

INFINITESIMAL QUANTUM \mathfrak{gl}_n AND LITTLE q -SCHUR ALGEBRAS

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ABSTRACT. We first follow De Concini and Kac [3] to give a presentation for the infinitesimal quantum \mathfrak{gl}_n , $u_k(n)$, and then reconstruct or realize $u_k(n)$ in two different ways following the Beilinson–Lusztig–MacPherson geometric setting approach [2]. Thus, we obtain three new bases for $u_k(n)$. In the second part of the paper, we use $u_k(n)$ to introduce the little q -Schur algebra $u_k(n, r)$ as a subalgebra of the q -Schur algebra $U_k(n, r)$. The symmetry structure of a little q -Schur algebra is then investigated through the construction of various bases of monomial, BLM and PBW types for $u_k(n)$ and q -Schur algebras. We also obtain a formula for the dimension of $u_k(n, r)$.

1. INTRODUCTION

Let $\mathbf{U} = \mathbf{U}(n)$ be the quantum enveloping algebra of \mathfrak{gl}_n over $\mathbb{Q}(v)$ defined by a Drinfeld–Jimbo presentation. Using a geometric setting for q -Schur algebras, A. Beilinson, G. Lusztig and R. MacPherson [2] reconstructed \mathbf{U} as a “limit” of q -Schur algebras. They first constructed an algebra \mathbf{K} over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ without identity, and then took its completion algebra $\widehat{\mathbf{K}}$ with identity over $\mathbb{Q}(v)$. They proved that, inside $\widehat{\mathbf{K}}$, there is a subspace \mathbf{V} which is also a subalgebra of $\widehat{\mathbf{K}}$ isomorphic to \mathbf{U} . By a similar construction for q -Schur algebras, they constructed explicitly the epimorphism ζ_r from $\mathbf{U}(n)$ to the q -Schur algebra $\mathbf{U}(n, r)$.

The intimate relation between $\mathbf{U}(n)$ and $\mathbf{U}(n, r)$ has been further explored in [9]. Viewing a q -Schur algebra as a “little quantum \mathbf{U} ”, B. Parshall and the first author presented $\mathbf{U}(n, r)$ with $3(n-1)$ generators along the work of Beilinson–Lusztig–MacPherson (cf. a different approach in [6] for a presentation with $3(n-1) + 1$ generators), and introduced a monomial basis theory for q -Schur algebras and their related subalgebras such as Borel subalgebras and Hecke algebras.

In this paper, we will present a parallel theory for the infinitesimal quantum group $u_k = u_k(n)$ associated to \mathbf{U} and its associated little q -Schur algebras. Here k is a field containing a primitive root ε of 1 of odd order l , and u_k is defined at ε . We shall consider the k -algebra $\mathbf{K}_k = \mathbf{K} \otimes_{\mathcal{A}} k$ by specializing v to ε and its completion algebra $\widehat{\mathbf{K}}_k$, and construct a finite dimensional subspace \mathcal{W} as the counterpart of \mathbf{V} over k . We then prove that \mathcal{W} is a subalgebra of $\widehat{\mathbf{K}}_k$ and is isomorphic

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to u_k via a Drinfeld–Jimbo type presentation for u_k . We also prove that the k -algebra \mathcal{W} is isomorphic to the algebra \mathcal{K}' constructed in [2, §6].

The second part of the paper is devoted to investigating the image $u_k(n, r)$ of the restriction of $\zeta_{r,k} := \zeta_r \otimes 1$ to $u_k(n)$. We call it a *little q -Schur algebra*. This algebra inherits many nice properties from u_k and q -Schur algebras including certain nice bases of monomial, BLM and PBW types. In particular, its dimension is the dimension of the q -Schur algebra $\mathbf{U}(n, r)$ “modulo l ”. More precisely, if Ξ_r denotes the set of all $n \times n$ matrices over \mathbb{N} whose entries sum to r , then $\dim \mathbf{U}(n, r) = \#\Xi_r$. We shall prove that $\dim u_k(n, r) = \#\bar{\Xi}_r$ where $\bar{\Xi}_r$ is the image of Ξ_r under the map sending a matrix (a_{ij}) over \mathbb{Z} to the matrix (\bar{a}_{ij}) over $\mathbb{Z}_l := \mathbb{Z}/l\mathbb{Z}$. We expect that there should be a similar relation at the representation level. In a forthcoming paper, we shall study representations of a little q -Schur algebra. For a comparison between $u_k(n, r)$ and the infinitesimal q -Schur algebras investigated in [7] and [4], see [10].

