MULTI-PARAMETER CELLS OF FINITE COXETER GROUPS

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ABSTRACT. Cells of Coxeter groups are certain equivalence classes defined by the Kazhdan-Lusztig type basis of the associated Hecke algebra with a set of parameters. In this paper, we prove that, for finite Coxeter groups, cells arising from a multi-parameter Hecke algebra are determined by those arising from a Hecke algebra with parameters which are powers of a single parameter.

1. INTRODUCTION

Let (W, S) be a Coxeter system. For each $s \in S$, let u_s be an indeterminant such that $u_s = u_t$ if $s, t \in S$ are W-conjugate. Let $\mathcal{A} = \mathbb{Z}[\Gamma]$ be the group ring, where Γ is the abelian group generated by u_s for all $s \in S$. Let $\lhd$ be a total (or linear) order on Γ, which is compatible with the group structure of Γ. In [7] (see [6] for the equal parameter case) Lusztig introduced the canonical basis with respect to $\lhd$ for the Hecke algebra $\mathcal{H} = \mathcal{H}_{\mathcal{A}}$ over $\mathcal{A}$ associated to W (with parameters $\{u_s\}_{s\in S}$). Via such a basis, W can be divided into certain equivalence classes called generalized cells relative to $\lhd$ (or simply $\lhd$-cells), which play an important role in the representation theory of Hecke algebras. If Γ is not a cyclic group, these cells are called multi-parameter cells in this paper.

Consider a group homomorphism $\sigma : \Gamma \to \mathbb{Z}$. It induces a homomorphism $\tilde{\sigma} : \mathcal{A} \to \mathcal{Z} := \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ such that $\tilde{\sigma}(u_s) = q^{\sigma(u_s)/2}$ and let $\mathcal{H}^{\sigma} = \mathcal{H}^{\sigma}_{\mathcal{Z}} = \mathcal{H} \otimes \mathcal{Z}$. We shall assume that σ is admissible (see Definition 3.1 below), and call the generalized cells (or σ-cells) arising from the canonical basis for $\mathcal{H}^{\sigma}$ the Kazhdan-Lusztig cells if $\sigma(u_s) = \sigma(u_t)$ for all $s, t \in S$, or unequal parameter cells otherwise.

The unequal-parameter cells have been studied by many authors. For example, in [7], Lusztig showed that all the left σ-cell representations are simple if the Hecke algebras of type B_n have unequal parameters q, $q^{3/2}$ or q and $q^{\frac{1}{2}}$. In [4], σ-cells were used to study certain endomorphism algebras. Also, beautiful relations between certain σ-cells and Kazhdan-Lusztig cells were discovered in [1]. Moreover, σ-cells are the main objects discussed in the recent book [9].

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However, it seems not clear for us to study the representations of Hecke algebras with multi-parameters via cells. This is because the multi-parameter cells are harder to be described. This motivates the authors to propose the following questions.

Question 1.1. (a) *For any total order $\lhd$ on Γ, is there a specialization σ such that the multi-parameter $\lhd$-cells coincide with σ-cells?*

(b) *For any specialization σ, is there a total ordering on Γ such that the resulting $\lhd$-cells agree with the σ-cells?*

In this paper, we show that, for a finite Coxeter group, Question 1.1(a) is true. However, Question 1.1(b) is not true in general. We will provide some examples to illustrate it.

The contents of this paper are organized as follows. In §2, we recollect some definitions and results on multi-parameter cells. In §3, we prove that multi-parameter $\lhd$-cells of W coincide with σ-cells for some specialization σ. In §4, we describe all multi-parameter $\lhd$-cells for dihedral groups and multi-parameter $\lhd$-cells for the Weyl group of type B_3 in order to make a comparison with the unequal-parameter σ-cells described in [7]. Our example shows that Question 1.1(b) is not true in general.

2. MULTI-PARAMETER CELLS

In this section, we recollect some definitions and results. We shall restrict our attention to finite Coxeter groups although some of definitions and results in this section are valid for infinite Coxeter groups.

Let (W, S) be a finite Coxeter system where $S = \{s_1, s_2, \dots, s_n\}$ is the set of Coxeter generators. For any $s_i \in S$, let u_{s_i} be an indeterminate such that $u_{s_i} = u_{s_j}$ if s_i and s_j are W-conjugate. Write $u_i = u_{s_i}$ for brevity. Let $\Gamma = \langle u_1, u_2, \dots, u_n\rangle$ be the free abelian group generated by $u_1, u_2, \dots, u_n$, and let $A = \mathbb{Z}[\Gamma]$ be the group ring of Γ. Then the Hecke algebra $\mathcal{H}$ associated to W is an associative algebra with free A-basis $\{T_w\}_{w\in W}$ satisfying the following multiplicative rules:

$$\begin{cases} (T_{s_i} - u_i^2)(T_{s_i} + 1) = 0, & \text{if } s_i \in S, \\ T_y T_w = T_{yw}, & \text{if } l(yw) = l(y) + l(w). \end{cases}$$

Here l is the length function on (W, S).

Let $u_w = \prod_{j=1}^k u_{i_j}$ if $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression of w. It is known that u_w is independent of the choice of a reduced expression of w. Putting $\widetilde{T}_w = u_w^{-1} T_w$, we obtain a new basis $\{\widetilde{T}_w\}_{w\in W}$ for $\mathcal{H}$. Following [7], let $^-$ be the $\mathbb{Z}$-linear involution on the ring $\mathcal{A}$ satisfying $\bar{\gamma} = \gamma^{-1}$ for all $\gamma \in \Gamma$, and extend it to $\mathcal{H}$ by setting

$$\overline{\sum_w a_w \widetilde{T}_w} = \sum_w \overline{a}_w \widetilde{T}_{w^{-1}}^{-1},\ a_w \in \mathcal{A}.$$

For any $x, y \in W$, let $R_{x,y} \in \mathcal{A}$ be defined by $\widetilde{T}_{y^{-1}}^{-1} = \sum_{x \in W} R_{x,y} \widetilde{T}_x$. Clearly, $R_{x,y} \neq 0$ implies $x \leqslant y$ where $\leqslant$ is the Bruhat-Chevalley order on W, and for $sy < y$, we have

$$\begin{aligned}
\widetilde{T}_{y^{-1}}^{-1} = \widetilde{T}_s^{-1} \widetilde{T}_{(sy)^{-1}}^{-1} &= \sum_{z \leqslant sy} R_{z,sy} \widetilde{T}_s \widetilde{T}_z + (u_s^{-1} - u_s) \sum_{z \leqslant sy} R_{z,sy} \widetilde{T}_z \\
&= \sum_{z \leqslant sy, sz > z} R_{z,sy} \widetilde{T}_{sz} + (u_s^{-1} - u_s) \sum_{z \leqslant sy} R_{z,sy} \widetilde{T}_z \\
&+ \sum_{z \leqslant sy, sz < z} \Big(R_{z,sy} \widetilde{T}_{sz} + (u_s - u_s^{-1}) \widetilde{T}_z \Big)
\end{aligned}$$

