REPRESENTATIONS OF FINITE LIE ALGEBRAS

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ABSTRACT. The notion of Frobenius morphisms for finite dimensional associative algebras is introduced in [2] and applied there to the study of representations of finite dimensional algebras over finite fields. In this paper, we develop a parallel theory for Lie algebras. Thus, a connection between representations over finite fields $\mathbb{F}_q$ and over their algebraic closures $k = \overline{\mathbb{F}}_q$ is established analogously. This enables us to understand irreducible representations of a Lie algebra over $\mathbb{F}_q$ through those over $k$. We further show that Frobenius morphisms can be used to easily determine the $\mathbb{F}_q$-forms of classical simple Lie algebras, and hence reobtain a classical result given in [15]. Finally, we illustrate the theory with the example of classifying simple modules of $\mathfrak{sl}(2, \mathbb{F}_q)$.

1. Introduction

Over the last fifty years, the study on representations of finite dimensional (restricted) Lie algebras over an algebraically closed field of positive characteristics has made significant advances; see, e.g., the surveys [7, 10]. However, there seems to be no systematic work on representations of finite Lie algebras (i.e., finite dimensional Lie algebras over a finite field $\mathbb{F}_q$). In this paper, we attempt to tackle the problem by investigating representations of Lie algebras with Frobenius morphisms over the algebraic closure $k$ of $\mathbb{F}_q$.

In [2, 3], B. Deng and the first author investigated the representation theory of finite dimensional $k$-algebras $A$ with Frobenius morphisms $F$. Through this theory, representations of finite algebras (over a finite field) are embedded into the representation category of $A$ as $F$-stable representations. Thus, this provides an efficient approach to completely determine representations of all finite algebras over finite fields. We will mimic the approach given in [2] to develop a general theory which directly links representations of $\mathfrak{g}$ over $k$ with those of $\mathfrak{g}^F$ over $\mathbb{F}_q$. We then apply this to restricted Lie algebras through their reduced enveloping algebras $U_\chi(\mathfrak{g})$ defined by linear functions $\chi \in \mathfrak{g}^*$ of $\mathfrak{g}$. Like the partition of simple $\mathfrak{g}$-modules in terms of their $p$-characters, we partition simple $\mathfrak{g}^F$-modules by forming the full subcategory $\mathfrak{g}^F-\text{mod}_\chi$ associated to the $F$-orbit $\chi$ of their $p$-characters $\chi$. We characterize the objects in $\mathfrak{g}^F-\text{mod}_\chi$ in terms of $F_\chi$-stable modules in a full subcategory of $\mathfrak{g}^F-\text{mod}$ (Theorem 4.9). Here $r$ is the $F^r$-period of $\chi$. Through this process, we see immediately that there is no upper bound for irreducible representations of $\mathfrak{g}^F$. When we further restrict our attention to classical (simple) Lie algebras $\mathfrak{g}$, we are able to classify all $\mathbb{F}_q$-forms of $\mathfrak{g}$ through the action of the automorphism group Aut($\mathfrak{g}$) of $\mathfrak{g}$ on the set of all Frobenius morphisms on $\mathfrak{g}$.

We point out that Frobenius morphisms on a connected reductive group $G$ and their induced action on the Lie algebra of $G$ have been considered in [13] (see also [12]) in the study of Fourier transforms of invariant functions on finite simple Lie algebras.

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We organize the paper as follows. In §2, we introduce Lie algebras with Frobenius morphisms and their basic properties, and in §3, we discuss the Frobenius twist functor on Lie algebra representations. Thus, we introduce $F$-orbits and $F$-periods of representations, $F$-stable representations and the construction of indecomposable $F$-stable representations. We apply in §4 the general theory developed in previous sections to representations of finite restricted Lie algebras. Further applications are given to classical simple (restricted) Lie algebras in §5. We determine all Frobenius morphisms on such a restricted Lie algebra up to conjugacy classes in the automorphism group of the corresponding Dynkin diagram, and thus, determine all the $\mathbb{F}_q$-forms of the Lie algebra. Finally, as an example, we complete in §6 the classification of irreducible representations of $\mathfrak{sl}(2, \mathbb{F}_q)$.

Throughout, let $q$ be a given power of a prime $p$, $\mathbb{F}_q$ the finite field of $q$ elements and $k$ the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let $f : k \to k$ be the field automorphism sending $a$ to $a^q$. Then $\mathbb{F}_q$ identifies the fixed-point subfield $k^f = \{ \lambda \in k \mid f(a) = a \}$. All module categories of Lie or associative algebras are finite dimensional module categories.

2. Lie algebras with Frobenius morphisms

Let $V$ and $W$ be two vector spaces over $k$. An abelian group homomorphism (resp. isomorphism) $g$ from $V$ to $W$ is called a $q$-semilinear map (resp. isomorphism) if $g(av) = a^qg(v) = f(a)g(v)$ for all $v \in V$ and $a \in k$. A $q$-semilinear automorphism $F$ on $V$ satisfying, for any $v \in V$, $F^n(v) = v$ for some $n > 0$, is called a $(q)$-Frobenius map. The smallest $r$ satisfying $F^r(v) = v$ is called a period of $v$ (under $F$), denoted by $\varphi(v) = \varphi_F(v)$.

A Frobenius morphism $F$ on $V$ determines an $\mathbb{F}_q$ subspace—the fixed point space:

$$V^F = \{ v \in V \mid F(v) = v \}$$

which forms an $\mathbb{F}_q$-structure on $V$. (An $\mathbb{F}_q$ subspace $V_0$ of $V$ is called an $\mathbb{F}_q$-structure on $V$ if the natural homomorphism $V_0 \otimes \mathbb{F}_q k \to V$ is a $k$-space isomorphism.) Conversely, given an $\mathbb{F}_q$-structure on $V$, there is a natural Frobenius morphism $F = \text{id}_{V_0} \otimes f$ on $V = V_0 \otimes k$ sending $v \otimes a$ to $v \otimes a^q$ such that $V_0 = V^F$.

Note that a $q$-semilinear automorphism on an infinite dimensional $k$-space is not necessarily a Frobenius map.

**Remarks 2.1.** (1) For $k$-spaces $V, W$ with Frobenius map $F_V, F_W$, respectively, there is an induced $q$-semilinear automorphism

$$F_{(V,W)} : \text{Hom}_k(V,W) \to \text{Hom}_k(V,W); f \mapsto F_{(V,W)}(f) = F_W \circ f \circ F_V^{-1}.$$ 

If $V$ is infinite dimensional with a basis $\{v_i\}_{i \in \mathbb{N}}$, one may construct an $f$ such that $f(v_i) = a_iw$ where $0 \neq w \in W$ is fixed and $a_i \in \mathbb{F}_{q^{i+1}} \setminus \mathbb{F}_q$. Since $F_W^n(a_iw) = a_iw$ implies $n > i$, it follows that, $F^n_{(V,W)}(f) \neq f$ for all $n > 0$. This also shows that, if $V$ is finite dimensional, then $F_{(V,W)}$ is a Frobenius map.

(2) Let $V$ be of finite dimensional with a Frobenius map $F$, and let $V^*$ be the dual space of $V$. Then $F^* = F_{(V,k)}$ is a Frobenius map on $V^*$ and, for $\chi \in V^*$, $\chi = F^*(\chi) = f \circ \chi \circ F^{-1}$ if and only if $\chi(V^F) \subseteq \mathbb{F}_q$. Thus, by restriction, we may identify the $\mathbb{F}_q$-space $(V^*)^F$ as $(V^F)^*$. Thus, in general, a simple application of Lang-Steinberg’s theorem\(^1\) shows that a $q$-semilinear automorphism on a **finite dimensional** vector space is necessarily a Frobenius map (see, e.g., [3, 2.2]). Thus, if $V$ is finite dimensional, then an abelian group automorphism $F$ on $V$ is a Frobenius map if and only if

$$F(av) = f(a)F(v) \quad \text{for all } a \in k, v \in V.$$

\(^1\)This theorem asserts that if $G$ be a connected affine algebraic group and $\Phi$ a surjective endomorphism of $G$ with a finite number of fixed points, then the map $L : g \mapsto g^{-1}\Phi(g)$ from $G$ onto itself is surjective.
This motivates the following.

**Definition 2.2.** (1) Let \( \mathfrak{g} \) be a Lie algebra over \( k \). A Frobenius map \( F = F_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g} \) is called a Frobenius morphism on \( \mathfrak{g} \) if

\[
F[xy] = [F(x)F(y)] \quad \text{for all } x, y \in \mathfrak{g}.
\]

(2) Let \( \mathfrak{g} \) be a Lie algebra over \( k \) with a Frobenius morphism \( F \) and let \( V \) be a \( \mathfrak{g} \)-module. A Frobenius map \( F_V : V \to V \) is called a module Frobenius map if

\[
F_V(xv) = F(x)F_V(v) \quad \text{for all } x \in \mathfrak{g}, v \in V.
\]

Clearly, a Frobenius morphism \( F \) on \( \mathfrak{g} \) is also a Lie algebra \( \mathbb{F}_q \)-automorphism, and the \( \mathbb{F}_q \)-structure \( \mathfrak{g}^F \) on \( \mathfrak{g} \) is itself a Lie algebra over \( \mathbb{F}_q \). If a \( \mathfrak{g} \)-module \( V \) admits a module Frobenius map \( F_V \), then \( V^{F_V} \) is a \( \mathfrak{g}^F \)-module.

Note that, if \( \mathfrak{g} = A^- \) is the Lie algebra obtained from an associative algebra \( A \) with the Lie product \( [xy] = xy - yx \) for all \( x, y \in A \), then any Frobenius morphism on \( A \) in the sense of [2] induces a Frobenius morphism on \( A^- \).

**Example 2.3.** Let \( V \) be a finite dimensional \( k \)-space with a Frobenius map \( F_V \). Then the induced Frobenius map \( F' = F_{V,V} \) on \( \text{End}_k(V) \) gives rise to an algebra isomorphism ([2, 2.4]), and

\[
\text{End}_k(V)^{F'} \cong \text{End}_{\mathfrak{g}^F}(V^{F_V}).
\]

Putting \( \mathfrak{gl}(V) = \text{End}_k(V)^{-} \), we obtain a Lie algebra isomorphism \( \mathfrak{gl}(V)^{F'} = \mathfrak{gl}(V^{F_V}) \). If, in addition, \( V \) admits a module Frobenius map \( F_V \), then restriction gives the algebra isomorphism

\[
\text{End}_{\mathfrak{g}}(V)^{F'} \cong \text{End}_{\mathfrak{g}^F}(V^{F_V}).
\]

On the other hand, a Frobenius morphism on a Lie algebras naturally induces a Frobenius morphism on a certain associative algebra—its universal enveloping algebra.

**Lemma 2.4.** Let \( \mathfrak{U}(\mathfrak{g}) \) be the universal enveloping algebra of \( \mathfrak{g} \). Then a Frobenius morphism \( F \) on \( \mathfrak{g} \) induces naturally a Frobenius morphism \( F \) on \( \mathfrak{U}(\mathfrak{g}) \) satisfying \( \mathfrak{U}(\mathfrak{g})^F = \mathfrak{U}(\mathfrak{g}^F) \).

