \geqslant 1$,

$$\Lambda(n,r;m) = \{\lambda \in \Lambda(n,r) \mid 0 \leqslant \lambda_2, \ldots, \lambda_m < l', 0 \leqslant \lambda_{m+1}, \ldots, \lambda_n < l\}.$$

Then $\#\Lambda(n,r;1) = \#\overline{\Lambda(n,r)}_l$ and $\#\Lambda(n,r;n) = \#\overline{\Lambda(n,r)}_{l'}$. By Theorem 7.1, we have $\dim u_k(n,r)^{\pm} = \#\Lambda(N,r;1)$, where $N = \binom{n}{2}+1$, while, by Theorem 7.2, $\dim u_k(n,r) = \#\Lambda(n^2,r;n)$. More explicitly, we have the following dimension formulas.

Theorem 7.3 ([8],[13]). *We have, for* $1 \leqslant m \leqslant n$,

$$\#\Lambda(n,r;m) = \sum_{s\geqslant 0}(-1)^{s+t}\binom{m-1}{s}\binom{n-m}{t}\binom{n+r-tl-sl'-1}{n-1}.$$

In particular, we have

(1) $\dim u_k(n,r)^0 = \sum_{s\geqslant 0}(-1)^s\binom{n-1}{s}\binom{n+r-sl'-1}{n-1}$;
(2) $\dim u_k(n,r)^+ = \sum_{s\geqslant 0}(-1)^s\binom{N-1}{s}\binom{N+r-sl-1}{N-1}$ *with* $N = \binom{n}{2}+1$;
(3) $\dim u_k(n,r) = \sum_{s,t\geqslant 0}(-1)^{s+t}\binom{n-1}{s}\binom{n^2-n}{t}\binom{n^2+r-tl-sl'-1}{n^2-1}$.

There is a simpler description of the number $\dim u_k(n,r)$ if $l' = l$ is odd. For any $A = (a_{i,j}) \in M_n(\mathbb{Z})$, let $\bar{A} = \overline{(a_{i,j})} = (\bar{a}_{i,j})$ where $\bar{a}_{i,j} \in \mathbb{Z}_l$, and let $\overline{\Xi(n,r)} = \{\bar{A} \mid A \in \Xi(n,r)\}$. Then it is easy to see that all the sets $\overline{\Xi(n,r)}$, $\Lambda(n^2,r;1)$ and $\overline{\Lambda(n^2,r)}_l$ have the sane cardinality. Hence we have

$$\dim u_k(n,r) = \#\overline{\Xi(n,r)} = \sum_{s\geqslant 0}(-1)^s\binom{n^2-1}{s}\binom{n^2+r-sl-1}{n^2-1}.$$

Remark 7.4. Infinitesimal Schur/q-Schur algebras were introduced in [5], [3] as the dual algebras of the homogeneous components of the infinitesimal thickening (by the torus) of the Frobenius kernel of the quantum coordinate algebra of GL_n. It is proved in [12] that the subalgebra of $U(n,r)_k$ generated by the little q-Schur algebra $u_k(n,r)$ and $\left[\begin{smallmatrix}\mathtt{k}_i;0\\ t\end{smallmatrix}\right]$ ($1 \leqslant i \leqslant n$, $t \in \mathbb{N}$) is isomorphic to the infinitesimal q-Schur algebra investigated in [3].

8. PRESENTING $\mathbf{V}(\infty, r)$

Recall from §4 the $\mathbb{Q}(v)$-algebra $\mathbf{V}(\infty,r)$. By Theorem 4.1(3) for $\eta = \mathbb{Z}$, it is generated by the elements

$$\mathsf{e}_i = E_{i,i+1}(\mathbf{0},r),\ \mathsf{f}_i = E_{i+1,i}(\mathbf{0},r),\ \text{and } \mathsf{k}_i = 0(\boldsymbol{e}_i, r)$$

for all $i \in \mathbb{Z}$, where $\boldsymbol{e}_i \in \mathbb{Z}^{\infty}$ has 1 as the ith component and 0 elsewhere.

For any $\lambda \in \mathbb{N}^{\infty}$, since λ has finite support, the product (cf. (6.2.1))

$$\mathsf{k}_\lambda = \prod_{i=-\infty}^{\infty}\begin{bmatrix}\mathsf{k}_i;0\\ \lambda_i\end{bmatrix},$$

is well-defined, and a result similar to Lemma 6.3 holds. We also introduce the products $\mathsf{e}^{(A^+)}$ and $\mathsf{f}^{(A^-)}$. Bases similar to those given in Theorem 6.5 exist for $\mathbf{V}(\infty, r)$.

Lemma 8.1 ([7, 6.7-8]). *Each of the following sets forms a basis for* $\mathbf{V}(\infty, r)$*:*

(1) $\mathcal{N}_\infty = \Big\{\mathsf{e}^{(A^+)}\mathsf{k}^{\mathbf{j}}\mathsf{f}^{(A^-)} \;\Big|\; A \in \Xi^{\pm}(\infty), \mathbf{j} \in \mathbb{N}^\infty, |\mathbf{j}| + |A| \leqslant r\Big\}$;
(2) $\mathcal{B}_\infty = \big\{A(\mathbf{j}, r) \;\big|\; A \in \Xi^{\pm}(\infty), \mathbf{j} \in \mathbb{N}^\infty, |\mathbf{j}| + |A| \leqslant r\big\}$.

We remark that a basis similar to the PBW basis 6.5(3) can also be constructed. In order to avoid the dependence of the sequence $\mathbf{i}$ defined in (6.4.1), we may simply use the PBW basis constructed in [10, 4.6].

We now present $\mathbf{V}(\infty, r)$ by generators and relations; cf. Theorem 6.1. The proof of the following theorem relies on the basis $\mathcal{N}_\infty$.