Consequently,

$$R_{x,y} = \begin{cases} R_{sx,sy}, & \text{if } sx < x \text{ and } sy < y, \\ (u_s^{-1} - u_s) R_{x,sy} + R_{sx,sy}, & \text{if } sx > x \text{ and } sy < y. \end{cases} \tag{2.1}$$

Note that the polynomial $R_{x,y}$ is the polynomial $\bar{R}^*_{x,y}$ defined in [7], and $\bar{R}_{x,y} = \epsilon_x \epsilon_y R_{x,y}$, where $\epsilon_w = (-1)^{l(w)}$.

Lemma 2.1. *For any $x, y \in W$ with $x \leqslant y$, if $\prod_{i=1}^n u_i^{a_i}$ is a monomial appearing in $R_{x,y}$, then $\sum_{i=1}^n |a_i| \leqslant l(y) - l(x)$. Moreover, if the inequality becomes an equality, then $R_{x,y}$ has the highest term $\epsilon_y \epsilon_x u_y u_x^{-1}$ and the lowest term $u_x u_y^{-1}$.*

Proof. By (2.1), $R_{x,x} = R_{e,e} = 1$. Hence the result follows. Suppose $x < y$. We prove the result by induction on $2l(y) - l(x)$.

Since $x < y$, we have $2l(y) - l(x) \geqslant l(y) + 1 \geqslant 2$. If $2l(y) - l(x) = 2$, then $y \in S$ and $x = e$. Thus, the results follow immediately from the equality $R_{e,s} = u_s^{-1} - u_s$ with highest term $\epsilon_s u_s$ and lowest term u_s^{-1}. Assume $2l(y) - l(x) > 2$. Then

(1) $2l(sy) - l(sx) < 2l(y) - l(x)$ if $sx < x$ and $sy < y$, and
(2) $2l(sy) - l(sx) < 2l(sy) - l(x) < 2l(y) - l(x)$, if $sx > x$ and $sy < y$.

In the first case, we have by (2.1) $R_{x,y} = R_{sx,sy}$. Since $l(y) - l(x) = l(sy) - l(sx)$, all results follow from induction. In the second case, since $l(sy) - l(sx) < l(sy) - l(x) < l(y) - l(x)$, we have $\sum_{i=1}^n |a_i| \leqslant l(y) - l(x)$ by (2.1) and induction assumption. If the inequality is an equality, then the highest and lowest terms must appear in $(u_s^{-1} - u_s) R_{x,sy}$. Thus, by induction, they are of the forms $-u_s \epsilon_x \epsilon_{sy} u_x^{-1} u_{sy} = \epsilon_y \epsilon_x u_y u_x^{-1}$ and $u_s^{-1} u_x u_{sy}^{-1} = u_x u_y^{-1}$, as desired. □

Consider a total order $\lhd$ on Γ which is *compatible* with the group structure of Γ in the sense that, if $a \lhd b$ and $c \lhd d$, then $ac \lhd bd$. Such an order is said to be *admissible* if all $u_i \in \Gamma$ are strictly bigger than the identity element 1 of Γ. Let $\lhd$ be a fixed admissible total ordering on Γ. Thus, if $1 \lhd \gamma$, then $\bar{\gamma} = \gamma^{-1} \lhd 1$. Put $\Gamma_+ = \{\gamma \in \Gamma | \, 1 \lhd \gamma\}$, $\Gamma_- = \Gamma_+^{-1}$ and $\Gamma_0 = \{1\}$. Let $\mathbb{Z}[\Gamma_-]$ (resp. $\mathbb{Z}[\Gamma_+]$) be the $\mathbb{Z}$-submodule of $\mathbb{Z}[\Gamma]$ spanned by Γ_- (resp. Γ_+).

Fix a linear ordering on W: $w_1, w_2, \dots, w_\nu$ with $\nu = |W|$ such that $w_i \leqslant w_j$ implies $i \leqslant j$. Let $R = (R_{ij})$ be the $\nu \times \nu$ matrix with $R_{ij} = R_{w_i, w_j}$. It is known that $R_{i,i} = 1$ for all i, and $R_{i,j} = 0$ unless $w_i \leqslant w_j$. Therefore, R is a pomatrix in the sense of [3, 1.2].[1] Furthermore, $R\bar{R} = I$, the identity matrix. So the matrix R is a 1-pomatrix in the sense of [3, 1.2][2]. We now endow Γ a total ordering so that R can be uniquely written as a product of an ∞-pomatrix P and the 0-pomatrix $\bar{P}^{-1}$.

Lemma 2.2. *Maintain the notation introduced above. There exists a unique upper matrix* $P^{\triangleleft} = (P^{\triangleleft}_{i,j})$ *with* $P^{\triangleleft}_{i,j} = P^{\triangleleft}_{w_i,w_j} \in \mathbb{Z}[\Gamma_-]$, $P^{\triangleleft}_{i,i} = P^{\triangleleft}_{w_i,w_i} = 1$ *such that* $R = P^{\triangleleft}\bar{P}^{\triangleleft^{-1}}$.