**Proof.** Let \( x_1, x_2, \ldots \) be an ordered basis for \( \mathfrak{g} \), and \( x_1^{n_1} \cdots x_i^{n_i} \) (\( i_1 < \cdots < i_l \)) be the associated PBW basis for \( \mathfrak{U}(\mathfrak{g}) \). Extending \( F \) to \( \mathfrak{U}(\mathfrak{g}) \) by sending \( x_1^{n_1} \cdots x_i^{n_i} \) to \( F(x_1)^{n_1} \cdots F(x_i)^{n_i} \), we see easily that it is a Frobenius morphism on \( \mathfrak{U}(\mathfrak{g}) \). To see the last equality, we simply take the basis to be a basis for \( \mathfrak{g}^F \). \( \square \)

The notion of Frobenius morphisms on Lie algebras is also closely related the \( \mathbb{F}_q \)-forms of Lie algebra. Recall that a \( k' \)-subalgebra \( \mathfrak{g}' \) of \( \mathfrak{g} \), where \( k' \) is a subfield of a field \( k \), is called a \( k' \)-form (or \( k' \)-structure) of \( \mathfrak{g} \) if the natural homomorphism from \( \mathfrak{g} \otimes_{k'} k \) to \( \mathfrak{g} \) is an isomorphism of Lie algebras. The determination of \( k' \)-forms of \( \mathfrak{g} \) when \( k' \) is small (in size) is important to understand the structure of \( \mathfrak{g} \). We refer to [15, IV], [19, V] and [18, §2] for a general theory on forms and its extension to algebraic groups. Clearly, an \( \mathbb{F}_q \)-form \( \mathfrak{g}_0 \) of \( \mathfrak{g} \) can always be interpreted as a \( \mathfrak{g}^F \) for some Frobenius morphism on \( \mathfrak{g} \). This motivates the following definition.

**Definition 2.5.** Let \( \mathfrak{S}_q(\mathfrak{g}) \) be the set of all Frobenius morphisms on \( \mathfrak{g} \). We define an equivalence relation \( \sim \) on \( \mathfrak{S}_q(\mathfrak{g}) \) by setting, for \( F_1, F_2 \in \mathfrak{S}_q(\mathfrak{g}), F_1 \sim F_2 \) if and only if \( \mathfrak{g}^{F_1} \cong \mathfrak{g}^{F_2} \) as Lie \( \mathbb{F}_q \)-algebras.

Thus, the determination of \( \mathbb{F}_q \)-forms of \( \mathfrak{g} \) is equivalent to the determination of equivalence classes of Frobenius morphisms. On the other hand, the automorphism group \( \text{Aut}(\mathfrak{g}) \) of \( \mathfrak{g} \) acts naturally on \( \mathfrak{S}_q(\mathfrak{g}) \): for any \( \sigma \in \text{Aut}(\mathfrak{g}) \), it’s easy to see by direct verification that \( \sigma \circ F := \sigma^{-1} \circ F \circ \sigma \) is still a Frobenius morphism on \( \mathfrak{g} \). The following result describes the equivalence classes in terms of the \( \text{Aut}(\mathfrak{g}) \)-orbits.
Lemma 2.6. Let $F_1$ and $F_2$ be two Frobenius morphisms on $\mathfrak{g}$.

(a) $F_1 = F_2$ if and only if $\mathfrak{g}^{F_1} = \mathfrak{g}^{F_2}$;
(b) $F_1 \sim F_2$ if and only if $F_1 = \sigma \ast (F_2)$ for some $\sigma \in \text{Aut}(\mathfrak{g})$.

Proof. (a) The "only if" part is clear. Conversely, suppose $\mathfrak{g}^{F_1} = \mathfrak{g}^{F_2} = \mathfrak{g}_0$. Choose a basis $\{x_1, \ldots, x_l\}$ for $\mathfrak{g}_0$ (and hence for $\mathfrak{g}$). For any $x \in \mathfrak{g}$ with $x = \sum a_i x_i = F_2(x)$, proving $F_1 = F_2$.

(b) Suppose $F_1 \sim F_2$. By definition, there is an $\mathbb{F}_q$-isomorphism $\sigma$ from $\mathfrak{g}_1 := \mathfrak{g}^{F_1}$ to $\mathfrak{g}_2 := \mathfrak{g}^{F_2}$. Then $\sigma$ can be extended naturally to an isomorphism of $\mathfrak{g}$, which is still denoted by $\sigma$. We claim $F_1 = \sigma \ast F_2$. Indeed, this is equivalent by (a) to $\mathfrak{g}_1 = \mathfrak{g}^{\sigma \ast F_2}$, or to $\mathfrak{g}_1 \subset \mathfrak{g}^{\sigma^{-1} F_2 \sigma}$ by a dimensional comparison. The latter can be easily verified. Conversely, if $F_1 = \sigma \ast F_2$ for some $\sigma \in \text{Aut}(\mathfrak{g})$, then $x \in \mathfrak{g}_1$ if and only if $\sigma(x) \in \mathfrak{g}_2$, which implies that $\sigma$ is an $\mathbb{F}_q$-isomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$. Hence, $F_1 \sim F_2$.

We shall determine the equivalence classes of Frobenius morphisms on a classical Lie algebra in §5.

Remark 2.7. (1) If we modify the space structure on $V$, then a Frobenius map on $V$ can be described as a certain linear isomorphism to the new structure.

For each $k$-space $V$ and $r \geq 1$, let $V^{(r)}$ be the new vector space obtained from $V$ by a base change via $F$: $V^{(r)} = V \otimes_{\mathbb{F}_q} k$. Thus, putting $v^{(r)} = v \otimes 1$, we have

$$(u + v)^{(r)} = u^{(r)} + v^{(r)}, \quad (av)^{(r)} = a^{q^r} v^{(r)}.$$ 

So we have $V^{(r)} = V$ as abelian groups together with a new scalar multiplication

$$a \cdot v = q^r a \cdot v.$$ 

Let $\tau_{V,r} : V \to V^{(r)}$ be the $\mathbb{F}_q$-linear isomorphism sending $v$ to $v^{(r)}$ and let $\tau_V = \tau_{V,1}$. It is clear that $F : V \to V$ is a Frobenius morphism if and only if $F \circ \tau_V^{-1} : V \to V^{(1)}$ is a linear isomorphism.

(2) For a $k$-linear map $\phi : V \to W$, the map $\phi^{(r)} := \phi \otimes 1 : V^{(r)} \to W^{(r)}$ is again a $k$-linear map. Thus, we obtain an exact additive functor $( )^{(r)}$ from the category of $k$-spaces onto itself. Note that, using the identification $V = V^{(r)}$ and $W = W^{(r)}$ (with new scalar multiplications), we have $\phi = \phi^{(r)}$.

(3) If $\mathfrak{g}$ is a Lie algebra over $k$, then so is $\mathfrak{g}^{(1)}$, and $\tau_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}^{(1)}$ becomes an $\mathbb{F}_q$-algebra isomorphism. Clearly, a map $F : \mathfrak{g} \to \mathfrak{g}$ is a Frobenius morphism if and only if $\tau_{\mathfrak{g}} \circ F^{-1} : \mathfrak{g} \to \mathfrak{g}^{(1)}$ is a Lie algebra isomorphism.

(4) We shall see in the next section, not every $\mathfrak{g}$-module $V$ admits a module Frobenius map. If this is the case, then we call $V$ an $F$-stable module.

Let $\mathfrak{g} \text{-mod}^F$ denote the category whose objects are $F$-stable modules $V$ with a fixed module Frobenius map $F_V$ and whose morphism sets are defined by

$$\text{Hom}_{\mathfrak{g} \text{-mod}^F} (V, W) = \{ \theta \in \text{Hom}_\mathfrak{g} (V, W) \mid \theta \circ F_V = F_W \circ \theta \}.$$ 

As for associative algebras, we have the following category equivalence.

Theorem 2.8. The category $\mathfrak{g} \text{-mod}^F$ is equivalent to the category $\mathfrak{g}^F \text{-mod}$ of finite dimensional $\mathfrak{g}^F$-modules. In particular, there is a one-to-one correspondence between isoclasses of irreducible (resp., indecomposable) $\mathfrak{g}^F$-modules and isoclasses of irreducible (resp., indecomposable) $F$-stable $\mathfrak{g}$-modules.

\footnote{The Lie algebra $\mathfrak{g}^{(1)}$ has appeared in [6, p.1058].}
Proof. This follows immediately from the category equivalence between $U(\mathfrak{g})^{\mathfrak{g}}\text{-mod}$ and $U(\mathfrak{g})\text{-mod}^{F}$ given in [2, Thm3.2] and the category identification between $U(\mathfrak{g})\text{-mod}$ (resp. $U(\mathfrak{g})^{\mathfrak{g}}\text{-mod}$) and $\mathfrak{g}\text{-mod}$ (resp. $\mathfrak{g}^{\mathfrak{g}}\text{-mod}$).

3. Frobenius (twisting) functor on $\mathfrak{g}\text{-mod}$

We have seen above that a Frobenius morphism on $\mathfrak{g}$ induces naturally a Frobenius morphism on the universal enveloping algebra $U(\mathfrak{g})$. Thus, using the category identification between $U(\mathfrak{g})\text{-mod}$ and $\mathfrak{g}\text{-mod}$, we may transfer much of the theory on associative algebras developed in [2] (except the part only for finite dimensional algebras) to a theory on Lie algebras. However, for completeness and future applications, it is worthwhile to present the theory independently.

We start with twisting $\mathfrak{g}$-modules via a Frobenius morphisms $F$ on $\mathfrak{g}$.

**Definition 3.1.** Let $\mathfrak{g}$ be a Lie algebra over $k$ with a Frobenius morphism $F$ and let $V$ be a $\mathfrak{g}$-module defined by the Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V) := \text{End}_k(V)^{-}$. This gives a Lie algebra homomorphism $\rho^{(1)} : \mathfrak{g}^{(1)} \rightarrow \text{End}_k(V^{(1)})$. Thus, the composition $\rho^{[1]}$ of the following maps defines a $\mathfrak{g}$-module structure on $V^{(1)}$ with the following new action

$$x \cdot (v^{(1)}) = (F^{-1}(x)v)^{(1)}, \quad \forall x \in \mathfrak{g}, v \in V.$$  

We denote this module by $V^{[1]}$ and call it the Frobenius twist of $M$. Note that 3.1.1 defines the representation $\rho^{[1]} : U(\mathfrak{g}) \rightarrow \text{End}_k(V^{[1]})$ with

$$\rho^{[1]}(x) = \tau_{\mathfrak{g}} \circ \rho(F^{-1}(x)) \circ \tau_{\mathfrak{g}}^{-1}.$$  

Regarding $V^{[1]}$ as a $U(\mathfrak{g})$-module, we have the following commutative diagram:

$$
\begin{array}{ccc}
U(\mathfrak{g}) & \xrightarrow{\rho^{[1]}} & \text{End}_k(V^{[1]}) \\
\downarrow \tau_{U(\mathfrak{g})} \circ F^{-1} & & \downarrow \tau_{U(\mathfrak{g})} \circ F^{-1} \\
U(\mathfrak{g})^{(1)} & \xrightarrow{\rho^{(1)}} & \text{End}_k(V^{(1)})
\end{array}
$$

(3.1.2)

If $f : V \rightarrow W$ is a $\mathfrak{g}$-module homomorphism, then the $k$-linear map $f^{(1)} : V^{(1)} \rightarrow W^{(1)}$ becomes a $\mathfrak{g}$-module homomorphism $V^{[1]} \rightarrow W^{[1]}$ which is denoted by $f^{[1]}$ in the sequel. Thus, we obtain a functor

$$(\cdot)^{[1]} = (\cdot)^{[1]}_{A\text{-mod}} : A\text{-mod} \rightarrow A\text{-mod}.$$  

This functor will be called the Frobenius (twist) functor on $\mathfrak{g}\text{-mod}$. Clearly, it is a category equivalence.

Inductively, we can define the $s$-fold Frobenius twists $V^{[s]} := (V^{[s-1]})^{[1]}$ of $V$ and $f^{[s]} = (f^{[s-1]})^{[1]}$ for $s \geq 1$, where $V^{[0]} = V$ and $f^{[0]} = f$ by convention. Further, we can define $V^{[-1]}$ to be the $\mathfrak{g}$-module $W$ such that $V = W^{[1]}$ and similarly for $f^{[-1]}$, and inductively for $V^{[s]}$ and $f^{[s]}$ for $s < 0$.