Theorem 8.2 ([7, 6.6]). *The algebra* $\mathbf{V}(\infty, r)$ *over* $\mathbb{Q}(\upsilon)$ *has the following presentations with generators*

$$\mathsf{e}_i,\ \mathsf{f}_i,\ \mathsf{k}_i,\ i \in \mathbb{Z}$$

and relations: for $i, j \in \mathbb{Z}$,

(1) $\mathsf{k}_i\mathsf{k}_j = \mathsf{k}_j\mathsf{k}_i$;
(2) $\prod_{i\in\mathbb{Z}}[\mathsf{k}_i; t_i]! = 0$ *for all* $\mathbf{t} = (t_i) \in \mathbb{N}^\infty$ *with* $|\mathbf{t}| = \sum_{i\in\mathbb{Z}} t_i = r + 1$,
(3) $\mathsf{k}_i\mathsf{e}_j = \upsilon^{\delta_{i,j}-\delta_{i,j+1}}\mathsf{e}_j\mathsf{k}_i$, $\mathsf{k}_i\mathsf{f}_j = \upsilon^{\delta_{i,j+1}-\delta_{i,j}}\mathsf{f}_j\mathsf{k}_i$;
(4) $\mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i$, $\mathsf{f}_i\mathsf{f}_j = \mathsf{f}_j\mathsf{f}_i$ *when* $|i - j| > 1$;
(5) $\mathsf{e}_i^2\mathsf{e}_j - (\upsilon + \upsilon^{-1})\mathsf{e}_i\mathsf{e}_j\mathsf{e}_i + \mathsf{e}_j\mathsf{e}_i^2 = 0$ *when* $|i - j| = 1$;
(6) $\mathsf{f}_i^2\mathsf{f}_j - (\upsilon + \upsilon^{-1})\mathsf{f}_i\mathsf{f}_j\mathsf{f}_i + \mathsf{f}_j\mathsf{f}_i^2 = 0$ *when* $|i - j| = 1$;
(7) $\mathsf{e}_i\mathsf{f}_j - \mathsf{f}_j\mathsf{e}_i = \delta_{ij}\frac{\widetilde{\mathsf{k}}_i - \widetilde{\mathsf{k}}_i^{-1}}{\upsilon - \upsilon^{-1}}$, *where* $\widetilde{\mathsf{k}}_i = \mathsf{k}_i\mathsf{k}_{i+1}^{-1}$;

We rewrite e_i etc., as $\mathsf{e}_{i,r}$, etc. if different r's are under consideration.

Corollary 8.3. *For* $r \in \mathbb{N}$*, there is an epimorphism from* $\mathbf{V}(\infty, r+1)$ *to* $\mathbf{V}(\infty, r)$ *by sending* $\mathsf{e}_{i,r+1}$ *to* $\mathsf{e}_{i,r}$*,* $\mathsf{f}_{i,r+1}$ *to* $\mathsf{f}_{i,r}$ *and* $\mathsf{k}_{i,r+1}$ *to* $\mathsf{k}_{i,r}$ *for* $i \in \mathbb{Z}$.

We shall see in the next section that $\mathbf{V}(\infty, r)$ is isomorphic to $\mathbf{U}(\infty, r)$. Thus, we obtain the so-called transfer maps from $\mathbf{U}(\infty, r + 1)$ to $\mathbf{U}(\infty, r)$ (cf. [23]) which is useful in the study of the polynomial representation category $\mathcal{C}_r$ in §11.

Let $V(\infty, r) = \xi_r(U(\infty))$ be the integral $\mathcal{Z}$-form of $\mathbf{V}(\infty, r)$, where ξ_r is defined in Theorem 4.1(3). Then we have the following integral basis for $V(\infty, r)$; cf. Theorem 6.4.

Proposition 8.4 ([7, 6.11]). *The set*

$$\mathcal{M}_\infty = \{\mathsf{e}^{(A^+)}\mathsf{k}_\lambda\mathsf{f}^{(A^-)} \mid A \in \Xi^{\pm}(\infty), \lambda \in \mathbb{N}^\infty, \lambda \geqslant \boldsymbol{\sigma}(\lambda), |\lambda| \leqslant r\}.$$

forms a basis for $V(\infty, r)$*, where* $\lambda \geqslant \boldsymbol{\sigma}(\lambda)$ *means* $\lambda_i \geqslant \sigma_i(A)$ *for all* i.

Moreover, the subset $\{\mathsf{e}^{(A^+)}\mathsf{k}_{\boldsymbol{\sigma}(A)}\mathsf{f}^{(A^-)} \mid A \in \Xi(\infty, r)\}$ *of* $\mathcal{M}_\infty$ *forms a basis for* $\mathcal{K}(\infty, r)$.

Note that the last assertion is seen from [7, (5.5.1)].

Remarks 8.5. (1) Since the relation (S7) from Theorem 6.1 does not hold in $\mathbf{V}(\infty, r)$, there is no natural homomorphism from the q-Schur algebra $\mathbf{U}([-n, n], r)$ to $\mathbf{V}(\infty, r)$. However, there is a natural homomorphism from a Borel subalgebra $\mathbf{U}([-n, n], r)^{\geqslant 0}$ or $\mathbf{U}([-n, n], r)^{\leqslant 0}$ into $\mathbf{V}(\infty, r)$ (sending 1 to 1). Thus, the identity element 1 of $\mathbf{V}(\infty, r)$ is a finite sum of orthogonal idempotent elements labelled by the elements in the set $\Lambda([-n,n],r)$. This fact is useful in the proof of Theorem 8.2.

(2) It is interesting to point out that the Doty-Giaguinto type presentation described in Theorem 6.1(1) does not exist for $\mathbf{V}(\infty,r)$.

9. Infinite q-Schur algebras and their elementary structure

In this section, we first introduce the q-Schur algebra $\mathcal{S}(\eta,r)$ at η. We will mainly focus on the structure of infinite q-Schur algebras.

Let $\mathcal{H}$ be the Hecke algebra over $\mathcal{Z}$ associated with the symmetric group $\mathfrak{S}_r$ on r letters. Then $\mathcal{H}$ as a $\mathcal{Z}$-algebra has a basis $\{\mathcal{T}_w\}_{w\in\mathfrak{S}_r}$ subject the relations (cf. Theorem 6.7): for all $w\in\mathfrak{S}_r$ and $s\in S:=\{(i,i+1)\mid 1\leqslant i\leqslant n-1\}$,

$$\mathcal{T}_s\mathcal{T}_w=\begin{cases}\mathcal{T}_{sw}, & \text{if } sw>w,\\ (v-v^{-1})\mathcal{T}_w+\mathcal{T}_{sw}, & \text{if } sw<w.\end{cases}$$

Let $\boldsymbol{\mathcal{H}}=\mathcal{H}\otimes\mathbb{Q}(v)$.