Proof. We claim that $P^{\triangleleft}_{i,j}$ are the unique solution to the system

$$x_{i,j} = \sum_{\substack{k \\ w_i \leqslant w_k \leqslant w_j}} R_{i,k}\bar{x}_{k,j}$$

with $x_{i,i} = 1$ and $x_{i,j} \in \mathbb{Z}[\Gamma_-]$ for $w_i < w_j$. Thus, $R = P^{\triangleleft}\bar{P}^{\triangleleft^{-1}}$. The proof of the claim is entirely similar to [2, 1.2] (cf. [8, 7.10]): write $i \leqslant j$ if $w_i \leqslant w_j$ and suppose $P^{\triangleleft}_{k,j}$ are known for all k, $i < k \leqslant j$. Then

$$\begin{aligned}\sum_{\substack{k, \\ i<k\leqslant j}} R_{i,k}\bar{P}^{\triangleleft}_{k,j} &= \sum_{\substack{k, \\ i<k\leqslant j}} R_{i,k} \sum_{\substack{k', \\ k\leqslant k'\leqslant j}} \bar{R}_{kk'}P^{\triangleleft}_{k',j} = \sum_{\substack{k', \\ k\leqslant k'\leqslant j}} \sum_{\substack{k, \\ i<k\leqslant j}} R_{i,k}\bar{R}_{kk'}P^{\triangleleft}_{k',j} \\ &= \sum_{\substack{k', \\ k\leqslant k'\leqslant j}} (-\bar{R}_{ik'})P^{\triangleleft}_{k'j} = -\sum_{\substack{k, \\ i<k\leqslant j}} \overline{R_{i,k}\bar{P}^{\triangleleft}_{k,j}}\end{aligned}$$

Now $P^{\triangleleft}_{i,j} \in \mathbb{Z}[\Gamma_-]$ is the unique solution satisfying

$$P^{\triangleleft}_{i,j} - \bar{P}^{\triangleleft}_{i,j} = \sum_{\substack{k, \\ i<k\leqslant j}} R_{i,k}\bar{P}^{\triangleleft}_{k,j}.$$

□

Note that $P^{\triangleleft}_{x,w}$ is a multi-variable polynomial and is the polynomial $P^{*}_{x,w}$ discussed in [7]. They can be calculated recursively by the following formulas:

$$\begin{cases} P^{\triangleleft}_{w,w} = 1, \text{ for any } w \in W, \\ P^{\triangleleft}_{x,w} - \bar{P}^{\triangleleft}_{x,w} = \sum_{x<y\leqslant w} R_{x,y}P^{\triangleleft}_{y,w}. \end{cases} \tag{2.2}$$

Also, we have

$$\sum_{x\leqslant z\leqslant y} \epsilon_x\epsilon_z P^{\triangleleft}_{x,z}P^{\triangleleft}_{w_0y,w_0z} = \delta_{x,y} \text{ for all } x, y \in W,$$

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[1] A pomatrix is a matrix $A = (a_{ij})_{i,j\in I}$ over a poset I satisfying $a_{ii} = 1$ and $a_{ij} = 0$ unless $i \leqslant j$.

[2] A 1-pomatrix R is a pomatrix R such that $R\bar{R} = I$. An ∞-pomatrix R (resp. 0-pomatrix) is a pomatrix such that every (i, j)-th entry $R_{ij} \in \mathbb{Z}[\Gamma_-]$ (resp. $R_{ij} \in \mathbb{Z}[\Gamma_+]$).

where w_0 is the longest element in W.

The decomposition $R = P^{\lhd}\bar{P}^{\lhd^{-1}}$ given in Lemma 2.2 is called an *antipode* decomposition. Every admissible total orderings $\lhd$ gives rise to such a decomposition. However, the number of such decompositions is finite as we shall see below.

Definition 2.3. Two admissible total orders $\lhd$ and $\lhd'$ on Γ are said to be equivalent if $P^{\lhd} = P^{\lhd'}$.

Proposition 2.4. *The number of non-equivalent admissible total orders on* Γ *is finite.*

Proof. First, we claim that, if a monomial $\prod_{i=1}^n u_i^{a_i}$ appears in $P^{\lhd}_{x,w}$ for $x < w$, then the inequality $\sum_{i=1}^n |a_i| \leqslant l(w)-l(x)$, independent of the total order $\lhd$. We prove it by induction on $l(w)-l(x)$. Indeed, if $l(w)-l(x) = 1$, then a reduced expression of x can be obtained from w by dropping a factor $s \in S$. In this case, $P^{\lhd}_{x,y} = u_s^{-1}$, and our claim follows. Assume now $l(w) - l(x) > 1$. By Lemma 2.1, we have $\sum_{i=1}^n |b_i| \leqslant l(y) - l(x)$ if $\prod_{i=1}^n u_i^{b_i}$ is a monomial appearing in $R_{x,y}$ for any $x < y \leqslant w$. Since $y > x$, we have $l(w) - l(y) < l(w) - l(x)$. By inductive hypothesis, we have $\sum_{i=1}^n |c_i| \leqslant l(w) - l(y)$ if $\prod_{i=1}^n u_i^{c_i}$ appears in $P^{\lhd}_{y,w}$ for $y < w$. Therefore, by (2.2), any monomial $\prod_{i=1}^n u_i^{a_i}$ of $P^{\lhd}_{x,w} - \bar{P}^{\lhd}_{x,w}$ satisfies $\sum_{i=1}^n |a_i| \leqslant l(w) - l(x)$ (independent of $\lhd$). Since $P^{\lhd}_{x,w} \in \mathbb{Z}[\Gamma_-]$ and $\bar{P}^{\lhd}_{x,w} \in \mathbb{Z}[\Gamma_+]$, our claim follows immediately.

By the claim, for given $x \leqslant w$, the number of such polynomials $P^{\lhd}_{x,w}$ is finite. Since $P^{\lhd}_{x,w} = 0$ if $x \nleqslant w$ and $P^{\lhd}_{x,x} = 1$ for all $x \in W$, there are only finitely many antipode decompositions of R. Consequently, the number of non-equivalent total orderings on Γ is finite. ☐

With the matrix $P^{\lhd}$, we may define the *canonical basis* $\{C^{\lhd}_w \mid w \in W\}$ for Hecke algebra $\mathcal{H}$ where $C^{\lhd}_w = \sum_{y \leqslant w} P^{\lhd}_{y,w}\widetilde{T}_y$. Such a basis was first introduced by Lusztig in [7][3]. He also gave the following formula:

$$C^{\lhd}_s C^{\lhd}_w = \begin{cases} (u_s + u_s^{-1})C^{\lhd}_w, & \text{if } sw < w, \\ C^{\lhd}_{sw} + \sum_{z<w, sz<z} M^{s,\lhd}_{z,w} C^{\lhd}_z, & \text{if } sw > w, \end{cases} \tag{2.3}$$

where $M^{s,\lhd}_{z,w}$ is defined by the condition that $\sum_{z \leqslant y < w, sy<y} P^{\lhd}_{z,y} M^{s,\lhd}_{y,w} - u_s P^{\lhd}_{y,w}$ is a $\mathbb{Z}$-linear combinations of elements in Γ_-. Note that $\bar{M}^{s,\lhd}_{z,w} = M^{s,\lhd}_{z,w}$ since $\bar{C}^{\lhd}_w = C^{\lhd}_w$.