We may twist a $\mathfrak{g}$-module $V$ via a Frobenius map $F_V$ on $V$: define a new $\mathfrak{g}$-module $V^{[F_V]}$, called the $F_V$-twist of $V$, such that $V^{[F_V]} = V$ as a vector space together with a new ($F$-twisted) $\mathfrak{g}$-action

$$x \ast v := F_V(F^{-1}(x)F^{-1}_V(v)) \text{ for all } x \in \mathfrak{g}, v \in V.$$
If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\rho^{[F\mathcal{F}]} : \mathfrak{g} \to \mathfrak{gl}(V^{[F\mathcal{F}]}$) denote the corresponding representations, then

$$\rho^{[F\mathcal{F}]}(x) = F_{(V)}(\rho(F^{-1}(x))) = F_{V} \circ \rho(F^{-1}(x)) \circ F_{V}^{-1}$$

for all $x \in \mathfrak{g}$,

where $F_{(V)}$ is the induced Frobenius map on $\mathfrak{gl}(V)$ (cf. Example 2.1(1)). Note that, if $F_{V}$ is a module Frobenius map, then $x * v = xv$ and so $V^{[F\mathcal{F}]} = V$ as $\mathfrak{g}$-modules. Moreover, one checks easily that the linear isomorphism $\tau \circ F_{V}^{-1} : V^{[F\mathcal{F}]} \to V[1]$ is in fact a $\mathfrak{g}$-module isomorphism. Thus, we have already proved the following.

**Lemma 3.2.** Up to isomorphism, the $F_{V}$-twist $V^{[F\mathcal{F}]}$ is independent of the choice of the Frobenius map $F_{V}$ on $V$, and is isomorphic to $V[1]$.

Inductively, we may also define $s$-fold $F_{V}$-twist $V^{[F\mathcal{F}]^{s}} = (V^{[F\mathcal{F}]}^{s-1})^{[F\mathcal{F}]}$ for each integer $s > 1$. Clearly, we have $V^{[F\mathcal{F}]^{s}} = V^{[F\mathcal{F}]^{s-1}}$; the latter is the $F_{V}^{s}$-twist of $V$ whose $\mathfrak{g}$-action is defined with respect to $F_{V}^{s}$ on $\mathfrak{g}$:

$$(3.2.1) \quad x * s v := F_{V}^{s}\left(F^{-s}(x)F_{V}^{-s}(v)\right) \quad \text{for all } x \in \mathfrak{g}, v \in V.$$

Clearly, $*_{s} = *$ for $s = 1$.

A $\mathfrak{g}$-module $V$ is called $F$-periodic if $V \cong V^{[r]}$ for some $r \geq 1$. The minimal number satisfying $V \cong V^{[r]}$ is called the $F$-period of $V$, denoted by $\varphi(V) = \varphi_{F}(V)$. Clearly, if $\varphi(V) = r$, then $V, V^{[1]}, \ldots, V^{[r-1]}$ are pair-wise non-isomorphic. We call the list $V, V^{[1]}, \ldots, V^{[r-1]}$, where $r = \varphi(V)$, the $F$-orbit of $V$.

The $F$-period measures the distance for a $\mathfrak{g}$-module to possess a module Frobenius map.

**Proposition 3.3.** Let $V$ be a $\mathfrak{g}$-module. Then $\mathfrak{g}$ is $F$-periodic if and only if there exists a Frobenius map $F_{V}$ on $V$ and an integer $r \geq 1$ such that $F_{V}^{r}$ is a module Frobenius map on $V$ (with respect to $F_{V}$ on $\mathfrak{g}$). In particular, if $\mathfrak{g}$ is a finite dimensional Lie algebra, then every $\mathfrak{g}$-module is $F$-periodic.

**Proof.** The “if” part is clear since $V = V^{[F\mathcal{F}]} \cong V^{[r]}$. The converse is seen as follows. Suppose $\phi : V^{[r]} \to V$ is a given isomorphism, and recall the map $\tau = \tau_{V, r} : V \to V^{[r]}$. Then $F = \phi \circ \tau$ is a $q^{r}$-Frobenius map on $V$, and by (3.1.1) we have, for any $x \in \mathfrak{g}, v \in V$,

$$F'(am) = (\phi \circ \tau)(am) = \phi((am)^{r}) = \phi(F^{r}(a) \bullet m^{(r)}) = F^{r}(a)(\phi \circ \tau)(m) = F^{r}(a)F^{r}(m).$$

So $F'$ is a module $q^{r}$-Frobenius map. Now, by using a basis for $F^{r}$, we may define a $q$-Frobenius morphism $F_{V}$ satisfying $F_{V}^{r} = F'$.

In case $\mathfrak{g}$ is finite dimensional, then, for any Frobenius map $F_{V}$ on $V$, $F_{V}^{r}$ is a module Frobenius map for some $r$. To see this, take $k$-basis $\{x_{1}, \ldots, x_{s}\}$ for $\mathfrak{g}$ and $k$-basis $\{v_{1}, \ldots, v_{t}\}$ for $V$, such that for $x = \sum a_{i}x_{i} \in \mathfrak{g}$ and $v = \sum b_{j}v_{j} \in V$

$$F(\sum a_{i}x_{i}) = \sum a_{i}^{q}x_{i}, \quad \text{and} \quad F_{V}(\sum b_{j}v_{j}) = \sum b_{j}^{q}v_{j}.$$  

Write $x_{i}v_{j} = \sum c_{ijl}v_{l}$, where $c_{ijl} \in k$. Then there is an integer $r$ such that all $a_{i}, b_{j}, c_{ijl}$’s are in $\mathbb{F}_{q} r$. Thus, $F^{r}(xv) = xv = F_{g}(x)F^{r}(v)$. \hfill \square

**Corollary 3.4.** Let $V$ be a $\mathfrak{g}$-module with a Frobenius map $F_{V}$.

(a) $V$ is $F$-stable if and only if $V \cong V^{[1]}$.

(b) If $V$ is $F$-periodic, then there exists an $F$-stable module $W$ such that $V$ is isomorphic to a direct summand of $W$.

**Proof.** The statement (a) follows directly from the proposition above. If $\varphi(V) = r$, then $\varphi(W) = 1$ where

$$W := V \oplus V^{[1]} \oplus \cdots \oplus V^{[r-1]}.$$  

So $W$ is $F$-stable by (a), and $V$ is a direct summand of $W$, proving (b). \hfill \square
The proof of 3.3 provides a module Frobenius map $F_W$ on $W$ which depends on a selection of a $g$-module automorphism $\phi : W^{[1]} \to W$. So this does not provide much information on the $g^F$-module $W^{F_W}$. The following construction of a module Frobenius map for an $F$-periodic module is given in [2, §5].

Let $V$ be $F$-periodic with $F$-period $r = \varphi(V)$. Then, by Prop 3.3, there is a Frobenius map $F_V$ on $V$ such that $F_V$ is a module Frobenius map on $V$ with respect to $F^r$. This means $V^{[F_V]} = V$ as a $g$-module. Thus, $V, V^{[F_V]}, \ldots, V^{[F_V^{-1}]}$ are pairwise non-isomorphic. Let

\[(3.4.1)\] 
$\tilde{V} = V \oplus V^{[F_V]} \oplus \cdots V^{[F_V^{-1}]}$

and define a Frobenius map $\tilde{F}_V : \tilde{V} \to \tilde{V}$ by

\[(3.4.2)\] 
$\tilde{F}_V(v_0, v_1, \ldots, v_{r-1}) = (F_V(v_{r-1}), F_V(v_0), \ldots, F_V(v_{r-2})).$

Since for $x \in g$ and $v = (v_0, v_1, \ldots, v_{r-1}) \in \tilde{V}$ (noting (3.2.1))

$$xv = (xv_0, F_V(F^{-1}(x)F_V^{-1}(v_1)), \ldots, F_V^{r-1}(F^{-r+1}(x)F_V^{-r+1}(v_{r-1}))),$$

it follows that

$$\tilde{F}_V(xv) = (F(x)F_V(v_{r-1}), F_V(xv_0), F_V^2(F^{-1}(x)F_V^{-1}(v_1)), \ldots)$$

$$= (F(x)F_V(v_{r-1}), F(x) \ast_1 F_V(v_0), F(x) \ast_2 F_V(v_1), \ldots)$$

$$= F(x)\tilde{F}_V(v).$$

Hence, $\tilde{F}_V$ is a module Frobenius map and $\tilde{V}$ is $F$-stable. Thus, $\tilde{V}^F := \tilde{V}\tilde{F}_V$ is a $g^F$-module.

**Theorem 3.5.** Let $V$ be an $F$-periodic $g$-module with $r = \varphi(V)$, and let $F_V$ be a Frobenius map on $V$ such that $F_V$ is a module Frobenius map with respect to $F^r$. Let $\tilde{F} = \tilde{F}_V$ be the module Frobenius map on $\tilde{V}$ defined in (3.4.2).

(a) If $V$ is irreducible, then $\tilde{V}^F$ is an irreducible $g^F$-module, and $\text{End}_{g^F}(\tilde{V}^F) \cong F^{q^r}$.

(b) If $V$ is indecomposable, then $\tilde{V}^F$ is an indecomposable $g^F$-module.

Moreover, every indecomposable or irreducible $g^F$-module can be obtained in this way.

**Proof.** We give a proof for the irreducible case. The proof for the indecomposable case is similar, involving care of the radical of endomorphism algebras; see [2, 5.1] for details in this case.

Let $V$ be an irreducible $g$-module. Then $V, V^{[F_V]}, \ldots, V^{[F_V^{-1}]}$ are all irreducible and no two of them are isomorphic. Suppose $\tilde{V}^F$ has a proper submodule $0 \neq X \subset \tilde{V}^F$. Then $X \otimes k$ is a proper $F$-stable submodule of $\tilde{V}$. This implies that $(X \otimes k) \cap V^{[F_V]}$ is a proper (nonzero) submodule of $V^{[F_V]}$ for some $i$, a contradiction. To see the isomorphism, observe

$$\text{End}_g(\tilde{V}) \cong \text{End}_g(V) \oplus \cdots \text{End}_g(V^{[r-1]}) \cong k \oplus \cdots \oplus k \ (r \text{ copies}).$$

So we identify an element $f \in \text{End}_g(\tilde{V})$ with an $r$-tuple $(a_0, a_1, \ldots, a_{r-1})$ where $a_i \in k$. Now, it is direct to check that the induced Frobenius morphism $F' := F'_V(\tilde{V}, \tilde{V})$ of $\tilde{F}$ is given by

$$F'(a_0, a_1, \ldots, a_{r-1}) = (a_0^q, a_1^q, \ldots, a_{r-2}^q).$$

Hence,

$$\text{End}_g(\tilde{V})^{F'} \cong F^{q^r}.$$

Now, the isomorphism $\text{End}_g(\tilde{V})^{F'} \cong \text{End}_{g^F}(\tilde{V}^F)$ (see 2.3) gives the required result. The proof for the last statement is straightforward.
Corollary 3.6. If \( g \) is a finite-dimensional Lie algebra, then there is a one-to-one correspondence between the isoclasses of indecomposable (resp. irreducible) \( g^F \)-modules and the \( F \)-orbits of the isoclasses of indecomposable (resp. irreducible) \( g \)-modules.

This establishes a close connection between simple representations of a finite Lie algebra \( g_0 \) over the finite field \( F_q \) and of its \( k \)-extension \( g = g_0 \otimes k \) over the closed algebraically closed field \( k = \mathbb{F}_q \).

4. Restricted Lie Algebras and their Representations

From now on, we will assume that \( g \) is a finite dimensional restricted Lie algebra. Thus, \( g \) is endowed with a \( p \)-map \( \langle p \rangle \) satisfying certain conditions\(^3\) (see [9, p.185] for more details). Suppose \( F \) is a Frobenius morphism on \( g \). Call \( F \) a restricted Frobenius morphism if it satisfies the additional condition \( F(x^{(p)}) = F(x)^{(p)} \).