Let η be a consecutive segment of $\mathbb{Z}$. Let Ω_η be a free $\mathcal{Z}$-module with basis $\{\omega_i\}_{i\in\eta}$ and let $\boldsymbol{\Omega}_\eta=\Omega_\eta\otimes\mathbb{Q}(v)$. $\mathcal{H}$ acts on $\Omega_\eta^{\otimes r}$ from the right by "place permutations":

$$(\omega_{i_1}\cdots\omega_{i_r})\mathcal{T}_{(j,j+1)}=\begin{cases}\omega_{i_1}\cdots\omega_{i_{j+1}}\omega_{i_j}\cdots\omega_{i_r}, & \text{if } i_j<i_{j+1};\\ v\omega_{i_1}\cdots\omega_{i_r}, & \text{if } i_j=i_{j+1};\\ (v-v^{-1})\omega_{i_1}\cdots\omega_{i_r}+\omega_{i_1}\cdots\omega_{i_{j+1}}\omega_{i_j}\cdots\omega_{i_r}, & \text{if } i_j>i_{j+1}.\end{cases}$$

The algebras

$$\boldsymbol{\mathcal{S}}(\eta,r):=\mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\eta^{\otimes r}),\qquad \mathcal{S}(\eta,r):=\mathrm{End}_{\mathcal{H}}(\Omega_\eta^{\otimes r})$$

are called *q-Schur algebras at η*. Note that $\boldsymbol{\mathcal{S}}(n,r):=\boldsymbol{\mathcal{S}}(\eta,r)$ for $\eta=[1,n]$ is the q-Schur algebra we discussed before. Clearly,, if η is finite, $\boldsymbol{\mathcal{S}}(\eta,r)\cong\boldsymbol{\mathcal{S}}(|\eta|,r)$ and $\boldsymbol{\mathcal{S}}(\eta,r)\cong\mathcal{S}(\eta,r)\otimes_{\mathcal{Z}}\mathbb{Q}(v)$. When $\eta=\mathbb{Z}$, the algebras $\mathcal{S}(\infty,r):=\mathcal{S}(\mathbb{Z},r)$ and $\boldsymbol{\mathcal{S}}(\infty,r)$ are called *infinite q-Schur algebras.*

Since the $\boldsymbol{\mathcal{H}}$-action commutes with the action of $\mathbf{U}(\eta)$, it follows that $\mathbf{U}(\eta,r)\subseteq\boldsymbol{\mathcal{S}}(\eta,r)$. Hence, we obtain algebra homomorphisms

$$\zeta_r:\mathbf{U}(\eta)\to\boldsymbol{\mathcal{S}}(\eta,r),\qquad \zeta_r|_{U(\eta)}:U(\eta)\to\mathcal{S}(\eta,r). \tag{9.0.1}$$

These homomorphisms are surjective whenever η is finite. In particular, we have $U(n,r)=\mathcal{S}(n,r)$ and $\mathbf{U}(n,r)=\boldsymbol{\mathcal{S}}(n,r)$ for all $n,r\geqslant 1$.

Since $\Omega_{[-n,n]}^{\otimes r}$ is a direct summand of the $\mathcal{H}$-submodule $\Omega_{[-n-1,n+1]}^{\otimes r}$ for all $n\geqslant 1$, it follows that we may embed $\mathcal{S}([-n,n],r)$ as a centralizer subalgebra of $\mathcal{S}([-(n+1),n+1],r)$. Hence, we obtain a direct limit system $\{\mathcal{S}([-n,n],r)\}_{n\geqslant 1}$. By Lemma 3.1(2), we have

$$\mathcal{K}(\infty,r)\cong\varinjlim_n\mathcal{S}([-n,n],r),\qquad \boldsymbol{\mathcal{K}}(\infty,r)\cong\varinjlim_n\boldsymbol{\mathcal{S}}([-n,n],r).$$

This isomorphism implies immediately the second isomorphism in the following.

Proposition 9.1. *The $\mathcal{H}$-module $\Omega_\infty^{\otimes r}$ is isomorphic to the $\mathcal{H}$-module $\oplus_{\lambda\in\Lambda(\infty,r)}x_\lambda\mathcal{H}$. Thus, we have the following algebra isomorphisms*

$$\mathcal{S}(\infty,r)\cong\prod_{\mu\in\Lambda(\infty,r)}\bigoplus_{\lambda\in\Lambda(\infty,r)}\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H})$$

$$\mathcal{K}(\infty,r)\overset{\theta}{\cong}\bigoplus_{\mu\in\Lambda(\infty,r)}\bigoplus_{\lambda\in\Lambda(\infty,r)}\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H}).$$

Here the multiplication in $\prod_{\mu\in\Lambda(\infty,r)}\bigoplus_{\lambda\in\Lambda(\infty,r)}\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H})$ is given as follows: $\big(\sum_\nu g_{\nu,\tau}\big)_\tau\big(\sum_\lambda f_{\lambda,\mu}\big)_\mu=\big(\sum_\nu\sum_\lambda g_{\nu,\lambda}f_{\lambda,\mu}\big)_\mu$, where $f_{\lambda,\mu},g_{\lambda,\mu}\in\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H})$ for all λ,μ.

Similar results hold with $\mathcal{S}$, Ω, $\mathcal{H}$, $\mathcal{K}$ replaced by $\boldsymbol{\mathcal{S}}$, $\boldsymbol{\Omega}$, $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{K}}$ respectively.