The canonical basis $\{C^{\lhd}_w\}$ is used to define cells *relative to* the total order $\lhd$, or simply $\lhd$-cells. As defined in [6], let $\leqslant_{L,\lhd}$ be the pre-order relation on W generated by the relation " $x \leqslant_{L,\lhd} y$ if there is $s \in S$ such that $C^{\lhd}_x$ appears in the expression of $C^{\lhd}_s C^{\lhd}_y$ with non-zero coefficient." In other words, $x \leqslant_{L,\lhd} y$ if there is a sequence $x_1 = x, x_2, \cdots, x_k = y$ such that, for every $i = 1, 2, \cdots, k-1$, either $x_i = sx_{i+1} > x_{i+1}$ or $M^{s,\lhd}_{x_i,x_{i+1}} \neq 0$ for some $s \in S$. We denote by $x \leqslant_{R,\lhd} y$ if $x^{-1} \leqslant_{L,\lhd} y^{-1}$. Let $\leqslant_{LR,\lhd}$ be the pre-order on W generated by $\leqslant_{L,\lhd}$ and $\leqslant_{R,\lhd}$. For any $x \in \{L, R, LR\}$ and

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[3] The element $C^{\lhd}_w$ is denoted C^*_w there.

$w, w' \in W$, $w \sim_{x,\lhd} w'$ if $w \leqslant_{x,\lhd} w' \leqslant_{x,\lhd} w$. Thus $\sim_{L,\lhd}$, $\sim_{R,\lhd}$ and $\sim_{LR,\lhd}$ are equivalent relations on W. The corresponding equivalence classes are called left, right and two-sided cells of W relative to the given total order $\lhd$, or simply, left, right and two-sided *$\lhd$-cells*. If Γ is *not* cyclic, *$\lhd$-cells* are called *multi-parameter cells*; if Γ is cyclic and the total order $\lhd$ is the natural order, then $\lhd$-cells are called *Kazhdan-Lusztig cells* if all u_s are equal, or *unequal-parameter cells* otherwise.

Recall from [7] that, if Γ is cyclic (thus, $\Gamma \cong \mathbb{Z}$), then $u_s = q^{i_s/2}$ for some integer i_s and indeterminate $q^{\frac{1}{2}}$. In this case, the Hecke algebra is a one-parameter Hecke algebra and the natural order on $\mathbb{Z}$ gives a unique decomposition $R = P\bar{P}^{-1}$, where $P = (P_{ij})$ is an ∞-pomatrix in the sense that P is an upper unitriangular matrix with $P_{ij} \in q^{-1/2}\mathbb{Z}[q^{-1/2}]$ for $i < j$. Using P, we may introduce a new basis $C_{w_j} = \sum_{i \leqslant j} P_{ij}\widetilde{T}_{w_i}$ (see [3, §2-3]), fixed under $^-$. Thus, the Coxeter group is divided into (single-parameter) cells, which were introduced in [6] when $i_s = i_t$ for all $s, t \in S$, and in [7] otherwise. These cells[4] are uniquely defined by the given one-parameter Hecke algebra. Such a Hecke algebra is a specialization $\mathcal{H}^\sigma$ of $\mathcal{H}$ via a map $\sigma : \Gamma \to \mathbb{Z}$. It is natural to relate $\lhd$-cells with the cells arising from $\mathcal{H}^\sigma$.

3. A COMPARISON ON MULTI-PARAMETER AND UNEQUAL-PARAMETER CELLS

In order to compare two kinds of cells, we first have to look at the specialization of the canonical basis for $\mathcal{H}$ relative to $\lhd$. We hope that image of any canonical basis element in $\mathcal{H}$ is a canonical basis element in $\mathcal{H}^\sigma$. This requires that the specialization σ preserve the orderings. Recall from [6] that the definition of cells in one-parameter case uses the natural ordering on the cyclic group. So we have the following definition.

Definition 3.1. An abelian group homomorphism $\sigma : \Gamma \to \mathbb{Z}$, sending u_s to i_s, is called *admissible* if $i_s > 0$ for all $s \in S$ and $\sum a_s i_s < \sum a'_s i_s$ whenever $\prod u_s^{a_s} \lhd \prod u_s^{a'_s}$ in Γ.

Cells in a finite Coxeter group W are determined by the cells in the irreducible components of W. Without loss of generality, we may assume that W is irreducible (i.e., the corresponding Coxeter graph is connected [5]). We can assume that W is of type B_n, C_n, F_4, or $I(m)$ $(m > 3)$, since the Hecke algebras of types A_{n-1}, D_n, E_6, E_7, E_8, H_3 and H_4 have equal parameters. Consequently, the Hecke algebra $\mathcal{H}$ associated to W has two parameters, say, u and v, and the group $\mathbb{Z} \times \mathbb{Z} \cong \Gamma$ under the "exponential" isomorphism $\boldsymbol{e} : (a, b) \mapsto u^a v^b$ for $a, b \in \mathbb{Z}$. If σ is an admissible specialization of Γ as given in Def. 3.1, then $i = \sigma(1, 0) > 0$ and $j = \sigma(0, 1) > 0$ and σ induces a map $\sigma : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ with $\sigma(a, b) = a\sigma(1, 0) + b\sigma(0, 1)$, $a, b \in \mathbb{Z}$. Note that

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[4] The corresponding cell representations in this case have also been investigated in [4] in the context of stratifications of Hecke endomorphism algebras.

$\sigma(a,b) < \sigma(a',b')$ if $u^av^b \lhd u^{a'}v^{b'}$. For any $k \in \mathbb{Z}$, let

$$L_k = \sigma^{-1}(k) = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid \sigma(x,y) = k, \forall x,y \in \mathbb{Z}\}.$$

Define "half 0-lines" $L_0^+ = \{(a,b) \in L_0 | \ a > 0\}$ and $L_0^- = \{(a,b) \in L_0 | \ a < 0\}$. Note that, if $(a,b) \in \Gamma_0$, then $ab < 0$.