Lemma 4.1. Suppose \( F \) is a Frobenius morphism on a restricted Lie algebra \((g, \langle p \rangle)\).

(a) \( F \) is restricted if and only if \( g^F \) is a restricted subalgebra of \( g \) over \( F_q \).

(b) If \( F \) is restricted, then \((g^n, \langle p^n \rangle)\)Frobenius morphism \( F^n \) is also restricted.

Proof. The statement (b) is clear. We now prove (a). If \( F \) is a restricted morphism, then, for any \( x \in g^F \), \( F(x^{(p)}) = F(x)^{(p)} = x^{(p)} \). Hence \( x^{(p)} \in g^F \). So \( g^F \) is really a restricted subalgebra.

Conversely, suppose \( g^F \) is a restricted subalgebra. Fix a basis of \( g \) in \( g^F \) : \( x_i, i = 1, 2, \ldots, l \). Then \( F(x_i^{(p)}) = x_i^{(p)} \) for all \( i \). For any \( x \in g \), we write it uniquely as \( x = \sum_{i \in I} a_i x_i \) where \( a_i \neq 0 \) and \( I \) is a subset of \( \{1, 2, \ldots, l \} \). We now prove \( F(x^{(p)}) = F(x)^{(p)} \) by induction on \( |I| \). Clearly, for \( |I| = 1 \), we have \( F((a_i x_i)^{(p)}) = a_i^{(p)} x_i^{(p)} = F(a_i x_i)^{(p)} \). For \( |I| > 1 \), write \( x = \sum_{s \in S} a_s x_s + \sum_{t \in T} b_t x_t \), where \( I \) is a disjoint union of non-empty subsets \( S \) and \( T \), and \( A = \sum_{s \in S} a_s x_s \) and \( B = \sum_{t \in T} b_t x_t \). Since \( (A + B)^{(p)} = A^{(p)} + B^{(p)} + \Lambda_p(A, B) \), where \( \Lambda_p(A, B) \) is an \( \mathbb{F}_p \)-linear combination of monomials \( \text{ad} C_1 \text{ad} C_2 \cdots \text{ad} C_{p-1} \) with each \( C_i \in \{A, B\} \), it follows that

\[
F((A + B)^{(p)}) = F(A^{(p)}) + F(B^{(p)}) + F(\Lambda_p(A, B)),
\]

\[
F(A^{(p)} + B^{(p)}) + \Lambda_p(F(A), F(B)).
\]

By induction, \( F(A^{(p)}) = F(A)^{(p)} \) and \( F(B^{(p)}) = F(B)^{(p)} \). Meanwhile, \( F \) is a Lie \( \mathbb{F}_q \)-automorphism, and hence satisfies \( F(\Lambda_p(A, B)) = \Lambda_p(F(A), F(B)) \). Therefore, we have \( F(x^{(p)}) = F(x)^{(p)} \) for any \( x \in g \), as desired. \( \square \)

Definition 4.2. (1) For \( \chi \in g^* \) define \( U_\chi(g) = U(g)/J_\chi \), where \( J_\chi \) is the ideal generated by \( x^p - x^{(p)} - \chi(x)^p \) for all \( x \in g \). Each \( U_\chi(g) \) is called a reduced enveloping algebra of \( g \). If \( \chi = 0 \), then \( U_0(g) \) is just the restricted enveloping algebra, which is called the \( u \)-algebra of \( g \) in [9, p.192].

(2) If \( V \) is a \( g \)-module defined by \( \rho : g \to \mathfrak{gl}(V) \) and \( \chi \in g^* \) then we say that \( V \) has \( p \)-character \( \chi \) if \( \rho(x)^p - \rho(x^{(p)}) = \chi(x)^p id_V \) for all \( x \in g \). The \( g \)-modules with \( p \)-character 0 correspond to the restricted representations \( \rho \) which satisfy: \( \rho(x)^p = \rho(x^{(p)}) \) for all \( x \in g \).

We list a few facts about reduced enveloping algebras and \( p \)-characters (see [20, 17]).

Lemma 4.3. (a) If \( x_1, \ldots, x_l \) is a basis of \( g \) then the algebra \( U_\chi(g) \) has basis \( x_1^{p_{i_1}} \cdots x_l^{p_{i_l}}, 0 \leq n_i \leq p \) for all \( i = 1, 2, \ldots, l \). In particular, all \( U_\chi(g) \) have the same underlying vector space.

(b) Every simple \( g \)-module has a \( p \)-character.

\(^3\)We didn’t use the traditional notation \( [p] \) for a \( p \)-map in this paper to avoid possible confusion with the Frobenius functor \( [p] \).
(c) The bijective correspondence \( \{ \mathfrak{g} \text{-modules} \} \rightarrow \{ U(\mathfrak{g}) \text{-modules} \} \) induces for each \( \chi \in \mathfrak{g}^* \) a bijective correspondence:

\[
\{ \text{modules of } p\text{-character } \chi \} \rightarrow \{ U_\chi(\mathfrak{g}) \text{-modules} \}.
\]

In particular, the \( \mathfrak{g} \text{-module } U_\chi(\mathfrak{g}) \) itself has \( p\text{-character } \chi \) and corresponds to the regular representation of \( U_\chi(\mathfrak{g}) \).

Part (b) is seen as follows. For any \( x \in \mathfrak{g} \), \( \xi(x) = x^p - x^{(p)} \) is in the center \( Z(\mathfrak{g}) \) of \( U(\mathfrak{g}) \), and the map \( \xi : U(\mathfrak{g}) \rightarrow Z(\mathfrak{g}) \) is \( p\)-semilinear. So, if \( V \) is a simple \( \mathfrak{g} \text{-module} \), then \( \xi(x) \) acts on \( V \) by a scalar. This scalar can be written as \( \chi(x)^p \) where \( \chi : \mathfrak{g} \rightarrow k \) (see [17, 5.2.5]). The semilinearity of \( \xi \) implies that \( \chi \in \mathfrak{g}^* \). Thus, \( \chi \) is the \( p\text{-character} \) of \( V \).

We now bring Frobenius morphisms into the theory. Let \( F \) be a restricted Frobenius morphism on \( \mathfrak{g} \), and for \( \chi \in \mathfrak{g}^* \), let

\[
\chi^{[1]} = F^*(\chi) = f \circ \chi \circ F^{-1}.
\]

Thus, \( \chi^{[1]}(x) = (\chi(F^{-1}x))^q \) for \( x \in \mathfrak{g} \). Inductively, we can define \( \chi^{[r]} = (\chi^{[r-1]})^{[1]} \) for \( r > 1 \). Note that one should not confuse \( \chi^{[1]} \) with the notation \( \chi^{(1)}(= \chi) \) introduced in 2.7(2).

**Proposition 4.4.** Suppose \( F \) is a restricted Frobenius morphism on the restricted Lie algebra \( \mathfrak{g} \). Then \( F \) induces an \( \mathbb{F}_q \text{-algebra isomorphism } F : U_\chi(\mathfrak{g}) \rightarrow U_{\chi^{[1]}}(\mathfrak{g}) \) which is \( q\)-semilinear. In particular, \( F \) is a Frobenius map on their common underlying \( k\text{-space}. \)

**Proof.** Clearly, \( F \) induces a Frobenius morphism on the associated \( k\text{-algebra } U(\mathfrak{g}) \) sending the basis elements \( x_1^{n_1} \cdots x_l^{n_l} \) to \( F(x_1)^{n_1} \cdots F(x_l)^{n_l} \). Putting \( \xi_\chi(x) := x^p - x^{(p)} - \chi(x)^p \) for \( x \in U(\mathfrak{g}) \), we have

\[
F(\xi_\chi(x)) = F(x)^p - F(x^{(p)}) - \chi(x)^p = F(x)^p - F(x^{(p)}) - (f(\chi(x)))^p = F(x)^p - F(x^{(p)}) - (\chi^{[1]}(F(x)))^p = \xi_{\chi^{[1]}}(F(x)) \in J_{\chi^{[1]}}.
\]

Thus, \( F \) carries the ideal \( J_\chi \) into the ideal \( J_{\chi^{[1]}} \), and hence induces an injective map from \( U_\chi(\mathfrak{g}) \) onto \( U_{\chi^{[1]}}(\mathfrak{g}) \). Now, the result follows from a dimensional comparison. \( \square \)

Recall from Remark 2.7(1) that the map \( \tau_\mathfrak{g} : \mathfrak{g} \rightarrow \mathfrak{g}^{(1)} \). Clearly, it induces an \( \mathbb{F}_q \text{-algebra isomorphism } \)

\[
\tau_\mathfrak{g} : U_\chi(\mathfrak{g}) \rightarrow U_{\chi^{(1)}}(\mathfrak{g}) = U_{\chi^{[1]}}(\mathfrak{g}).
\]

**Corollary 4.5.** The \( q\)-semilinear isomorphism \( F \) from \( U_\chi(\mathfrak{g}) \) to \( U_{\chi^{[1]}}(\mathfrak{g}) \) induces a \( k\text{-algebra isomorphism } \tau_\mathfrak{g} \circ F^{-1} : U_{\chi^{[1]}}(\mathfrak{g}) \rightarrow U_{\chi^{(1)}}(\mathfrak{g}) \). In particular, it restricts to a \( \mathfrak{g} \text{-module isomorphism } U_{\chi^{[1]}}(\mathfrak{g}) \cong U_{\chi^{(1)}}(\mathfrak{g}) \). Hence we have \( \varphi(U_\chi(\mathfrak{g})) = \varphi(\chi) \).

**Proof.** The first assertion is clear. Since \( U_\chi(\mathfrak{g})^{[1]} = U_{\chi^{(1)}}(\mathfrak{g}) = U_{\chi^{[1]}}(\mathfrak{g}) \) by definition, restriction gives the required \( \mathfrak{g} \text{-module isomorphism}. \) (Alternatively, both \( U_{\chi^{[1]}}(\mathfrak{g}) \) and \( U_{\chi^{(1)}}(\mathfrak{g}) \) have the same underlying space, and for \( x \in \mathfrak{g} \) and \( y \in U_\chi(\mathfrak{g})^{[1]} \),

\[
x * y = F(F^{-1}(x)F^{-1}(y)) = xy.
\]

Thus, the identity map is the required \( \mathfrak{g} \text{-module isomorphism}. \) \( \square \)

**Corollary 4.6.** Suppose \( V \) is a \( \mathfrak{g} \text{-module with } p\text{-character } \chi. \) Then:

(a) \( V^{[1]} \) is a \( \mathfrak{g} \text{-module with } p\text{-character } \chi^{[1]}; \)

(b) If \( V \) is \( F\)-stable then \( \chi^{[1]} = \chi; \)


(c) \( \varphi(\chi) | \varphi(V) \).

**Proof.** The commutative diagram (3.1.2) gives rise to a commutative diagram

\[
\begin{array}{ccc}
U_{\chi}^{[1]}(g) & \xrightarrow{\rho^{[1]}} & \text{End}_k(V^{[1]}) \\
\tau_{U(g)} \circ F^{-1} & & \\
U_{\chi}(g) \oplus & \rho^{(1)} & \text{End}_k(V)^{(1)} \\end{array}
\]

So (a) follows immediately, and (b) and (c) follow from (a). \( \square \)

If \( r = \varphi(\chi) > 1 \), then the induced Frobenius map \( F \) on the \( g \)-module \( U_{\chi}(g) \) in 4.4 is not a module Frobenius map. However, by the construction in (3.4.1) and (3.4.2), \( F \) gives rise to a module Frobenius morphism \( \tilde{F}_{\chi} \) on

\[
\tilde{U}_{\chi}(g) := U_{\chi}(g) \oplus U_{\chi}^{[1]}(g) \oplus \cdots \oplus U_{\chi}^{[r-1]}(g).
\]

Since the \( g \)-action on \( \tilde{U}_{\chi}(g) \) gives in fact the regular representation of the \( k \)-algebra \( \tilde{U}_{\chi}(g) \), the module Frobenius map \( \tilde{F}_{\chi} \) is really a Frobenius morphism on the algebra \( \tilde{U}_{\chi}(g) \). On the other hand, it is clear that \( \tilde{U}_{\chi}(g) = \tilde{U}_{\chi}^{[1]}(g) \). So the algebra \( \tilde{U}_{\chi}(g) \) depends only on the \( F \)-orbit \( \chi = (\chi, \chi^{[1]}, \cdots, \chi^{[r-1]}) \) of \( \chi \). Thus, we will simply write \( U_{\chi}(g) \) for \( \tilde{U}_{\chi}(g) \), and \( F_{\chi} \) for \( \tilde{F}_{\chi} \) in the sequel.