The second isomorphism can also be made more explicit. For any $\lambda\in\Lambda(\infty,r)$, let $\mathfrak{S}_\lambda$ be the corresponding Young subgroup, and let $\mathfrak{D}_\lambda$ be the set of distinguished right $\mathfrak{S}_r$-coset representatives. We also put $\mathfrak{D}_{\lambda\mu}=\mathfrak{D}_\lambda\cap\mathfrak{D}_\mu^{-1}$. This is the set of distinguished double $(\mathfrak{S}_\lambda,\mathfrak{S}_\mu)$-coset representatives. We also define, for any $w\in\mathfrak{D}_{\lambda,\mu}$, a map

$$\phi_{\lambda\mu}^w:\oplus_{\lambda\in\Lambda(\infty,r)}x_\lambda\mathcal{H}\to\oplus_{\lambda\in\Lambda(\infty,r)}x_\lambda\mathcal{H}$$

by setting $\phi_{\lambda\mu}^w(x_\nu h)=\delta_{\mu,\nu}\sum_{x\in\mathfrak{S}_\lambda w\mathfrak{S}_\mu}h$. It is well-known (see, e.g., [15, (1.3.10)]) that there is a bijection

$$\jmath:\{(\lambda,w,\mu)\mid\lambda,\mu\in\Lambda(\infty,r),w\in\mathfrak{D}_{\lambda,\mu}\}\longrightarrow\Xi(\infty,r)$$

Corollary 9.2. *The isomorphism θ is induced by $\jmath$. In other words, if $\jmath(\lambda,w,\mu)=A$, then $\theta(\phi_A)=\phi_{\lambda\mu}^w$.*

We will identify the two bases under θ.

Let $\varpi\in\Lambda(\infty,r)$ such that $\varpi_i=1$ if $1\leqslant i\leqslant r$ and $\varpi_i=0$ otherwise. The following describe some elementary structure of $\mathcal{S}(\infty,r)$. Part (1) follows from Lemma 6.3, and Part (4) shows that the Hecke algebra $\mathcal{H}$ is always a centralizer subalgebra of $\mathcal{S}(\infty,r)$.

Lemma 9.3. *Let $\lambda,\mu\in\Lambda(\infty,r)$.*

1. $\mathsf{k}_\lambda=[\mathrm{diag}(\lambda)]=\phi_{\mathrm{diag}(\lambda)}=\phi_{\lambda\lambda}^1$ *for all* $\lambda\in\Lambda(\infty,r)$.
2. $\mathcal{S}(\infty,r)\mathsf{k}_\lambda=\mathcal{K}(\infty,r)\mathsf{k}_\lambda$;
3. $\mathsf{k}_\lambda\mathcal{S}(\infty,r)\mathsf{k}_\mu=\mathsf{k}_\lambda\mathcal{K}(\infty,r)\mathsf{k}_\mu=\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H})$;
4. *We have* $\mathcal{H}\cong\mathsf{k}_\varpi\mathcal{S}(\infty,r)\mathsf{k}_\varpi\cong\mathrm{End}_{\mathcal{S}(\infty,r)}(\Omega_\infty^{\otimes r})\cong\mathrm{End}_{U(\infty)}(\Omega_\infty^{\otimes r})$.

Similar results hold for the algebras over $\mathbb{Q}(v)$.

We continue to use boldface fonts for objects over the field $\mathbb{Q}(v)$. The identification of $\mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\infty^{\otimes r})$ with $\prod\limits_{\mu\in\Lambda(\infty,r)}\bigoplus\limits_{\lambda\in\Lambda(\infty,r)}\mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu\boldsymbol{\mathcal{H}},x_\lambda\boldsymbol{\mathcal{H}})$ gives rise to a natural injective algebra homomorphism $\bar\zeta_r:\boldsymbol{\mathcal{K}}(\infty,r)\to\mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\infty^{\otimes r})$. By 9.1 and the definition of $\widehat{\boldsymbol{\mathcal{K}}}(\infty,r)$ we know $\bar\zeta_r$ induces naturally an injective algebra homomorphism

$$\bar\zeta_r:\widehat{\boldsymbol{\mathcal{K}}}(\infty,r)\to\mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\infty^{\otimes r}),$$

sending $\sum_{A\in\Xi(\infty,r)}\beta_A[A]=\sum_{\mu\in\Lambda(\infty,r)}(\sum_{\substack{co(A)=\mu\\A\in\Xi(\infty,r)}}\beta_A[A])$ to $\big(\sum_{\substack{co(A)=\mu\\A\in\Xi(\infty,r)}}\beta_A[A]\big)_{\mu\in\Lambda(\infty,r)}$. We will view $\widehat{\boldsymbol{\mathcal{K}}}(\infty,r)$ as a subalgebra of $\mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\infty^{\otimes r})$.

Recall from Theorem 4.1(3) the algebra homomorphism $\xi_r:\mathbf{U}(\infty)\to\mathbf{V}(\infty,r)$. We are now ready to identify $\mathbf{U}(\infty,r)$ with $\mathbf{V}(\infty,r)$.

Theorem 9.4 ([7, 7.4])**.** *The map* $\zeta_r : \mathbf{U}(\infty) \to \mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\infty^{\otimes r})$ *factors through* $\bar{\zeta}_r : \mathbf{V}(\infty, r) \to \mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\infty^{\otimes r})$*. In other words, we have* $\zeta_r = \bar{\zeta}_r \circ \xi_r$*. Hence,* $\mathbf{U}(\infty, r) = \mathbf{V}(\infty, r)$*.*

Remark 9.5. (1) The injective map $\bar{\zeta}_r : \widehat{\boldsymbol{\mathcal{K}}}(\infty, r) \to \mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\infty^{\otimes r})$ is not surjective; see [7, Thm 7.4]. Thus, unlike the (finite) q-Schur algebra case, the homomorphism ζ_r from $\mathbf{U}(\infty)$ to the infinite q-Schur algebra $\boldsymbol{\mathcal{S}}(\infty, r)$ is not surjective, i. e., the algebra $\mathbf{U}(\infty, r)$ is a proper subalgebra of $\boldsymbol{\mathcal{S}}(\infty, r)$.

(2) It can be proved (see [7, §8]) that the epimorphism $\zeta_r : \mathbf{U}(\infty) \to \mathbf{U}(\infty, r)$ can be extended to an algebra epimorphism $\hat{\zeta}_r : \widehat{\boldsymbol{\mathcal{K}}}(\infty) \to \widehat{\boldsymbol{\mathcal{K}}}(\infty, r)$ by sending $\sum_{A \in \widetilde{\Xi}(\infty)} \beta_A[A]$ to $\sum_{A \in \Xi(\infty, r)} \beta_A[A]$. Thus, we obtain an algebra homomorphism $\zeta_r : \widehat{\boldsymbol{\mathcal{K}}}(\infty) \to \mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\infty^{\otimes r})$.