We organize the paper as follows. We recall the definition and basic results on the infinitesimal quantum \mathfrak{gl}_n , u_k , in §2, and describe its presentation by using a construction in [3]. In §3, we review the Beilinson-Lusztig-Macpherson construction of \mathbf{U} and q -Schur algebras, and in §4, we introduce the algebra \mathcal{W} . The isomorphisms between \mathcal{W} , u_k and the quantum group \mathcal{K}' defined in [2, §6] are proved in Theorem 5.5. In §6, new monomial bases for q -Schur algebras are introduced, and as an application, we also constructed the bases conjectured in [6, 3.2]. The little q -Schur algebra $u_k(n, r)$ is defined and their basic properties are discussed in §7. We construct in §8 various bases for $u_k(n, r)$ and display them in Table 8.6, indicating how they stand in relation to the bases for $\mathbf{U}(n)$, $\mathbf{U}(n, r)$ and $u_k(n)$. In §9, we derive several dimension formulas for little q -Schur algebras and their associated subalgebras. Finally, in the last section, we investigate the structure of the Borel subalgebras of a little q -Schur algebra.

Throughout, let v be an indeterminate and let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Let k be a field containing an l th primitive root ε of 1 with $l > 1$ odd. Specializing v to ε , k will be viewed as an \mathcal{A} -module. Let $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$.

2. LUSZTIG’S INFINITESIMAL QUANTUM \mathfrak{gl}_n

Most of the results in this section is known. For those we couldn’t find an appropriate reference, we provide a brief proof. We first recall the definition of the quantum enveloping algebra of \mathfrak{gl}_n (see, e.g., [8]).

Definition 2.1. The quantum enveloping algebra of \mathfrak{gl}_n is the algebra \mathbf{U} over $\mathbb{Q}(v)$ presented by generators

$$E_i, F_i \quad (1 \leq i \leq n-1), \quad K_j, K_j^{-1} \quad (1 \leq j \leq n)$$

and relations $(1 \leq i, j \leq n-1, 1 \leq i', j' \leq n)$

$$(a) \quad K_{i'} K_{j'} = K_{j'} K_{i'}, \quad K_{i'} K_{i'}^{-1} = 1;$$

- (b) $K_{i'}E_j = v^{\epsilon(i',j)}E_jK_{i'}$, where $\epsilon(j,j) = 1$, $\epsilon(j+1,j) = -1$, and $\epsilon(i',j) = 0$ otherwise;
- (c) $K_{i'}F_j = v^{-\epsilon(i',j)}F_jK_{i'}$ with $\epsilon(i',j)$ as in (b) above;
- (d) $E_iE_j = E_jE_i$, $F_iF_j = F_jF_i$ when $|i-j| > 1$;
- (e) $E_iF_j - F_jE_i = \delta_{ij}\frac{\widetilde{K}_i - \widetilde{K}_i^{-1}}{v - v^{-1}}$, where $\widetilde{K}_i = K_iK_{i+1}^{-1}$;
- (f) $E_i^2E_j - (v + v^{-1})E_iE_jE_i + E_jE_i^2 = 0$ when $|i-j| = 1$;
- (g) $F_i^2F_j - (v + v^{-1})F_iF_jF_i + F_jF_i^2 = 0$ when $|i-j| = 1$.

Note that the subalgebra generated by the E_i , F_i and \widetilde{K}_i ($1 \leq i \leq n-1$) is isomorphic with the quantum enveloping algebra $\mathbf{U}(\mathfrak{sl}_n)$, and that we will henceforth denote the subalgebra by \mathbf{U}' .