Lemma 3.2. *Let σ be an admissible specialization of $\Gamma = \langle u,v\rangle$ given as in Def. 3.1.*

1. *We have either $\boldsymbol{e}(L_0^-) \subset \Gamma_+$ and $\boldsymbol{e}(L_0^+) \subset \Gamma_-$, or $\boldsymbol{e}(L_0^-) \subset \Gamma_-$ and $\boldsymbol{e}(L_0^+) \subset \Gamma_+$.*
2. *If $(a,b) \in L_k$, $(c,d) \in L_{k'}$ and $u^av^b \rhd u^cv^d$, then $k \geqslant k'$. When $k = k'$, we have*
   - (a) *$a < c$ if and only if $\boldsymbol{e}(L_0^-) \subset \Gamma_+$;*
   - (b) *$a > c$, if and only if $\boldsymbol{e}(L_0^+) \subset \Gamma_+$.*

Proof. If (1) were false, then there would be $(a,b) \in L_0^-$ and $(a_1,b_1) \in L_0^+$ such that $u^av^b, u^{a_1}v^{b_1} \in \Gamma_+$. However, since $(u^av^b)^{b_1} = (u^{a_1}v^{b_1})^b$, $b_1 < 0$ and $b > 0$, $(u^av^b)^{b_1} \in \Gamma_+ \cap \Gamma_- = \emptyset$, a contradiction, forcing (1) to be true.

Since σ is admissible and $u^av^b \rhd u^cv^d$, $k = \sigma(a,b) \geqslant k' = \sigma(c,d)$ and $\boldsymbol{e}(a-c, b-d) \in \Gamma_+$. If $k = k'$, then $(a-c, b-d) \in L_0$. Therefore, $a < c$ (resp. $a > c$) if and only if $(a-c, b-d) \in L_0^-$ (resp. $\in L_0^+$). Now, (a) and (b) follow immediately from (1). □

This lemma suggests us to introduce some special admissible orders on Γ.

Definition 3.3. An admissible total ordering $\lhd$ on Γ is called a *parallel-line ordering* and write $u^av^b \lhd u^{a_1}v^{b_1}$ for $(a,b) \in L_k$ and $(a_1,b_1) \in L_{k'}$ if one of the following conditions holds true.

1. either $k < k'$, or $k = k'$ and $a < a_1$.
2. either $k < k'$, or $k = k'$ and $a > a_1$.

The following result shows that the parallel-line orderings exhaust the representatives for the equivalence classes of admissible orderings (see Def. 2.3).

Theorem 3.4. *Any admissible total ordering $\lhd$ on Γ is equivalent to a parallel-line ordering. Moreover, there are infinite number of parallel-line orders, which are equivalent to $\lhd$.*

Proof. Let $R = P^{\lhd}\bar{P}^{\lhd^{-1}}$ be the unique antipode decomposition with respect to $\lhd$. Let

$$X = \{(-a,b) \in \mathbb{Z}^{>0}\times\mathbb{Z}^{>0} | \ u^av^b \ \text{ appears in } P^{\lhd}_{x,y} \text{ for some } x,y \in W\},$$
$$Y = \{(a,-b) \in \mathbb{Z}^{>0}\times\mathbb{Z}^{>0} | \ u^av^b \ \text{ appears in } P^{\lhd}_{x,y} \text{ for some } x,y \in W\}.$$

Since W is a finite Coxeter group, both X and Y are finite sets.

First, assume $X \neq \emptyset$ and $Y \neq \emptyset$. Write

$$k = \max\{\frac{b}{-a} | \ (-a,b) \in X\} \ \text{ and } \ l = \min\{\frac{-b}{a} | \ (a,-b) \in Y\}.$$

We claim $k < l$. Indeed, if this is not the case, then there are $(-a_0, b_0) \in X$ and $(a_0', -b_0') \in Y$ such that $k = \frac{b_0}{-a_0} \geqslant l = \frac{-b_0'}{a_0'}$. Since $P^{\lhd}_{x,y} \in \mathbb{Z}[\Gamma_-]$, $u^{a_0} \lhd v^{-b_0}$ and $u^{a_0'}v^{b_0'} \in \Gamma_-$. Consequently,

$$u^{-a_0b_0'} = (u^{a_0})^{-b_0'} \lhd (v^{-b_0})^{-b_0'} = v^{b_0b_0'}, \text{ and } u^{a_0b_0'} \rhd v^{-b_0b_0'} \rhd u^{a_0'b_0}.$$

We have $a_0b_0' \geqslant a_0'b_0$. This together with $a_0'b_0 \geqslant a_0b_0'$ implies $a_0'b_0 = a_0b_0'$. Hence $(u^{a_0}v^{b_0})^{b_0'} = 1 \notin \Gamma_+$, a contradiction.

Let $\sigma : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a specialization satisfying $k < \sigma(1,0)/\sigma(0,1) < l$ and $\sigma(1,0), \sigma(0,1) \in \mathbb{Z}^{>0}$. Let $\lhd'$ be the parallel-line order defined by σ as in Definition 3.3. To show $\lhd'$ and $\lhd$ are equivalent, we need to prove that $P^{\lhd} = P^{\lhd'}$. By the uniqueness of the decomposition of R-matrix, we only need to prove that any monomial u^av^b appearing in $P^{\lhd}_{x,y}, x, y \in W$ satisfies $u^av^b \lhd' 1$.

Suppose u^av^b is a monomial in $P^{\lhd}_{x,y}$ for $x, y \in W$. Then we have $a, b < 0$ or $(-a, b) \in X$ or $(a, -b) \in Y$. Obviously, $u^av^b \lhd' 1$ if $a, b < 0$. Assume $(-a, b) \in X$. Then $\frac{b}{-a} \leqslant k < \frac{\sigma(1,0)}{\sigma(0,1)}$, and hence, $\sigma(a, b) < 0$. Thus $u^av^b \lhd' 1$. Similarly, $\sigma(a, b) < 0$ if $(a, -b) \in Y$, and hence $u^av^b \lhd' 1$. So we conclude that any monomial of $P^{\lhd}_{x,y}$, $x, y \in W$ is in the set Γ_-' defined by $\lhd'$. Hence $P^{\lhd} = P^{\lhd'}$ and $\lhd$ and $\lhd'$ are equivalent.

We set $k = 0$ (resp. $l = +\infty$) if $X = \emptyset$ and $Y \neq \emptyset$ (resp. $X \neq \emptyset$ and $Y = \emptyset$). If $X = Y = \emptyset$, we set $k = 0$ and $l = +\infty$. This case occurs only when all total orders on Γ are equivalent. For each of these cases, one can prove the first assertion similarly.

Since there are infinitely many rational numbers between k and l, it follows that there are infinitely many parallel-line orders which are equivalent to $\lhd$. This proves the second assertion. □

For any admissible $\sigma : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, let $m_\sigma = \sigma(1,0)/\sigma(0,1)$. Let $\tilde{\sigma}$ be the induced homomorphism from $\mathbb{Z}[\Gamma]$ to $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$.