(3) It is possible to introduce a new algebra $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$ which has the infinite q-Schur algebra as a homomorphic image. Let $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$ be the vector space of all formal (possibly infinite) $\mathbb{Q}(v)$-linear combinations $\sum_{A \in \widetilde{\Xi}(\infty)} \beta_A[A]$ which have the following properties: for any $\mathbf{x} \in \mathbb{Z}^\infty$,

$$\text{the set } \{A \in \widetilde{\Xi}(\infty) \mid \beta_A \neq 0,\ \mathrm{col}\,(A) = \mathbf{x}\} \text{ is finite.} \tag{9.5.1}$$

In other words, for any $\mu \in \mathbb{Z}^n$, the sum $\sum_{A \in \widetilde{\Xi}(\infty)} \beta_A[A] \cdot [\mathrm{diag}(\mu)]$ is finite. We can define the product of two elements $\sum_{A \in \widetilde{\Xi}(\infty)} \beta_A[A]$, $\sum_{B \in \widetilde{\Xi}(\infty)} \gamma_B[B]$ in $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$ to be $\sum_{A,B} \beta_A\gamma_B[A] \cdot [B]$. This defines an associative algebra structure on $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$. This algebra has an identity element $\sum_{\lambda \in \mathbb{Z}^\infty} [\mathrm{diag}(\lambda)]$: the sum of all $[D]$ with D a diagonal matrix in $\widetilde{\Xi}(\infty)$. One proves that the map $\zeta_r^\dagger : \widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty) \to \mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\infty^{\otimes r})$ sending $\sum_{A \in \widetilde{\Xi}(\infty)} \beta_A[A]$ to $\sum_{A \in \Xi(\infty, r)} \beta_A[A]$ is an epimorphism; see [7, 8.9].

10. Highest weight representations of $\mathbf{U}(\infty)$

In this section, we discuss the "standard" representation theory of $\mathbf{U}(\infty)$. This includes the category $\mathcal{C}$ of weight modules, the category $\mathcal{C}^{hi}$ of weight modules with highest weights, the category $\mathcal{C}^{int}$ of integrable modules and the category $\mathcal{O}$, all of which are full subcategories of the category $\mathbf{U}(\infty)$-**Mod** of $\mathbf{U}(\infty)$-modules.

Let $X(\infty) = \{\lambda = (\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}\}$ be the weight lattice, and let

$$X^+(\infty) = \{\lambda \in X(\infty) \mid \lambda_i \geqslant \lambda_{i+1} \text{ for all } i \in \mathbb{Z}\}$$

be the set of dominant weights. For $i \in \mathbb{Z}$, let as before $\boldsymbol{e}_i = (\cdots, 0, \underset{i}{1}, 0 \cdots)$ and $\alpha_i = \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$. Then $R(\infty) = \{\boldsymbol{e}_i - \boldsymbol{e}_j \mid i \neq j\}$ is the root system of $\mathfrak{gl}_\infty$, and $R^+(\infty) = \{\boldsymbol{e}_i - \boldsymbol{e}_j \mid i < j\}$ is the associated positive system. Let $\Pi(\infty) = \{\alpha_i \mid i \in \mathbb{Z}\}$ be the set of all simple roots, and let $\leqslant$ be the partial ordering on $X(\infty)$ defined by setting, for all $\lambda, \mu \in X(\infty)$, $\mu \leqslant \lambda$ if $\lambda - \mu \in \mathbb{N}\Pi(\infty)$.

For a $\mathbf{U}(\infty)$-module M and $\lambda \in X(\infty)$, let $M_\lambda = \{x \in M \mid K_i x = v^{\lambda_i} x \text{ for } i \in \mathbb{Z}\}$. If $M_\lambda \neq 0$, then λ is called a *weight* of M and M_λ is called a *weight space*; if $M = \oplus_{\lambda \in X(\infty)} M_\lambda$, then M is called a *weight module*.

Let $\mathrm{wt}(M) = \{\lambda \in X(\infty) \mid M_\lambda \neq 0\}$ denote the set of the weights of M. It is clear that, for a $\mathbf{U}(\infty)$-module M and $i \in \mathbb{Z}$, we have

$$E_i M_\lambda \subseteq M_{\lambda + \alpha_i} \quad \text{and} \quad F_i M_\lambda \subseteq M_{\lambda - \alpha_i}. \tag{10.0.2}$$

Let $\mathcal{C}$ denotes the full subcategory of $\mathbf{U}(\infty)$-**Mod** consisting of all weight modules, and let $\mathcal{C}^{hi}$ be the full subcategory of $\mathcal{C}$ whose objects are the weight $\mathbf{U}(\infty)$-module M with the property that, for any $x \in M$, there exists $n_0 \geqslant 1$ such that $u.x = 0$ whenever u is a monomial in the E_i's having degree at least n_0.

An *integrable* $\mathbf{U}(\infty)$-module M is a weight module satisfying that, for any $x \in M$ and any $i \in \mathbb{Z}$, there exists $n_0 \geqslant 1$ such that $E_i^n x = F_i^n x = 0$ for all $n \geqslant n_0$. Let $\mathcal{C}^{int}$ be the full subcategory of $\mathcal{C}$ consisting of integrable $\mathbf{U}(\infty)$-modules.

The category $\mathcal{O}$ is the full subcategory of $\mathcal{C}$ each of whose objects M has finite dimensional weight spaces and has weight set $\mathrm{wt}(M) \subseteq \bigcup_{i=1}^{s}(-\infty, \lambda^{(i)}]$ for some $\lambda^{(1)}, \cdots, \lambda^{(s)} \in X(\infty)$. Here, for $\lambda \in X(\infty)$, $(-\infty, \lambda] := \{\mu \in X(\infty) \mid \mu \leqslant \lambda\}$.

We first observe the following; see [7, §9].