We first characterize \( F_{\chi} \)-stable \( U_{\chi}(g) \)-modules via \( \tilde{U}_{\chi}(g) \)-modules.

**Lemma 4.7.** Let \( V \) be a \( g \)-module of \( p \)-character \( \chi \). Then the module \( \tilde{V} \) defined in (3.4.1) is naturally an \( F_{\chi} \)-stable \( U_{\chi}(g) \)-module.

**Proof.** Let \( V \) be a \( g \)-module with \( p \)-character \( \chi \). Then, by 4.6(c), its \( F \)-period \( s \) of \( V \) has a factor \( r = \varphi(\chi) \). By 3.3, we choose a Frobenius map \( \tilde{F}_V \) on \( V \) satisfying \( V^{[\tilde{F}_V]} = V \) as a \( g \)-module. Let

\[
V' = V \oplus V^{[\tilde{F}_V]} \oplus \cdots \oplus V^{[\tilde{F}_V^{[t-1]}]},
\]

where \( t = s/r \), and take the Frobenius map \( F_{\tilde{V}} = (F_V, \cdots, F_V) \) on \( V' \). Then \( V' \) is a \( U_{\chi}(g) \)-module with \( \varphi(V') = r \), and \( \tilde{V}' = V' \oplus V^{[\tilde{F}_V]} \oplus \cdots \oplus V^{[\tilde{F}_V^{[t-1]}]} \cong \tilde{V} \) is a \( \tilde{U}_{\chi}(g) \)-module satisfying \( V^{[\tilde{F}_V]} = V' \) (i.e., \( \tilde{F}_V \) is a Frobenius map on \( V' \) with respect to \( \tilde{F}_V \)). Thus, it induces a module Frobenius map \( \tilde{F}_{\tilde{V}} \) on \( \tilde{V}' \) and hence on \( \tilde{V} \). Therefore, \( \tilde{V} \) is \( F_{\chi} \)-stable. \( \square \)

The next result shows that these \( F_{\chi} \)-stable modules are all \( F_{\chi} \)-stable modules.

**Theorem 4.8.** A \( g \)-module \( W \) is an \( F_{\chi} \)-stable \( U_{\chi}(g) \)-module with a module Frobenius morphism \( F_W \) if and only if \( W \cong \tilde{V} \) for some \( g \)-module \( V \) of \( p \)-character \( \chi \) (i.e., a \( U_{\chi}(g) \)-module), and there is a Frobenius morphism \( \tilde{F}_V \) on \( V \) such that the isomorphism is compatible with \( F_W \) and \( \tilde{F}_V \) (that is, it induces isomorphism \( W^{F_W} \cong \tilde{V}^{\tilde{F}_V} \)).

**Proof.** The sufficient part follows from the above lemma. We now prove the necessary part.

Let \( e_0, e_1, \cdots, e_{r-1} \) be the central orthogonal idempotents of \( U_{\chi}(g) \) such that \( e_i U_{\chi}(g) \cong U_{\chi}^{[i]}(g) \). Then \( W = \oplus_{i=0}^{r-1} e_i W \) and \( F_{\chi}(e_i) = e_{i+1} \) where \( e_r = e_0 \). Since \( F_W \) is a module Frobenius map with respect to \( F_{\chi} \), it follows that \( F_W(e_i W) \subseteq e_{i+1} W \). By a comparison of dimensions, we must have \( F_W(e_i W) = e_{i+1} W \) for all \( i \). Let \( V = e_0 W \). Then \( V \) is a \( g \)-module with \( p \)-character \( \chi \), and \( V^{[\tilde{F}_V]} \cong F_W(V) \) for any Frobenius map \( \tilde{F}_V \) on \( V \), where \( \varphi^{[1]} = F_W \circ F_V^{-1} \) satisfying \( \varphi^{[1]}(x * v) =  

be the full subcategory consisting of $\varphi[i]$. In general, we have $V^{[F_{V}]_{\varphi[i]}} \cong F_{W}^{[V]}(V)$ where $\varphi[i] = F_{W}^{[V]} \circ F_{V}^{-i}$. Since $F_{V}^{r}(V) = V$, $F_{V}^{r}$ is a module Frobenius map in $V$ (with respect to $F^{r}$ on $g$). We choose $F_{V}$ on $V$ such that $F_{V}^{r} = F_{W}^{[V]}$. Then $F_{V}^{r}$ is a module Frobenius map with respect to $F^{r}$ on $g$ such that $V^{[F_{V}]} = V$ as a $g$-module (cf. 3.3), and $\phi^{[r]} = F_{W}^{[V]} \circ F_{V}^{-r} = id_{V} = \phi^{[0]}$. Now, the $U_{\chi}(g)$-module isomorphism

$$\phi^{[0]} + \phi^{[1]} + \cdots + \phi^{[r-1]} = \varphi : \tilde{V} \to W,$$

where $\tilde{V} = V \oplus V^{[F_{V}]} \oplus \cdots \oplus V^{[F_{V}^{r-1}]}$, is compatible with the Frobenius morphisms $\tilde{F}_{V}$ on $\tilde{V}$ and $F_{W}$ on $W$. \hfill $\square$

Let $g^{F_{\chi}}$ be the full subcategory consisting of $g^{F}$-modules $V_{0}$ such that the $g$-module $V = V_{0} \otimes k$ is an $F_{\chi}$-stable $U_{\chi}(g)$-modules. Thus, $V^{F_{\chi}}$ is a $U_{\chi}(g)^{F_{\chi}}$-module. We now characterize all $U_{\chi}(g)^{F_{\chi}}$-modules.

**Theorem 4.9.** Maintain the notation above, and let $U_{\chi}(g)^{F_{\chi}}$ be the abelian category of $F_{\chi}$-stable modules.

(a) There is a category equivalence

$$U_{\chi}(g)^{F_{\chi}} \text{-mod} \cong U_{\chi}(g) \text{-mod}^{F_{\chi}}.$$

(b) The category $U_{\chi}(g)^{F_{\chi}} \text{-mod}$ is a full subcategory of $g^{F_{\chi}} \text{-mod}$ where $r = \varphi(\chi)$.

(c) The (base change) functor

$$- \otimes_{F_{q}} F_{q}^{r} = \Theta : g^{F} \text{-mod} \to g^{F_{\chi}} \text{-mod}$$

restricts to a functor

$$\Theta : g^{F_{\chi}} \text{-mod}_{\chi} \to U_{\chi}(g)^{F_{\chi}} \text{-mod}.$$ 

**Proof.** The statement (a) follows from [2, 3.2].

We now prove (b). Clearly, for any $x = (x_{0}, x_{1}, \cdots, x_{r-1}) \in U_{\chi}(g)$, $F_{\chi}(x) = x$ if and only if $F_{V}(x_{i}) = x_{i}$ for all $i$. Thus, the embedding $g \to U_{\chi}(g)$ induces an embedding $g^{F_{\chi}} \to U_{\chi}(g)^{F_{\chi}}$. On the other hand, the epimorphism $U_{\chi}(g) \to U_{\chi}(g)$ induces an epimorphism $U_{\chi}(g)^{F_{\chi}} \to U_{\chi}(g)^{F_{\chi}}$, and hence an epimorphism $U_{\chi}(g)^{F_{\chi}} \to U_{\chi}(g)^{F_{\chi}}$ by Lemma 2.4. This proves (b).

It remains to prove (c). We first claim that, if $V$ is a $g$-module with $F$-period $s$ and $p$-character $\chi$ and $F_{V}$ is a Frobenius map on $V$ such that $F_{V}^{r}$ is a module Frobenius map on $V$, then the module $\tilde{V}^{(F_{V})^{r}}$ is a $U_{\chi}(g)^{F_{\chi}}$-module, where $F_{V}$ is defined in (3.4.2). Indeed, let $s = \varphi(V)$, $r = \varphi(\chi)$, $t = \frac{s}{r}$, and let $W = V \oplus V^{[F_{V}]} \cdots \oplus V^{[F_{V}^{r-1}]}$. Then $W$ is a $U_{\chi}(g)$-module with $\varphi(W) = t$, and $F_{W} = (F_{V}^{r}, \cdots, F_{V}^{r})$ is a Frobenius map on $W$ satisfying $W^{[F_{W}]} = W$—a condition required in the construction (3.4.1) and (3.4.2). But, $\tilde{W} = W \oplus W^{[F_{W}]} \cdots \oplus W^{[F_{W}^{r-1}]} = \tilde{V}$ is a $U_{\chi}(g)$-module, and $\tilde{F}_{W} = (\tilde{F}_{V})^{r}$ is a module Frobenius morphism on $\tilde{V}$ with respect to $F_{\chi}$. Therefore, $\tilde{V}^{(\tilde{F}_{V})^{r}} = \tilde{W}^{\tilde{F}_{W}}$ is a $U_{\chi}(g)^{F_{\chi}}$-module. Since every indecomposable $g^{F}$-module is of the form $\tilde{V}^{(F_{V})^{r}}$ for some $g$-module $V$ by 3.5, and $\tilde{V}^{F_{V}} \otimes F_{q}^{r} \cong \tilde{V}^{(F_{V})^{r}}$, it follows from the claim that the functor take an indecomposable $g^{F}$-module, and hence any $g^{F}$-module, to a $U_{\chi}(g)^{F_{\chi}}$-module, proving (c). \hfill $\square$

**Corollary 4.10.** If $\chi^{[1]} = \chi$, then $U_{\chi}(g) = U_{\chi}(g)$ and $F_{\chi} = F$. In particular, $U_{\chi}(g)^{F}$ is a homomorphic image of $U(g)^{F}$.
Recall from §2 that $\text{Aut}(g)$ is the automorphism group of $g$. Let $\text{Aut}(g)^{(p)}$ denote the subgroup of $\text{Aut}(g)$ consisting of all automorphisms $\sigma$ preserving the $p$-map, i.e. $\sigma(x^{(p)}) = \sigma(x)^{(p)}$ for all $x \in g$. Then each $\sigma \in \text{Aut}(g)^{(p)}$ induces an isomorphism of algebras: $U_\chi(g) \cong U_{\sigma(\chi)}(g)$, where $\sigma(\chi) = \chi \circ \sigma^{-1}$. Thus, the representation theory of $U_\chi(g)$ depends only on the $\text{Aut}(g)^{(p)}$-orbit of $\chi$. We now show that there is an analogous relation for finite Lie algebras.

Let $g_0$ be a finite Lie algebra over $\mathbb{F}_q$ and let $g = g_0 \otimes k$ be equipped with the restricted Frobenius morphism $F = \text{id}_{g_0} \otimes f$. Thus, $g^F = g_0$. For every $\sigma \in \text{Aut}(g)^{(p)}$ the Frobenius morphism $F' := \chi * F = \sigma^{-1} \circ F \circ \sigma$ is also restricted and, by the proof of 2.6, $\sigma$ induces a Lie algebra isomorphism $\sigma : g^F \sim g^{F'}$. This results a category isomorphism

$$(4.10.1) \quad (\sigma) : g^F\text{-mod} \sim g^{F'}\text{-mod},$$

by taking a $g^F$-module $V$ to the $g^{F'}$-module $V^\sigma$ defined by setting $V^\sigma = V$ as vector space and $x \cdot v = \sigma^{-1}(x)v$ for all $x \in g^{F'}$ and $v \in V$.