Corollary 3.5. *Let k, l be integers determined by $\lhd$ as in the proof above. If the admissible $\sigma : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ satisfies $k < m_\sigma < l$, then $x \leqslant_{L,\sigma} y$ implies $x \leqslant_{L,\lhd} y$.*

Proof. From the proof of Theorem 3.4, we have that the image of any monomial of $P^{\lhd}$ under $\tilde{\sigma}$ is of form $q^{i/2}$ with $i < 0$. By the uniqueness of antipode decomposition of the R-matrix, we have $\tilde{\sigma}(P^{\lhd}) = P^{\sigma}$, and hence $\tilde{\sigma}(B^{\lhd}_w) = B^{\sigma}_w$. Since $\tilde{\sigma}$ induces a ring homomorphism $\tilde{\sigma}$ from $\mathcal{H}$ to $\mathcal{H}^{\sigma}$, $\tilde{\sigma}(B^{\lhd}_xB^{\lhd}_y) = \tilde{\sigma}(B^{\lhd}_x)\tilde{\sigma}(B^{\lhd}_y) = B^{\sigma}_xB^{\sigma}_y$. Thus $\tilde{\sigma}(M^{s,\lhd}_{x,y}) = M^{s,\sigma}_{x,y}$ for any $x, y \in W$ where $M^{s,\lhd}_{x,y}$ is defined in (2.3) and $M^{s,\sigma}_{x,y}$ is defined with $\lhd$ replaced by σ in (2.3) for the Hecke algebra $\mathcal{H}^{\sigma}$. Thus $M^{s,\lhd}_{x,y} \neq 0$ if $M^{s,\sigma}_{x,y} \neq 0$. By the definition of the relation $\leqslant_L$, we obtain that $x \leqslant_{L,\sigma} y$ implies $x \leqslant_{L,\lhd} y$. □

Theorem 3.6. *For any $\lhd$ on Γ, there is a specialisation σ such that $x \sim_{L,\lhd} y$ if and only if $x \sim_{L,\sigma} y$.*

Proof. It follows from Theorem 3.4 that there are infinite number of specializations σ with $k < m_\sigma < l$ satisfying $\tilde{\sigma}(P^{\triangleleft}) = P^{\sigma}$, and $\tilde{\sigma}(M_{x,y}^{s,\triangleleft}) = M_{x,y}^{s,\sigma}$ for any $x, y \in W$, and the number of such m_σ is also infinite. Since W is a finite Coxeter group, there are only finitely many non-zero $M_{x,y}^{s,\triangleleft}$. Thus, if $M_{x,y}^{s,\triangleleft} \neq 0$, then the number of m_σ for which $\tilde{\sigma}(M_{x,y}^{s,\triangleleft}) = M_{x,y}^{s,\sigma} = 0$ is finite. Therefore, there exists a specialization σ such that $M_{x,y}^{s,\sigma} \neq 0$ whenever $M_{x,y}^{s,\triangleleft} \neq 0$. So $x \leqslant_{L,\sigma} y$ if $x \leqslant_{L,\triangleleft} y$. Now, the result follows immediately from Corollary 3.5. $\square$

Remark 3.7. The above theorem answers the question given in 1.1(a). At the same time, the proof above shows that the 1.1(b) may not be true. For a specialization σ which vanishes some M-polynomials, there could be no $\triangleleft$ on Γ for which both $\triangleleft$-cells and σ-cells agree. Explicit examples will be given in next section.

4. Examples of Multi-parameter cells

In this section, we shall describe $\triangleleft$-cells and σ-cells for dihedral groups and the Weyl group of type B_3. Our results suggest that the problem 1.1(b) is not true in general.

First, we recall [7, Proposition 4], which says that the polynomials $M_{x,y}^{s,\triangleleft}$'s satisfy the following recursive formula

$$M_{x,y}^{s,\triangleleft} = P_{sx,y}^{\triangleleft} - P_{x,sy}^{\triangleleft} + u_s P_{x,y}^{\triangleleft} - \sum_{x<z<y, sz<z} P_{x,z}^{\triangleleft} M_{z,y}^{s,\triangleleft}, \tag{4.1}$$

The following lemma can be found in [10, 1.14.13].

Lemma 4.1. *Let W be a finite Coxeter group.*

1. *For $x, y \in W$ with $x < sx \leqslant y$ and $sy < y$, we have $P_{x,y}^{\triangleleft} = u_s^{-1} P_{sx,y}^{\triangleleft}$.*
2. *If $x < y$ and $l(y) = l(x) + 1$, then a reduced expression of x can be obtained from y by dropping a factor $s \in S$. Let $t \in S$ be with $tx < x < y < ty$. Then*

$$M_{x,y}^{t,\triangleleft} = \begin{cases} 0, & \text{if } u_t < u_s, \\ 1, & \text{if } u_t = u_s, \\ u_t u_s^{-1} + u_s u_t^{-1}, & \text{if } u_t > u_s. \end{cases}$$

(I) The dihedral groups. Let $W = I(m)$ be the dihedral group with $S = \{s, t\}$. Then the order of the product st is m. We denote by $u_s = u$ and $u_t = v$.

Lemma 4.2. *Suppose that $\triangleleft$ is a total order on $\Gamma = \langle u, v \rangle$ such that $u \triangleleft v$. Then*

1. *$u_s^{-1} M_{x,y}^{s,\triangleleft} \in \mathbb{Z}[\Gamma_-]$ for all $x, y \in W$ and $s \in S$.*
2. *for any n with $1 \leqslant n \leqslant [m/2] - 1$, $P_{(st)^{n-1},(st)^n}^{\triangleleft} = u^{-1}v^{-1}$.*
3. *for any n with $1 \leqslant n \leqslant [m/2] - 2$, $M_{(ts)^n,(st)^{n+1}}^{t,\triangleleft} = 0$.*
4. *For any n with $0 \leqslant n \leqslant [m/2] - 3$, $M_{t(st)^n,(st)^{n+2}}^{t,\triangleleft} = 1$.*

Proof. (1) follows from Lemma 4.1(2) under the assumption $l(y)-l(x)=1$. In general, it follows from (4.1) and the inductive hypothesis. By Lemma 4.1(1), $P^{\lhd}_{e,st}=u^{-1}v^{-1}$. Thus (2) is true for $n=1$. Suppose $n>1$. We have

$$\begin{aligned}P^{\lhd}_{(st)^{n-1},(st)^n}&=uP^{\lhd}_{(st)^{n-1},t(st)^{n-1}}+P^{\lhd}_{t(st)^{n-2},t(st)^{n-1}}-M^{s,\lhd}_{(st)^{n-1},t(st)^{n-1}}\\&=uv^{-1}+P^{\lhd}_{t(st)^{n-2},t(st)^{n-1}}\\&=uv^{-1}+vP^{\lhd}_{t(st)^{n-2},(st)^{n-1}}+P^{\lhd}_{(st)^{n-2},(st)^{n-1}}-M^{t,\lhd}_{t(st)^{n-2},(st)^{n-1}},\\&=uv^{-1}+vu^{-1}+u^{-1}v^{-1}-(uv^{-1}+vu^{-1})\\&=u^{-1}v^{-1}.\end{aligned}$$