Proposition 10.1 ([7, 9.2])**.** *The category $\mathcal{O}$ is a full subcategory of $\mathcal{C}^{hi}$. Thus, we have the following flowchart for various chains of full subcategories:*

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{C}^{hi} \arrow[d, dash] \arrow[drr, dash] & & \mathcal{C}^{int} \arrow[d, dash] \\
\mathcal{O} \arrow[dr, dash] & & \mathcal{C}^{hi} \cap \mathcal{C}^{int} \arrow[dl, dash] \\
 & \mathcal{O} \cap \mathcal{C}^{int} &
\end{tikzcd}$$

For a $\mathbf{U}(\infty)$-module M, if there exist a $\lambda \in X(\infty)$ and a nonzero vector $x_0 \in M_\lambda$ such that $E_i x_0 = 0$, $K_i x_0 = v^{\lambda_i} x_0$ for all $i \in \mathbb{Z}$ and $\mathbf{U}(\infty) x_0 = M$, then M is called a *highest weight module.* The vector x_0 is called a highest weight vector. Using a standard argument (for finite dimensional highest weight modules), one sees easily that a highest weight $\mathbf{U}(\infty)$-module with highest weight λ is a weight module and $M = \oplus_{\mu \leqslant \lambda} M_\mu$. Moreover, $\dim M_\lambda = 1$ and M contains a unique maximal submodule.

For $\lambda \in X(\infty)$, let

$$M(\lambda) = \mathbf{U}(\infty)/(\sum_{i \in \mathbb{Z}} \mathbf{U}(\infty) E_i + \sum_{i \in \mathbb{Z}} \mathbf{U}(\infty)(K_i - v^{\lambda_i})),$$

which is called a *Verma module.* This is a highest weight module with the highest weight vector 1_λ, the image of the 1. Thus, $M(\lambda)$ has a unique irreducible quotient module $L(\lambda)$. Clearly, the modules $M(\lambda)$ and $L(\lambda)$ are all in the category $\mathcal{C}^{hi}$ and in the category $\mathcal{O}$.

We have the following classification of irreducible modules in the category $\mathcal{C}^{hi}$.

Theorem 10.2 ([7, 9.4])**.** *There is a bijection $\lambda \mapsto L(\lambda)$ between $X(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{C}^{hi}$ (resp., in the category $\mathcal{O}$).*

The classification of irreducible integrable modules seems very hard. However, it is possible to classify all irreducible integrable modules in $\mathcal{C}^{hi}$, i. e., all irreducible modules in $\mathcal{C}^{hi} \cap \mathcal{C}^{int}$. We shall see in the next section that there exist irreducible integrable modules which are not in $\mathcal{C}^{hi}$.

For $\lambda \in X^+(\infty)$, let $I(\lambda)$ be the submodule of $M(\lambda)$ generated by $F_i^{(\lambda_i-\lambda_{i+1}+1)}$ ($i \in \mathbb{Z}$). By the commutator formula [20, 4.1(a)]:

$$E_i^{(k)}F_i^{(l)} = \sum_{t=0}^{\min(k,l)} F_i^{(l-t)} \begin{bmatrix} \widetilde{K}_i; 2t-k-l \\ t \end{bmatrix} E_i^{(k-t)},$$

we deduce $I(\lambda) = \sum_{i\in\mathbb{Z}} \mathbf{U}^-(\infty)F_i^{(\lambda_i-\lambda_{i+1}+1)}1_\lambda$. Hence $I(\lambda)$ is a proper submodule of $M(\lambda)$. Let $\widetilde{L(\lambda)} = M(\lambda)/I(\lambda)$. By [21, 3.5.3], we have the following.

Lemma 10.3. *For $\lambda \in X^+(\infty)$, the module $\widetilde{L(\lambda)}$ is an integrable $\mathbf{U}(\infty)$-module. Hence, as a homomorphic image of $\widetilde{L(\lambda)}$, the module $L(\lambda)$ is an integrable $\mathbf{U}(\infty)$-module.*

We now have the following classification theorem whose proof requires the fact that, if M is a object in $\mathcal{C}^{int}$ and $0 \neq x_0 \in M_\lambda$ satisfying $E_i x_0 = 0$ for all $i \in \mathbb{Z}$, then $\lambda \in X^+(\infty)$.

Theorem 10.4 ([7, 9.6-7])**.** *The map $\lambda \mapsto L(\lambda)$ defines a bijection between $X^+(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{C}^{int} \cap \mathcal{C}^{hi}$. Hence, it also defines a bijection between $X^+(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{C}^{int} \cap \mathcal{O}$. Moreover we have $L(\lambda) \cong \widetilde{L(\lambda)}$ for all $\lambda \in X^+(\infty)$.*

Remarks 10.5. It can be proved (see [7, 9.7]) that, for $\lambda \in X^+(\infty)$, $L(\lambda)$ is isomorphic to the direct limit of finite dimensional irreducible $\mathbf{U}([-n,n])$-module $L(\lambda_{[-n,n]})$, where $\lambda_{[-n,n]} = (\lambda_i)_{-n\leqslant i\leqslant n}$ if $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$. However, for a non-dominant weight λ, we could not prove that a similar isomorphism holds.

The following result shows that there are not many finite dimensional weight $\mathbf{U}(\infty)$-modules. Let $\mathcal{C}^{fd}$ be the category of finite dimensional weight $\mathbf{U}(\infty)$-modules.

Theorem 10.6 ([7, 9.10])**.** *The category $\mathcal{C}^{fd}$ is a completely reducible category, and the modules $L(m\mathbf{1})$ ($m \in \mathbb{Z}$), where $\mathbf{1} = (\cdots,1,1,\cdots,1,\cdots) \in X(\infty)$, are all non-isomorphic irreducible $\mathbf{U}(\infty)$-modules in $\mathcal{C}^{fd}$.*

11. Polynomial type representations of $\mathbf{U}(\infty)$

Polynomial representations of $\mathbf{U}(n)$ are obtained via q-Schur algebra representations and are all highest weight modules. In this section, we will see that the $\mathbf{U}(\infty)$-modules arising from weight modules over infinite q-Schur algebras are in general not highest weight modules. Thus, we obtain a "nonstandard" representation theory for $\mathbf{U}(\infty)$.