**Lemma 4.11.** Let $g$ be a restricted Lie algebra with a restricted Frobenius morphism $F$ and let $\chi \in g^*$. Then every $\sigma \in \text{Aut}(g)^{(p)}$ induces a category isomorphism

$$(\sigma) : g^F\text{-mod}_\chi \sim g^{F'}\text{-mod}_{\chi'},$$

where $\chi' = \sigma(\chi)$ and $F' = \sigma * F = \sigma^{-1} \circ F \circ \sigma$.

**Proof.** We have seen that $\sigma$ induces isomorphism $U_\chi(g) \cong U_{\sigma(\chi)}(g)$, and hence isomorphism

$$\tilde{\sigma} : U_\chi(g) \sim U := U_{\sigma(\chi)}(g) \oplus U_{\sigma(\chi^{[1]})}(g) \oplus \cdots U_{\sigma(\chi^{[r-1]})}(g),$$

where $r = \varphi(\chi)$. If we put $\chi^{[1]}' = (F')^*(\chi) = \chi \circ F^{-1}$, then

$$\sigma(\chi^{[1]}) = \chi \circ F^{-1} \circ \sigma^{-1} = \chi \circ \sigma^{-1} \circ F' = \sigma(\chi)^{[1]}' = \chi'^{[1]}'.$$

This implies that $\sigma(\chi^{[i]}) = \chi'^{[i]}'$ for all $i$, and hence $\varphi_F(\chi) = \varphi_{F'}(\chi')$. Thus,

$$U = U_\chi \oplus U_{\chi^{[1]}} \oplus \cdots \oplus U_{\chi^{[r-1]}} = U_{\chi'},$$

and we obtain an algebra isomorphism $\tilde{\sigma} : U_\chi(g) \sim U_{\chi'}(g)$. This isomorphism is compatible with the Frobenius morphisms $F_\chi$ and $F'_{\chi'}$. Hence we obtain an algebra isomorphism

$$\tilde{\sigma} : U_\chi(g)^{F_{\chi}} \sim U_{\chi'}(g)^{F'_{\chi'}},$$

and thus, a category isomorphism

$$(\sigma)^\natural : U_\chi(g)^{F_{\chi}}\text{-mod} \sim U_{\chi'}(g)^{F'_{\chi'}}\text{-mod}.$$  

Now the category isomorphism $g^F\text{-mod}_{\chi} \cong g^{F'}\text{-mod}_{\chi'}$ follows from the following commutative diagram

$$
\begin{array}{c}
g^F\text{-mod}_{\chi} \xrightarrow{\Theta} U_\chi(g)^{F_{\chi}}\text{-mod} \\
\downarrow (\sigma)^\natural \\
g^{F'}\text{-mod}_{\chi'} \xrightarrow{\Theta} U_{\chi'}(g)^{F'_{\chi'}}\text{-mod}
\end{array}
$$

\[\square\]
Remark 4.12. (1) The irreducible representations of \( \mathfrak{g}^F \) are partitioned according to the full subcategories \( \mathfrak{g}^F \text{-mod}_\chi \). Thus, the algebras \( U^{\chi}_{\chi}(\mathfrak{g})^F \) are the counterpart of the algebras \( U^{\chi}(\mathfrak{g}) \) in the finite case.

(2) As is well known, all irreducible modules of a finite-dimensional Lie algebra \( \mathfrak{g} \) over an algebraically closed field of characteristic \( p > 0 \) must be finite-dimensional with an upper bound. In particular, if \( (\mathfrak{g}, \langle p \rangle) \) is a restricted Lie algebra then the upper bound is less than \( p^{\dim g} \). But, in the case of finite restricted Lie algebras over a finite field, the situation is completely different. There is no upper bound for the dimensions of irreducible modules.

5. \( \mathbb{F}_q \text{-forms of classical (restricted) Lie Algebras} \)

In this section, we assume that \( \mathfrak{g} \) is a classical simple Lie algebra over \( k \) with \( p \neq 2, 3 \). Thus, they are classified by Dynkin diagrams labelled by \( A_n (n \geq 1), B_n (n \geq 2), C_n (n \geq 3), D_n (n \geq 4), E_6, E_7, E_8, F_4, G_2 \) (see [15, II.10.1]). We may view such a \( \mathfrak{g} \) as a reduction modulo \( p \) via a Chevalley basis of the corresponding complex simple Lie algebra \( \mathfrak{g}^C \). Thus, \( \mathfrak{g} \) admits a basis arising from a Chevalley basis of \( \mathfrak{g}^C \), still called a Chevalley basis for \( \mathfrak{g}_k \) below. We shall use the idea of Frobenius morphisms on \( \mathfrak{g} \) to determine the \( \mathbb{F}_q \)-forms of classical Lie algebras, and hence to reproduce some classical result.

Fix a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) so that \( \mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha \) is the Cartan decomposition relative to \( \mathfrak{h} \).

Let \( \{ h_i, e_a | 1 \leq i \leq \text{dim}(\mathfrak{h}), \alpha \in R \} \) be the Chevalley basis for \( \mathfrak{g} \). Each \( e_a \) defines a (nilpotent) linear transformation \( \text{ad} e_a \) on \( \mathfrak{g} \) by sending \( y \in \mathfrak{g} \) to \( [e_a y] \). Thus, for any \( a \in k \), \( x_a(a) = \exp(\text{ad} ae_a) \) is a linear automorphism on \( \mathfrak{g} \), and the group \( G = G(\mathfrak{h}) \) generated by all \( x_a(a) \) is called the Chevalley group of \( \mathfrak{g} \) (constructed relative to \( \mathfrak{h} \)). Note that \( G \) is independent of the choice of \( \mathfrak{h} \) as \( k \) is algebraically closed [15, IV.1.2], and as an algebraic group, \( G \) is simple (and hence connected) and \( \mathfrak{g} = \mathfrak{Lie}(G) \) is the Lie algebra of \( G \) except that \( \mathfrak{g} \) is of type \( A_n \) with \( p \mid n + 1 \). Note that, in this exceptional case, \( \mathfrak{g} \cong \mathfrak{Lie}(G)' / \text{the center of} \mathfrak{Lie}(G)' \) with \( \mathfrak{Lie}(G)' = [\mathfrak{Lie}(G), \mathfrak{Lie}(G)] \) (cf. [8, 5.4]). In particular, \( \mathfrak{g} \) is a restricted Lie algebra.

Recall that for any \( \sigma \in \text{Aut}(\mathfrak{g}) \), \( \sigma \) can be written uniquely as \( \sigma = g \gamma \) where \( \gamma \) is a graph automorphism (defined by a graph automorphism of the corresponding Dynkin graph) and \( g \in G \). Furthermore, Steinberg proved (under the assumption \( p \neq 2, 3 \)) that \( \text{Aut}(\mathfrak{g}) = G \times \Gamma \) is a semi-direct product of the graph automorphism group \( \Gamma \) and \( G \), and \( G \) is a normal subgroup ([15, III.5.1]).

Let \( F_0 \) be the standard Frobenius morphism on \( \mathfrak{g} \) which fixes the Chevalley basis of \( \mathfrak{g} \). Then \( \mathfrak{g}_0 = \mathfrak{g}^{F_0} \) is a split classical Lie algebra over \( \mathbb{F}_q \) (of the same type as \( \mathfrak{g} \)) with respect to the Cartan subalgebra \( \mathfrak{h} \). (Thus, \( \mathfrak{g}_0 \) has a Cartan decomposition of the form \( \mathfrak{h}_0 = \mathfrak{h}^{F_0} + \sum_{\alpha} \mathfrak{g}_0^{F_0} \).)

A Frobenius morphism on \( \mathfrak{g} \) is called split if \( \mathfrak{g}^F \) is a classical simple Lie algebra. Let \( G * F_0 \) denote the \( G \)-orbit of \( F_0 \). Clearly, the elements in \( G * F_0 \) are split Frobenius morphisms.

**Lemma 5.1.**

(a) For any \( \sigma \in \text{Aut}(\mathfrak{g}) \), the fixed-point subalgebra of the Frobenius morphism \( \sigma * F_0 = \sigma^{-1} F_0 \sigma \) is the split Lie algebra \( \sigma^{-1} \mathfrak{g}_0 \) with respect to the Cartan subalgebra \( \sigma^{-1} \mathfrak{h} \).

(b) For any \( \gamma \in \Gamma \), \( \gamma^{-1} F_0 \gamma = F_0 \).

(c) For any \( \sigma \in \text{Aut}(\mathfrak{g}) \), there exists \( g \in G \) such that \( \sigma^{-1} F_0 \sigma = g^{-1} F_0 g \).

**Proof.** The statements (a) and (b) follow directly from the definitions, while (c) follows from (b) and Steinberg’s theorem on \( \text{Aut}(\mathfrak{g}) \). \( \square \)

**Lemma 5.2.** Let \( F_0 \) be the standard Frobenius morphism on \( \mathfrak{g} \).

---

4There is an exception in the type \( A_n \) case with \( p \mid n + 1 \). In this exceptional case, \( \mathfrak{g} \cong \mathfrak{g}_k / Z(\mathfrak{g}_k) \), where \( \mathfrak{g}_k := \mathfrak{g}_k \otimes_k k \) for the \( Z \)-form \( \mathfrak{g}_k \) of \( \mathfrak{g}_C \), spanned over \( Z \) by a Chevalley basis of \( \mathfrak{g}_C \), and \( Z(\mathfrak{g}_k) \) is one-dimensional center of \( \mathfrak{g}_k \) which is contained in all Cartan subalgebra of \( \mathfrak{g}_k \) [15, II.3].
(a) Let $\Phi : G \to G, g \mapsto F_0 g F_0^{-1}$. It is a surjective endomorphism of $G$ with a finite number of fixed points.

(b) For any $g \in G$, there exists $h \in G$ such that $g F_0 = h \ast F_0 = h^{-1} F_0 h$.

(c) Any Frobenius morphism $F$ on $g$ is a split Frobenius morphism followed by a graph automorphism: $F = \gamma \cdot F_0$ where $\gamma \in \Gamma$, $F_0' \in G \ast F_0$. Moreover, such a decomposition is unique.

Proof. We first observe that $\Phi$ is the restriction of the Frobenius morphism $F_{(\mathfrak{g}, 0)}$ on the affine space $\text{End}_k(\mathfrak{g})$. Since $G$ is generated by elements of $\text{Aut}(\mathfrak{g})$ of the form $x_\alpha(a e_\alpha) := \exp(\text{ad} a e_\alpha)$, where $\alpha \in R$ and $a \in k$, the surjectivity of $\Phi$ follows from the fact $\Phi(x_\alpha(a)) = x_\alpha(a^q)$ which is seen by acting $\Phi(x_\alpha(a))$ on the Chevalley basis (see, e.g., [1, p.64]). The fixed-point set $G^\Phi$ is a subset of $\text{End}_k(\mathfrak{g})^{F_0}$ and so it is finite, proving (a). Applying Lang-Steinberg’s theorem to $\Phi$ gives (b).

We now prove (c). Consider the map $F F_0^{-1}$ on $\mathfrak{g}$. This is an automorphism of $\mathfrak{g}$. Hence it can be uniquely written as $g g'$ where $\gamma \in \Gamma$ and $g \in G$. Thus $F = \gamma \cdot g F_0$. By part (b), there is an $h \in G$ such that $F_0' := g F_0 = h \ast F_0 \in G \ast F_0$, and so $F = \gamma \cdot F_0'$ has the required decomposition, proving the first assertion. Suppose now $F = \gamma' \cdot F_0''$ with $\gamma' \in \Gamma$ and $F_0'' = h' \ast F_0 \in G \ast F_0$ is another decomposition. Then $\gamma = F F_0^{-1} = \gamma'(F_0' F_0^{-1}) = \gamma' \gamma''$ for $g' = F_0'' F_0^{-1} \in \text{Aut}(\mathfrak{g})$. Furthermore $g' = F_0'' F_0^{-1} = h'^{-1} F_0 h' F_0^{-1} = h'^{-1} \Phi(h') \in G$, we must have $\gamma = \gamma'$ and $g = g'$, proving the uniqueness.