This proves (2). If $(ts)^n<z<(st)^{n+1}$ and $tz<z$, then $z=(ts)^nt$. By (4.1),

$$M^{t,\lhd}_{(ts)^n,(st)^{n+1}}-vP_{(ts)^n,(st)^{n+1}}+P_{(ts)^n,(ts)^nt}M^{t,\lhd}_{(ts)^nt,(st)^{n+1}}\in\mathbb{Z}[\Gamma_-].$$

Using (1) and Lemma 4.1, we have

$$\begin{aligned}&P_{(ts)^n,(ts)^nt}M^{t,\lhd}_{(ts)^nt,(st)^{n+1}}=v^{-1}M^{t,\lhd}_{(ts)^nt,(st)^{n+1}}\in\mathbb{Z}[\Gamma_-],\\&vP_{(ts)^n,(st)^{n+1}}=u^{-1}.\end{aligned}$$

Thus, $M^{t,\lhd}_{(ts)^n,(st)^{n+1}}\in\mathbb{Z}[\Gamma_-]$. Since $\bar{M}^{s,\lhd}_{z,w}=M^{s,\lhd}_{z,w}$, we have $M^{t,\lhd}_{(ts)^n,(st)^{n+1}}=0$, proving (3). By a direct computation, we have $M^{t,\lhd}_{(ts)^{n+1},(st)^{n+2}}=0$, $P^{\lhd}_{t(st)^n,t(st)^{n+1}}=u^{-1}v^{-1}+v^{-1}u$ and $M^{t,\lhd}_{t(st)^{n+1},(st)^{n+2}}=uv^{-1}+vu^{-1}$. Using (4.1) and noting that $\bar{M}^{s,\lhd}_{z,w}=M^{s,\lhd}_{z,w}$, we obtain (4). ☐

Proposition 4.3. *Let W be the dihedral group $I(m)$ with $S=\{s,t\}$ as its distinguished generator set. Let $u=u_s$ and $v=u_t$.*

1. *If $u=v$, then W has 3 Kazhdan-Lusztig two-sided cells $\{e\}\cup\{w_0\}\cup W\backslash\{e,w_0\}$. The last two-sided cell consists of two Kazhdan-Lusztig left cells $\{w\in W|\ ws<w<wt\}$ and $\{w\in W|\ wt<w<ws\}$.*
2. *For any total order $\lhd$ on Γ with $u\lhd v$, W has 5 two-sided $\lhd$-cells $\{e\}\cup\{w_0\}\cup\{s\}\cup\{sw_0\}\cup W\backslash\{e,s,sw_0,w_0\}$. The last two-sided $\lhd$-cell consists of two left $\lhd$-cells $\{w\in W|\ ws<w<wt, w\neq s\}$ and $\{w\in W|\ wt<w<ws, w\neq sw_0\}$.*
3. *For any total order $\lhd$ on Γ with $v\lhd u$, W has 5 two-sided $\lhd$-cells $\{e\}\cup\{w_0\}\cup\{t\}\cup\{tw_0\}\cup W\backslash\{e,t,tw_0,w_0\}$. The last two-sided cell consists of two left $\lhd$-cells $\{w\in W|\ ws<w<wt, w\neq tw_0\}$ and $\{w\in W|\ wt<w<ws, w\neq t\}$.*

Proof. If $u=v$, then the Hecke algebra $\mathcal{H}$ has equal parameters. In this case, the cells of W are Kazhdan-Lusztig cells. The result (1) is known (see, e.g. [8]). We only deal with (2). One can prove (3) by switching s to t.

First, we claim $\{s\}$ is a two-sided $\lhd$-cell. Otherwise, there exists $z\in W$ with $z\sim_{LR,\lhd}s$ such that either $s\leqslant_{L,\lhd}z$ or $s\leqslant_{L,\lhd}z^{-1}$. We need only deal with $s\leqslant_{L,\lhd}z$ since we can switch the role between z and z^{-1}.

By the definition of $\leqslant_{L,\lhd}$, either $s=sz>z$ or $M^{s,\lhd}_{s,z}\neq 0$.

In the first case, $z = e$, a contradiction since e and s are not in the same two-sided cell.

In the second case, we have, by [10, 1.20], $\{s\} = \mathcal{R}(s) \supset \mathcal{R}(z)$, where $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. Notice that $z \neq e$, we have $\mathcal{R}(z) = \{s\}$. On the other hand, $tz < z$. Otherwise, $tz > z$ and $sz > z$ which implies that $z = e$, a contradiction. Consequently, $z = (ts)^i$ for $1 \leqslant i \leqslant [m/2] - 1$. Using induction on $l(y) - l(x)$ and (4.1), one can prove $M^{s,\triangleleft}_{x,y} = 0$ if $sx < x < y < sy$ (see [10, 1.21]). In particular, $M^{s,\triangleleft}_{s,(ts)^i} = 0$, a contradiction. Thus $\{s\}$ is a two-sided $\triangleleft$-cell.

By the arguments similar to those in [6], one can verify $M^{s,\triangleleft}_{x,y} + \epsilon_x\epsilon_y M^{s,\triangleleft}_{yw_0,xw_0} = 0$. Thus, $w \sim_{L,\triangleleft} y$ if and only if $ww_0 \sim_{L,\triangleleft} yw_0$. In particular, $\{sw_0\}$ is a two-sided $\triangleleft$-cell. By [10, 1.17], we have $\{e\}$ and $\{w_0\}$ are two distinct two-sided $\triangleleft$-cells. Finally, we show that the other elements forms a two-sided $\triangleleft$-cell. By Lemmas 4.1(2) and 4.2(4), the elements $t, st, tst, \ldots, w_0st$ are in the same left $\triangleleft$-cell. So $tw_0, stw_0, \ldots, w_0stw_0$ are in the same left cell, too. Since $\mathcal{R}(s) \neq \mathcal{R}(t)$, s and t are not in the same left cells (see [10, 1.20]). Note that $t \sim_{R,\triangleleft} ts$, $W \backslash \{e, s, sw_0, w_0\}$ forms a two-sided $\triangleleft$-cell. This proves (2). □

Remark 4.4. Note that Lemma 4.2 holds if we use $q^{i/2}$ and $q^{j/2}$ with $i < j$ instead of u and v, respectively. Thus, the σ-cells (in unequal parameters case) of a dihedral group are given in Proposition 4.3(2) and (3) for $i < j$ and $i > j$ respectively. They agree with the cells described in [9]. Moreover, the Kazhdan-Lusztig cells in $I(m)$ do not coincide with any $\triangleleft$-cells.