Let r be a positive integer, and let $\mathcal{C}_r$ be the full subcategory of $\mathcal{C}$ consisting of $\mathbf{U}(\infty,r)$-modules which are weight modules when regarded as $\mathbf{U}(\infty)$-modules via $\zeta_r : \mathbf{U}(\infty) \to \mathbf{U}(\infty,r)$. We also define $\mathcal{C}^{pol}$ to be the full subcategory of $\mathcal{C}$ consisting of weight $\mathbf{U}(\infty)$-modules M such that $\mathrm{wt}(M) \subseteq \mathbb{N}^\infty$. (Recall that the elements of $\mathbb{N}^\infty$ have finite support.) We call the objects in $\mathcal{C}^{pol}$ *polynomial representations*.

There is a close connection between polynomial representations and weight $\boldsymbol{\mathcal{S}}(\infty,r)$-modules. By definition, an $\boldsymbol{\mathcal{S}}(\infty,r)$-module M is called a weight $\boldsymbol{\mathcal{S}}(\infty,r)$-module if $M = \oplus_{\lambda\in\Lambda(\infty,r)}\mathsf{k}_\lambda M$. (Recall from Lemma 9.3(1) that $\mathsf{k}_\lambda = \phi^1_{\lambda\lambda}$.) Let $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod** be the category of weight $\boldsymbol{\mathcal{S}}(\infty,r)$-modules. Since the quantum group $\mathbf{U}(\infty)$ maps to $\boldsymbol{\mathcal{S}}(\infty,r)$, every $\boldsymbol{\mathcal{S}}(\infty,r)$-module is naturally a $\mathbf{U}(\infty)$-module. Moreover, we have the following (which justifies the terminology of weight $\boldsymbol{\mathcal{S}}(\infty,r)$-modules).

Lemma 11.1. *Let M be a weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module regarded naturally as a $\mathbf{U}(\infty)$-module. Then, for any $\lambda \in \Lambda(n, r)$, $\mathsf{k}_\lambda M = M_\lambda$. Hence, M is a weight $\mathbf{U}(\infty)$-module. Moreover, M is also an integrable $\mathbf{U}(\infty)$-module.*

We may define the category $\boldsymbol{\mathcal{K}}(\infty, r)$-**mod** of weight $\boldsymbol{\mathcal{K}}(\infty, r)$-modules. By regarding $\mathbf{U}(\infty, r)$ as a subalgebra of $\boldsymbol{\mathcal{S}}(\infty, r)$, we may define $\mathbf{U}(\infty, r)$-**mod** similarly. Note that $\mathbf{U}(\infty, r)$-**mod** is a full subcategory of $\mathcal{C}_r$.

Since every weight $\boldsymbol{\mathcal{K}}(\infty, r)$-module induces naturally a weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module by using the fact that, for any $x \in \boldsymbol{\mathcal{S}}(\infty, r)$ and $\lambda \in \Lambda(\infty, r)$, $x\mathsf{k}_\lambda \in \boldsymbol{\mathcal{K}}(\infty, r)$, we have immediately the following category isomorphisms.

Lemma 11.2 ([7, 10.2])**.** *The category $\boldsymbol{\mathcal{K}}(\infty, r)$-**mod**, and hence the category $\mathbf{U}(\infty, r)$-**mod**, is isomorphic to the category $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod**.*

Thus, $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod** can be regarded as a subcategory of $\mathcal{C}$.

We want to classify the irreducible modules in $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod**. Let $\boldsymbol{\Omega}(\infty, r) := \boldsymbol{\mathcal{S}}(\infty, r)\mathsf{k}_\varpi$. Since $\boldsymbol{\mathcal{S}}(\infty, r)\mathsf{k}_\lambda = \boldsymbol{\mathcal{K}}(n, r)\mathsf{k}_\lambda$ for all $\lambda \in \Lambda(\infty, r)$ (see 9.3), it follows that $\boldsymbol{\Omega}(\infty, r)$ is a weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module. Moreover, it is an $(\boldsymbol{\mathcal{S}}(\infty, r), \boldsymbol{\mathcal{H}})$-bimodule.

Proposition 11.3. *The $(\boldsymbol{\mathcal{S}}(\infty, r), \boldsymbol{\mathcal{H}})$-bimodule $\boldsymbol{\Omega}(\infty, r)$ is isomorphic to $\boldsymbol{\Omega}_\infty^{\otimes r}$.*

For $\lambda, \mu \in \Lambda(\infty, r)$ we say that λ and μ are associated if μ can be derived from λ by reordering the parts of λ. Let $\Lambda^+(r) = \{\lambda \in \Lambda(\infty, r) \mid \lambda_i = 0 \text{ for } i \leqslant 0 \text{ and } \lambda_i \geqslant \lambda_{i+1} \text{ for } i \geqslant 1\}$. This is the set of all partitions of r. We define a map from $\Lambda(\infty, r)$ to $\Lambda^+(r)$ by sending λ to λ^+ where λ^+ is the unique element in $\Lambda^+(r)$ which is associated with λ.

For $\lambda \in \Lambda(\infty, r)$, let λ' be the partition dual to λ^+. Let $y_{\lambda'} = \sum_{w \in \mathfrak{S}_{\lambda'}} (-q)^{-l(w)} T_w$ and $z_\lambda = \phi^1_{\lambda\varpi} T_{w_\lambda} y_{\lambda'}$ where $q = \upsilon^2 \in \mathbb{Q}(\upsilon)$ and $w_\lambda \in \mathfrak{D}_{\lambda,\lambda'}$ satisfying $\mathfrak{S}_\lambda \cap w_\lambda \mathfrak{S}_{\lambda'} w_\lambda^{-1} = \{1\}$. Let $S^\lambda = z_\lambda \boldsymbol{\mathcal{H}}$ be the Specht module of $\boldsymbol{\mathcal{H}}$ associated with λ, and let $W(\infty, \lambda) = \boldsymbol{\mathcal{S}}(\infty, r) z_\lambda$. We call $W(\infty, \lambda)$ the *Weyl module* of $\boldsymbol{\mathcal{S}}(\infty, r)$.