Part (b) can be generalized to any Frobenius morphisms.

Corollary 5.3. Let $F$ be a Frobenius morphism on $\mathfrak{g}$. For any $g \in G$, there exists $h \in G$ such that $g F = h \ast F_0 h F_0^{-1}$.

Proof. By part (c) above, $F = \gamma \cdot g^{-1} F_0 g$ for some $\gamma \in \Gamma$ and $g \in G$. This implies $F x F_0^{-1} \in G$ for every $x \in G$. Thus, we obtain an surjective endomorphism $x \mapsto F x F_0^{-1}$ on $G$. Now the result follows from Lang-Steinberg’s theorem.

Remark 5.4. It is well-known that a result of Steinberg [16, §11] similar to Lemma 5.2(c) holds for a simple algebraic group $G$ over $k$ with a Frobenius morphism $F$ (cf. [11, §1]). Thus, one would derive Lemma 5.2(c) from Steinberg’s result by taking differential of morphisms.

We are now ready to describe equivalence classes of Frobenius morphisms on $\mathfrak{g}$.

Theorem 5.5. Let $\mathfrak{g}$ be a classical Lie algebra over the algebraically closed field $k$ of characteristic $p > 3$, and let $F_0$ be the standard Frobenius morphism on $\mathfrak{g}$.

(a) If $F = \gamma \cdot F_0'$ with $\gamma \in \Gamma$, $F_0' \in G \ast F_0$, then $F \sim \gamma \cdot F_0$.

(b) For any two Frobenius morphisms $F_1$ and $F_2$ on $\mathfrak{g}$ with $F_1 = \gamma_1 \cdot F_0'$ and $F_2 = \gamma_2 \cdot F_0''$, where $\gamma_1, \gamma_2 \in \Gamma$, $F_0', F_0'' \in G \ast F_0$,

$$F_1 \sim F_2 \text{ if and only if } \gamma_1, \gamma_2 \text{ are conjugate in } \Gamma.$$ 

Therefore, there is a one-to-one correspondence between the equivalence classes of Frobenius morphisms on $\mathfrak{g}$ and the conjugacy classes in $\Gamma$ of graph automorphisms on $\mathfrak{g}$.

Proof. (a) Write $F_0' = g^{-1} F_0 g$ and put $h = F_0' F_0^{-1}$. Then $h \in G$ and $F = \gamma \cdot h F_0 = (\gamma h \gamma^{-1})(\gamma \cdot F_0)$. Thus, $(\gamma h^{-1} \gamma^{-1}) F = \gamma \cdot F_0$. By Corollary 5.3, there exists $h' \in G$ such that $(\gamma h^{-1} \gamma^{-1}) F = h'^{-1} F h'$, and therefore, $F \sim \gamma \cdot F_0$ by Lemma 2.6.

(b) If $\gamma_1, \gamma_2$ are conjugate in $\Gamma$, then by Lemma 5.1(b), $\gamma_1 \cdot F_0 \sim \gamma_2 \cdot F_0$. Thus, part (a) implies $F_1 \sim F_2$. Conversely, suppose $F_1 \sim F_2$. Then we may assume by part (a) that $F_1 = \gamma_1 \cdot F_0$, $F_2 = \gamma_2 \cdot F_0$. By Lemma 2.6, $F_1 = \sigma^{-1} F_2 \sigma$ for some $\sigma \in \text{Aut}(\mathfrak{g})$. Write $\sigma = g \gamma$ for $g \in G$ and
\[ \gamma \in \Gamma. \] Then
\[ \sigma^{-1}F_2\sigma = (\sigma^{-1}\gamma_2\sigma)(\sigma^{-1}F_0\sigma) \]
\[ = \gamma^{-1}g^{-1}\gamma_2g\gamma(g^{-1}F_0g) \]
\[ = (\gamma^{-1}\gamma_2\gamma)(\gamma^{-1}\gamma_2^{-1}g^{-1}\gamma_2g\gamma)(g * F_0), \]

Since \( G \) is normal in \( \text{Aut}(g) \), it follows that \( g' = \gamma^{-1}\gamma_2^{-1}g^{-1}\gamma_2g\gamma \in G \). By Corollary 5.3, there is a \( h \in G \) such that \( g'(g * F_0) = h^{-1}(g * F_0)h = (gh) * F_0 \). Thus \( F_1 = \sigma^{-1}F_2\sigma = (\gamma^{-1}\gamma_2\gamma) \cdot (gh * F_0) \).

The uniqueness proved in Lemma 5.2(c) implies that \( \gamma_1 = \gamma^{-1}\gamma_2\gamma \) (and \( F_0 = gh * F_0 \)). Therefore, \( \gamma_1 \) and \( \gamma_2 \) are conjugate in \( \Gamma \). \( \square \)

**Corollary 5.6.** Suppose \( F \) is an arbitrary given \( q \)-Frobenius morphism on a classical Lie algebra \( g \).

(a) The \( \mathbb{F}_q \)-form \( g^F \) of \( g \) is a restricted subalgebra of \( g \).

(b) If \( F \sim \gamma \cdot F_0 \), then \( F^n \sim \gamma^n \cdot F_0^n \) and \( g^F \cong g^F \otimes_{\mathbb{F}_q} \mathbb{F}_q^n \).

**Proof.** We write \( F = \gamma \cdot g^{-1}F_0g \) for some \( \gamma \in \Gamma \) and \( g \in G \). Since the \( p \)-map on \( g \) is the \( p \)-th power map on \( g \), it follows that \( F(x^{(p)}) = F(x)^{(p)} \) for all \( x \in g \). So part (a) follows directly from Lemma 4.1(a). Part (b) is clear by noting that \( (\gamma \cdot F_0)^n = \gamma^n \cdot F_0^n \). \( \square \)

By counting the number of conjugacy classes in \( \Gamma \) and noting the relation between the \( \mathbb{F}_q \)-forms and Frobenius morphisms (Lemma 2.6), we immediately reproduce the following well-known theorem (see [15, IV.6.1]).

**Theorem 5.7.** The number of (non-isomorphic) \( \mathbb{F}_q \)-forms of a classical simple Lie algebra \( g \) over \( k \) with \( p > 3 \) is given as follows:

- **Types** \( A_1, B, C, G_2, F_4, E_7, E_8 \): one
- **Types** \( A_n(n > 1), D_n(n \geq 5), E_6 \): two
- **Type** \( D_4 \): three

**Remark 5.8.** (1) This is the main result in [15, §6], which is proved there by a different method involving in Galois semi-automorphisms and \( 1 \)-cohomology.

(2) According to the theorem, all forms of classical Lie algebras of type \( A_1, B, C, G_2, F_4, E_7, E_8 \) are split. For types \( A_n(n > 1), D_n(n \geq 5), E_6 \), and \( D_4 \), we may obtain (split or non-split) forms \( g_0 \) by taking \( g^F \) for \( F = \gamma \cdot F_0 \) where \( \gamma \) is a graph automorphism.

### 6. Representations of \( \mathfrak{sl}(2, \mathbb{F}_q) \)

In this last section, we give an example to illustrate our theory. We assume \( p > 2 \) for simplicity.

We first introduce some standard notation of a restricted Lie algebra arising from the Lie algebra of a (simple) algebraic group \( G \) over \( k \) (e.g., the Chevalley group of \( g \)). Let \( g = \mathfrak{Lie}(G) \). Then the adjoint action of \( G \) on \( g \) commutes with the \( p \)-map, and thus, \( G \subseteq \text{Aut}(g)^{\langle p \rangle} \). The associated coadjoint action of \( G \) on \( g^* \):

\[ G \times g^* \to g^*; (\sigma, \chi) \mapsto \sigma(\chi) := \chi \circ \sigma^{-1} \]

induces isomorphism \( U_\chi(g) \cong U_{\sigma(\chi)} \) for any \( \sigma \in G, \chi \in g^* \). Thus, up to isomorphism, \( U_\chi(g) \) depends only on the (coadjoint) \( G \)-orbit of \( \chi \) (see the discussion right after 4.10). Consider a fixed triangular decomposition of \( g \): \( g = n^- \oplus h \oplus n^+ \), where \( b = h \oplus n^+ \) is called a Borel subalgebra, and the subalgebra \( U_\chi(b) \) of \( U_\chi(g) \). Each \( \lambda \in h^* \) affords a 1-dimensional representation \( k_\lambda \), which lifts to a representation of \( b \) by making \( n^- \) act as 0. This defines a \( U_\chi(b) \)-module. (Thus, \( \lambda \) satisfies \( \lambda(h^\alpha) - \lambda(h_\alpha) = \chi(h_\alpha)^p \) for all simple roots \( \alpha \)). The induced modules

\[ Z_\chi(\lambda) = U_\chi(g) \otimes_{U_\chi(b)} k_\lambda, \]
are known as "baby Verma modules".

We now let $G = SL(2, k)$. Then $\mathfrak{g} = \mathfrak{sl}(2, k)$. If $F$ is the standard Frobenius morphism sending $(a_{ij})$ to $(a_{ij}^p)$, then $\mathfrak{g}^F = \mathfrak{sl}(2, \mathbb{F}_p)$. We use the standard basis for both $\mathfrak{g}$ and $\mathfrak{g}^F$:

$$
eq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and take $\mathfrak{b} = ke + kh$ and $\mathfrak{h} = kh$. Thus, $\mathfrak{h}^* \cong k$. The $p$-map on $\mathfrak{g}$ in this case is in fact the $p$-th power map. Thus, $e^{(p)} = 0$, $f^{(p)} = 0$ and $h^{(p)} = h$.

We remark that, by Theorem 5.7, every Frobenius morphism $F'$ on $\mathfrak{g} = \mathfrak{sl}(2, k)$ is equivalent to $F$. Thus, with the category isomorphism (4.10.1), it suffices to work with $F$.

The determination of simple modules for $\mathfrak{g}$ over all positive characteristics was well-known ([14] or [6]). Following [6, §2], there are three different kinds of coadjoint orbits in $\mathfrak{g}^*$, represented by the elements

0 (the restricted type), $e$ (the regular nilpotent type), $ah$, $0 \neq a \in k$, (the semisimple type)

in $\mathfrak{g}$. The corresponding $\chi$ are

1. $\chi_0 = 0$;

2. $\chi_e(e) = 0$, $\chi_e(h) = 0$ and $\chi_e(f) = 1$;

3. $\chi_{ah}(e) = 0$, $\chi_{ah}(h) = a$ and $\chi_{ah}(f) = 0$.

All irreducible $\mathfrak{g}$-modules can be classified according to the three types. Generally speaking, each simple $U_\chi(\mathfrak{g})$-module is the quotient $L_\chi(\lambda)$ of a baby Verma module $Z_\chi(\lambda)$ for some $\lambda \in k$ satisfying $\lambda^p - \lambda = \chi(h)^p$. The selection of $\lambda$ depends on the type of $\chi$.

Case 1: $\chi = 0$. This is the restricted enveloping algebra case. There are $p$ non-isomorphic irreducible $U_0(\mathfrak{g})$-modules $L_\chi(\lambda)$, $\lambda \in \{0, 1, \cdots, p - 1\}$. If $v_0$ is a vector in $L_{\chi}(\lambda)$ of weight $\lambda$, then we may choose a basis $\{v_i = \frac{1}{i!} f^i v_0 \mid i = 0, 1, \cdots, \lambda\}$ for $L_\chi(\lambda)$ with the following $\mathfrak{g}$-actions (see [7, §7.2, Ex.7.5]): for all $0 \leq i \leq \lambda$,

$$
(6.0.1) \begin{cases} \\
h.v_i = (\lambda - 2i)v_i, \\
e.v_i = (\lambda - i + 1)v_{i-1}, \\
f.v_i = (i + 1)v_{i+1}.
\end{cases}
$$

Here $v_{-1} = 0 = v_p$.