(II) The Weyl group of type $W(B_3)$. Let W be the finite Weyl group of type B_3 with $S = \{s_1, s_2, s_3\}$ as its distinguished generator set. The relations among s_1, s_2, s_3 are $s_1s_2s_1 = s_2s_1s_2$, $s_1s_3 = s_3s_1$, $s_2s_3s_2s_3 = s_3s_2s_3s_2$. Let $u = u_{s_1} = u_{s_2}$ and $v = u_{s_3}$. We assume $u \triangleleft v \triangleleft u^2$.

By [10, 1.17], $\{e\}$ and $\{w_0\}$ are two distinct two-sided $\triangleleft$-cells of W. Recall [10, 1.20], $\mathcal{R}(x) = \mathcal{R}(y)$ if $x \sim_{L,\triangleleft} y$. Therefore, we only need to decompose the subsets with a given $\mathcal{R}$-set into left $\triangleleft$-cells. On the other hand, since the conjugate map sending $w \in W$ to w_0ww_0 is an automorphism of W, which fixes the Dynkin diagram of $W(B_n)$, we have $w_0sw_0 = s$ for $s \in S$ and $\mathcal{R}(ww_0) = S\backslash\{s\}$ if $\mathcal{R}(w) = \{s\}$. Note that $x \sim_{L,\triangleleft} y$ if and only if $xw_0 \sim_{L,\triangleleft} yw_0$ (see the proof of Proposition 4.3). Thus, we only need to deal with the $\mathcal{R}$-sets $\{s_i\}$ for $i = 1, 2, 3$.

Proposition 4.5. *We have*

$$\begin{aligned}
&\{w \in W \mid \mathcal{R}(w) = \{s_1\}\} = X_{1,1} \cup X_{1,2},\\
&\{w \in W \mid \mathcal{R}(w) = \{s_2\}\} = X_{2,1} \cup X_{2,2} \cup X_{2,3} \cup X_{2,4},\\
&\{w \mid W \mid \mathcal{R}(w) = \{s_3\}\} = X_{3,1} \cup X_{3,2} \cup X_{3,3}
\end{aligned}$$

where

$X_{1,1} = \{s_1, s_2s_1\}, X_{1,2} = \{s_3s_2s_1, s_2s_3s_2s_1, s_1s_2s_3s_2s_1\},$

$X_{2,1} = \{s_2, s_1s_2\}, X_{2,2} = \{s_1s_3s_2, s_3s_2s_1s_3s_2, s_2s_1s_3s_2\}$

$X_{2,3} = \{s_3s_2, s_2s_3s_2, s_1s_2s_3s_2\}, X_{2,4} = \{s_1s_2s_1s_3s_2, s_1s_3s_2s_1s_3s_2, s_2s_1s_3s_2s_1s_3s_2\}.$

$X_{3,1} = \{s_3, s_2s_3, s_1s_2s_3\}, X_{3,2} = \{s_3s_2s_3\},$

$X_{3,3} = \{s_1s_3s_2s_3, s_2s_1s_3s_2s_3, s_3s_2s_1s_3s_2s_3\}.$

Proof. First, we consider the case $i = 1$. By Lemma 4.1(2), each $X_{1,j}, j = 1, 2$ is in a left $\lhd$-cell . We need to verify $X_{1,1}$ and $X_{1,2}$ are not in the same left $\lhd$-cell. If it were not the case, then there would be $x \in X_{1,1}$ and $y \in X_{1,2}$ such that either $x = sy > y$ or $M^{s,\lhd}_{x,y} \neq 0$ for some $s \in S$. However, the first result is impossible since $l(x) \leqslant 2$ and $l(y) \geqslant 3$. On the other hand, all possible $M^{s,\lhd}_{x,y}$ for $x \in X_{1,1}, y \in X_{1,2}$ are given as follows, which contradict the fact $M^{s,\lhd}_{x,y} \neq 0$.

$$M^{s_1}_{s_1,s_3s_2s_1} = M^{s_1,\lhd}_{s_1,s_2s_3s_2s_1} = M^{s_2,\lhd}_{s_2s_1,s_3s_2s_1} = M^{s_2,\lhd}_{s_2s_1,s_1s_2s_3s_2s_1} = 0.$$

Consequently, $X_{1,1}$ and $X_{1,2}$ are two distinct left $\lhd$-cells.

Suppose $i = 2$. By a direct computation,

$$M^{s_1}_{s_1s_3s_2,s_3s_2s_1s_3s_2} = M^{s_1}_{s_1s_2s_1s_3s_2,s_2s_1s_3s_2s_1s_3s_2} = u^2v^{-1} + vu^{-2}.$$

Thus $s_1s_3s_2 \leqslant_{L,\lhd} s_3s_2s_1s_3s_2$ and $s_1s_2s_1s_3s_2 \leqslant_{L,\lhd} s_2s_1s_3s_2s_1s_3s_2$. Using Lemma 4.1(2) again, we see that all the elements in $X_{2,j}$ with $j \in \{1, 2, 3, 4\}$ are in the same left $\lhd$-cells.

Suppose $i = 3$. Since $M^{s_1}_{s_1s_3s_2s_3,s_3s_2s_1s_3s_2s_3} = u^2v^{-1} + vu^{-2}$, by Lemma 4.1(2), each $X_{3,j}$ is in a left $\lhd$-cell.

One can check, for $i = 2, 3$ and $j \neq k$, $X_{i,j}$ and $X_{i,k}$ are not in the same left $\lhd$-cell by the argument similar to those for $i = 1$. □

One sees easily that all results hold if we specialize u and v to q and $q^{\frac{3}{2}}$, respectively, since we only use the inequality $u \lhd v \lhd u^2$ in the above proof. Thus the left $\lhd$-cells of $W(B_3)$ coincide with the left σ-cells of $W(B_3)$ where $\sigma(u) = q, \sigma(v) = q^{\frac{3}{2}}$.

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