The following results classify irreducible representations of three relevant categories.

Theorem 11.4. *Let $\lambda \in \Lambda(\infty, r)$.*

1. ([7, 10.4]) *$W(\infty, \lambda) \cong W(\infty, \lambda^+)$ as $\boldsymbol{\mathcal{S}}(\infty, r)$-modules;*
2. ([7, 10.8(1)]) *the set $\{W(\infty, \lambda) \mid \lambda \in \Lambda^+(r)\}$ forms a complete set of irreducible modules in $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod**.*
3. ([7, 11.3(1)]) *the set $\{W(\infty, \lambda) \mid \lambda \in \cup_{i=0}^r \Lambda^+(i)\}$ forms a complete set of irreducible modules in $\mathcal{C}_r$.*
4. ([7, 11.3(2)]) *the set $\{W(\infty, \lambda) \mid \lambda \in \cup_{i=0}^\infty \Lambda^+(i)\}$ forms a complete set of irreducible modules in $\mathcal{C}^{pol}$.*

The proof for Part (3) requires the following facts: (1) there is a surjective homomorphism, the transfer map, from $\mathbf{U}(\infty, r+1)$ to $\mathbf{U}(\infty, r)$ (see 8.3); (2) every irreducible polynomial representation of $\mathbf{U}(\infty)$ is an irreducible weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module for some r.

Finally, we mention the following result.

Theorem 11.5 ([7, 10.8(2)])**.** *Every weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module is completely reducible.*

It would be interesting to point out that the category of finite dimensional weight $\mathbf{U}(n)$-modules possesses a quite rich structure. It covers all finite dimensional $\boldsymbol{\mathcal{S}}(n, r)$-modules which form a major constituent. As a contrast, the category $\mathcal{C}^{fd}$ of finite dimensional $\mathbf{U}(\infty)$-representations is more or less trivial. However, the infinite dimensional $\mathbf{U}(\infty)$-module categories $\mathcal{C}^{hi}$, $\mathcal{C}^{int}$, $\mathcal{O}$ and $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod** have inherited some of the features from their (finite dimensional) $\mathbf{U}(n)$ counterparts. For example, the complete reducibility continue to hold in $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod**, and the irreducible objects in $\mathcal{C}^{hi} \cap \mathcal{C}^{int}$ are indexed by "dominant weights".

REFERENCES

[1] A.A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of* GL_n, Duke Math.J. **61** (1990), 655–677.

[2] C. De Concini, and V.G. Kac, *Representations of quantum groups at roots of* 1, in Operator algebras, Unitary Representations, Enveloping Algebras and Invariant Theory, A. Connes, M. Dulfo, A. Joseph and R. Rentshler (eds) (1990), 471–506.

[3] A. G. Cox, *On some applications of infinitesimal methods to quantum groups and related algebras*, PhD thesis, University of London, 1997.

[4] S. Doty and A. Giaquinto, *Presenting Schur algebras,* International Mathematics Research Notices IMRN, **36** (2002), 1907–1944.

[5] S.R. Doty, D.K. Nakano and K.M. Peters, *On Infinitesimal Schur algebras,* Proc. London Math. Soc. **72** (1996), 588–612.

[6] J. Du, *A note on the quantized Weyl reciprocity at roots of unity,* Alg. Colloq. **2**(1995), 363–372.

[7] J. Du and Q. Fu, *Quantum* $\mathfrak{gl}_\infty$*, infinite q-Schur algebras and their representations*, preprint, ArXiv:0708.2525.

[8] J. Du, Q. Fu and J.-p. Wang, *Infinitesimal quantum* $\mathfrak{gl}_n$ *and little q-Schur algebras*, J. Algebra, **287**(2005), 199–233.

[9] J. Du and B. Parshall, *Monomial bases for q-Schur algebras,* Trans. Amer. Math. Soc. 355 (2003), 1593–1620.

[10] J. Du and B. Parshall, *Linear quivers and the geometric setting for quantum* GL_n, Indag. Math. N. S. **13**(2003), 459–481.

[11] J. Du, B. Parshall and L. Scott, *Quantum Weyl reciprocity and tilting modules*, Comm. Math. Phys. **195**(1998), 321–352.

[12] Q. Fu, *A comparison between little and infinitesimal q-Schur algebras,* Comm. Algebra, **33**(2005), 2663–2682.

[13] Q. Fu, *Little q-Schur algebras at even roots of unity*, J. Algebra, **311**(2007), 202–215.

[14] V. Ginzburg and E. Vasserot, *Langlands reciprocity for affine quantum groups of type* A_n, Internat. Math. Res. Notices 1993, 67-85.

[15] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Math. and its Appl. no. 6, Addison-Wesley, 1981 London.

[16] M. Jimbo, *A q-analogue of* $U(gl(N+1))$*, Hecke algebras, and the Yang-Baxter equation*, Lett. Math. Phy. **11** (1986), 247-252.

[17] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184.

[18] G. Lusztig, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras,* J. Amer. Math. Soc. **3**(1990), 257–296.

[19] G. Lusztig, *Canonical bases arising from quantized enveloping algebras,* J. Amer. Math. Soc. **3** (1990), 447-498.

[20] G. Lusztig, *Modular representations and quantum groups,* Comtemp. Math. **82** (1989), 59-77.

[21] G. Lusztig, *Introduction to quantum groups*, Progress in Math. **110**, Birkhäuser, 1993.

[22] G. Lusztig, *Aperiodicity in quantum affine* $\mathfrak{gl}_n$, Asian J. Math. **3** (1999), 147-177.

[23] G. Lusztig, *Transfer maps for quantum affine* $\mathfrak{sl}_n$, In Representations and quantizations (Shanghai, 1998), China High. Educ. Press, Beijing (2000), 341-356.

[24] C. M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101**(1990), 583–592.

[25] M. Takeuchi, *Some topics on $GL_q(n)$*, J. Algebra **147** (1992), 379–410.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW SOUTH WALES, UNSW SYDNEY NSW 2052, AUSTRALIA.
E-mail address: `j.du@unsw.edu.au`

DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI, 20092, CHINA
E-mail address: `q.fu@hotmail.com`