If $\chi \neq 0$ is one of the last two types, then $Z_\chi(\lambda)$ has a basis $\{v_i = f^i \otimes v_0 \mid 0 \leq i < p\}$, where $v_0$ is a basis vector for $k_\lambda$, on which the action takes the form (see [10, 5.4]):

$$
(6.0.2) \begin{cases} \\
h.v_i = (\lambda - 2i)v_i, \\
e.v_i = i(\lambda - i + 1)v_{i-1} \text{ (here } v_{-1} := 0\text{)}, \\
f.v_i = \begin{cases} v_{i+1}, & \text{if } i < p-1, \\
\chi(f)^p v_0, & \text{if } i = p-1.
\end{cases}
\end{cases}
$$

Case 2: $\chi = \chi_e$. In this case, we have $L_\chi(\lambda) = Z_\chi(\lambda)$ for $\lambda = 0, 1, \cdots, p - 1$ (noting $p > 2$). Since $L_\chi(\lambda) \cong L_\chi(p - \lambda - 2)$, there are $(p+1)/2$ non-isomorphic irreducible modules.

Case 3: $\chi = \chi_{ah} (0 \neq a \in k)$. Then $U_\chi(\mathfrak{g})$ is semisimple and $L_\chi(\lambda) = Z_\chi(\lambda)$ for every root $\lambda$ of the equation $\lambda^p - \lambda = a^p$. 
We now use this classification to classify irreducible modules for $\mathfrak{g}^F = \mathfrak{sl}(2, \mathbb{F}_q)$. Let $S$ be a simple $\mathfrak{g}^F$-module. Then $S$ must lie in $\mathfrak{g}^F$-$\text{mod}_{\chi}$ for some $\chi \in \mathfrak{g}^*$. Thus, there exists a $\sigma \in SL(2, k)$ with $\chi' = \sigma(\chi)$ is one of the three types. Since there is a category isomorphism $\mathfrak{g}^F$-$\text{mod}_{\chi} \cong \mathfrak{g}^{F'}$-$\text{mod}_{\chi'}$ by Lemma 4.11, it follows that we may assume at the outset that $\chi$ is one of the three types.

For any simple module $L_\chi(\lambda)$ given above, let $F_L$ be the standard Frobenius morphism on $L_\chi(\lambda)$ which fixes every basis element $v_i$. Since the coefficients in (6.0.1) and (6.0.2) for the restricted or regular nilpotent type are all in $\mathbb{F}_p$, we immediately have the following.

Lemma 6.1. Let $\chi = 0$ or $\chi e$. Then $\chi^{[1]} = \chi$ and every simple module $L_\chi(\lambda)$ is $F$-stable. Thus, all $S_\chi(\lambda) := L_\chi(\lambda)^{F_L}$ for $0 \leq \lambda < p$ is a complete list of simple modules in the full subcategory $\mathfrak{g}^F$-$\text{mod}_{\chi}$.

We shall simply write $S_\chi(\lambda)$ for $S_{\chi_e}(\lambda)$ below.

We now assume that $\chi = \chi ah$ is of semisimple type and that $r = \varphi(\chi ah)$. Then $\chi(e) = 0 = \chi(f)$, $\chi(h) = a$ and $r$ is the smallest integer with $a^{q^r} = a$. In this case, $L_\chi(\lambda) = Z_\chi(\lambda)$, where $\lambda \in k$ satisfies $\lambda^p - \lambda = a^p$. These $p$ solutions $\lambda$'s determine $p$ pairwise non-isomorphic irreducible modules of $U_\chi(\mathfrak{g})$. Let $s$ be the smallest integer with $\lambda^{q^a} = \lambda$. Then

$$(a^{q^a})^p = (a^p)^{q^a} = (\lambda^a - \lambda)^{q^a} = \lambda^p - \lambda = a^p.$$  

Thus, $a^{q^a} = a$ and so $r \mid s$. Since $Z_\chi(\lambda)[i] \cong Z_{\chi^{[i]}}(\lambda^{q^i})$, it follows that $\varphi(Z_\chi(\lambda)) = s$. Putting $S_\chi(\lambda) = \hat{Z}_\chi(\lambda)^{\hat{F}}$, where

$$a = \{a, a^q, \ldots, a^{q^{r-1}}\} \quad \Lambda = \{\lambda, \lambda^q, \ldots, \lambda^{q^{r-1}}\} \quad (t = s/r),$$

we see from 3.5 that $S_\chi(\lambda)$ is a simple $\mathfrak{g}^F$-module. Note that $a$ as the $t$-orbit of $a$ corresponds to the $F$-orbit of $\chi ah$ and $\Lambda$ is the $t$-orbit of $\lambda$. Note also that every member in $\Lambda$ is a solution to $\lambda^p - \lambda = a^p$ and this solution set is a disjoint union of $F$-orbits of $\chi ah$ and $\Lambda$ which index the simple module in $\mathfrak{g}^F$-$\text{mod}_{\chi ah}$. Applying 3.5, we now can state the following classification theorem for simple $\mathfrak{g}^F$-modules.

Theorem 6.2. Assume $\text{char}(k) > 2$. Let $G = SL(2, k)$, $\mathfrak{g} = \mathfrak{sl}(2, k)$ and $F$ the standard Frobenius morphism sending $(a_{ij})$ to $(a_{ij}^q)$. Let $S$ be a simple $\mathfrak{g}^F$-module in $\mathfrak{g}^F$-$\text{mod}_{\chi}$ for some $\chi \in \mathfrak{g}^*$. Then, every $\sigma \in G$ gives rise to a category isomorphism

$$(\ )_\sigma : \mathfrak{g}^F$-$\text{mod}_{\chi} \to \mathfrak{g}^{F'}$-$\text{mod}_{\chi'},$$

where $\chi' = \sigma(\chi)$ and $F' = \sigma F = \sigma^{-1} \circ F \circ \sigma$ as given in 4.11.

(a) If $\chi$ is of the restricted type, then $S^\sigma \cong S_\chi(\lambda)$ for some $\lambda \in \{0, 1, \ldots, p - 1\}$ and $\sigma \in G$ with $\sigma(\chi) = 0$.

(b) If $\chi$ is of the regular nilpotent type, then $S^{\sigma} \cong S_\epsilon(\lambda)$ for some $\lambda \in \{0, 1, \ldots, p^{\frac{p+1}{2}}\}$ and $\sigma \in G$ with $\sigma(\chi) = \chi_e$; this is a baby Verma module over $\mathbb{F}_q$.

(c) If $\chi$ is of the semisimple type with $0 \neq a \in k$, then $S^\sigma \cong S_\Lambda(\lambda)$ for some $\Lambda \in \{\Lambda_{0,1}, \ldots, \Lambda_{n,1}\}$ and $\sigma \in G$ with $\sigma(\chi) = \chi ah$.

We now explicitly describe the action of $\mathfrak{sl}(2, \mathbb{F}_q)$ on $S_\chi(\Lambda)$ via a basis. Let

$$v_\Lambda := (v_0, v_0, \ldots, v_0) \in S_\chi(\Lambda).$$

Since the Frobenius morphism $F$ on $\mathfrak{sl}(2, k)$ fixes the generators $e, f, h$, we have, by (6.0.2) and (3.4.1), $f^q v_\Lambda = (v_i, \ldots, v_i)$ for all $0 \leq i \leq p - 1$, and $h^q v_\Lambda = (\lambda^q v_0, \lambda^q v_0, \ldots, \lambda^q v_0)$. Thus,

$$h^{q^j} v_\Lambda = (\lambda^{q^j} v_0, \lambda^{q^j+1} v_0, \ldots, \lambda^{q^j-1} v_0, \lambda^q v_0, \ldots, \lambda^{q^j-1} v_0) \quad (0 \leq j \leq s - 1)$$
If we put $v_{ij} = f^i h^q, v_{ij}$ (and note that $v_{00} \neq v_{00}$), then

$$v_{ij} = (\lambda^q v_{i1}, \lambda^{q+1} v_{i1}, \cdots, \lambda^{q-1} v_{i1}, \lambda q v_{i1}, \cdots, \lambda^{q-1} v_{i1}).$$

**Proposition 6.3.** The set $\{v_{ij} \mid 0 \leq i \leq p - 1, 0 \leq j \leq s - 1\}$ forms a basis for the simple sl$(2, F_q)$-module $S_2(\lambda)$. The action of $sl(2, F_q)$ is given by the following formulas:

1. $h.v_{ij} = c_{0j}v_{i0} + c_{1j}v_{i1} + \cdots + c_{s-1,i-1,s-1}v_{is} - 2iv_{ij}$
2. $e.v_{ij} = \begin{cases} 0 & \text{if } i = 0, \\ ic_{j0}v_{i-1,0} + ic_{j1}v_{i-1,1} + \cdots + ic_{j,s-1}v_{i-1,s-1} - i(i - 1)v_{i-1,j} & \text{if } i > 0, \\ \end{cases}$
3. $f.v_{ij} = \begin{cases} v_{i+1,j} & \text{if } i < p - 1, \\ 0 & \text{if } i = p - 1. \end{cases}$

**Proof.** The proof for the first assertion is straightforward by using the Van der Monde determinant in $\lambda, \lambda^q, \lambda^{q+1}, \cdots, \lambda^{q-1}$. We now derive the actions of $h, e, f$. By the definition of $s$, the set $\{\lambda, \lambda^q, \lambda^{q+1}, \cdots, \lambda^{q-1}\}$ of all roots of the minimal polynomial of $\lambda$ forms a $F_q$-basis for $F_q$. Write

$$\mu := \lambda^{q+1} = c_{0j} + c_{1j} \lambda^q + \cdots + c_{j,s-1} \lambda^{q-1}, j = 0, 1, \cdots, s - 1$$

where $c_{ij} \in F_q$ for all $0 \leq i \leq p - 1$ and $0 \leq j \leq s - 1$. Then by (6.0.2) and (3.2.1)

$$h.v_{ij} = (\lambda^q (\lambda - 2i)v_{i1}, \lambda^{q+1} (\lambda^q - 2i)v_{i1}, \cdots, \lambda^{q-1} (\lambda^{q-1} - 2i)v_{i1})$$

$$= (\mu v_{i1}, \mu^q v_{i1}, \cdots, \mu^{q-1} v_{i1}) - 2iv_{ij}$$

$$= \sum_{t=0}^{s-1} c_{jt} \lambda^{q t} v_{i1}, \sum_{t=0}^{s-1} c_{jt} \lambda^{q t+1} v_{i1}, \cdots, \sum_{t=0}^{s-1} c_{jt} \lambda^{q t+s-1} v_{i1} - 2iv_{ij}$$

$$= c_{0j}v_{i0} + c_{1j}v_{i1} + \cdots + c_{s-1,i-1,s-1}v_{is} - 2iv_{ij}.$$  

Similarly, in the second case, we have

$$e.v_{ij} = (\lambda^q i(\lambda - i + 1)v_{i-1,1}, \lambda^{q+1} i(\lambda^q - i + 1)v_{i-1,1}, \cdots, \lambda^{q-1} i(\lambda^{q-1} - i + 1)v_{i-1,1})$$

$$= (i\mu v_{i-1,1}, i\mu^q v_{i-1,1}, \cdots, i\mu^{q-1} v_{i-1,1}) - i(i - 1)v_{i-1,j}.$$  

Same computation as in the first case gives the required formula. The last case is clear. \(\square\)

**Remark 6.4.** The irreducible module $S_2(\lambda)$ is not necessarily a “highest weight” module for the finite Lie algebra $sl(2, F_q)$, which is different from the situation over an algebraically closed field.

**References**


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