Following [12, 13], let $U_{\mathcal{A}}$ (resp., $U_{\mathcal{A}}^+$, $U_{\mathcal{A}}^-$) be the \mathcal{A} -subalgebra of \mathbf{U} generated by all $E_i^{(m)}$, $F_i^{(m)}$, K_i and $\left[\begin{smallmatrix} K_i; 0 \\ t \end{smallmatrix} \right]$ (resp., $E_i^{(m)}$, $F_i^{(m)}$), where for $m, t \in \mathbb{N}$ and $c \in \mathbb{Z}$,

$$E_i^{(m)} = \frac{E_i^m}{[m]!}, \quad F_i^{(m)} = \frac{F_i^m}{[m]!}, \quad \text{and} \quad \left[\begin{smallmatrix} K_i; c \\ t \end{smallmatrix} \right] = \prod_{s=1}^t \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}}$$

with $[m]! = [1][2] \cdots [m]$ and $[i] = \frac{v^i - v^{-i}}{v - v^{-1}}$. Let $U_{\mathcal{A}}^0$ be the \mathcal{A} -subalgebra of \mathbf{U} generated by all K_i and $\left[\begin{smallmatrix} K_i; 0 \\ t \end{smallmatrix} \right]$. Then there is a triangular decomposition

$$U_{\mathcal{A}} = U_{\mathcal{A}}^+ U_{\mathcal{A}}^0 U_{\mathcal{A}}^- \cong U_{\mathcal{A}}^+ \otimes U_{\mathcal{A}}^0 \otimes U_{\mathcal{A}}^-$$

compatible with the PBW type \mathcal{A} -basis (see, e.g., [8, 2.2])

$$(2.1.1) \quad \prod_{\alpha \in R^+} E_{\alpha}^{(a_{\alpha})} \prod_{i=1}^n K_i^{\delta_i} \left[\begin{smallmatrix} K_i; 0 \\ t_i \end{smallmatrix} \right] \prod_{\alpha \in R^+} F_{\alpha}^{(b_{\alpha})} \quad (a_{\alpha}, b_{\alpha}, t_i \in \mathbb{N}, \delta_i \in \{0, 1\}).$$

Here R^+ is the positive root system of the Lie algebra \mathfrak{sl}_n indexed by all pair (i, j) with $1 \leq i < j \leq n$, and E_{α} , F_{α} are certain root vectors defined by a braid group action [12, 1.3].¹

Let k be a field containing an l th primitive root ε of 1 with l odd. Specializing v to ε , we obtain the \mathcal{A} -module k and the *quantum hyperalgebra* $U_k = U_{\mathcal{A}} \otimes_{\mathcal{A}} k$ over k . We will denote the images of $E_i^{(m)} \otimes 1$ etc. in U_k by the same letters. Since $\varepsilon^l - \varepsilon^{-l} = 0$, we have $E_i^l = 0 = F_i^l$ and $K_i^{2l} = 1$ in U_k (see [11, 4.4].)

Let \tilde{u}_k be the k -subalgebra of U_k generated by the elements E_i , F_i , $K_j^{\pm 1}$ for all i, j . Let \tilde{u}_k^+ , \tilde{u}_k^0 , and \tilde{u}_k^- be the k -subalgebras of \tilde{u}_k generated respectively by the elements E_i 's, $K_j^{\pm 1}$'s, and F_i 's. Inherited from (2.1.1), we have the triangular decomposition

$$(2.1.2) \quad \tilde{u}_k = \tilde{u}_k^+ \tilde{u}_k^0 \tilde{u}_k^- \cong \tilde{u}_k^+ \otimes \tilde{u}_k^0 \otimes \tilde{u}_k^-.$$

Lemma 2.2. *The set $\mathcal{X} = \left\{ \prod_{i=1}^n K_i^{N_i} \mid 0 \leq N_i \leq 2l-1 \right\}$ forms a basis for \tilde{u}_k^0 . In particular, $\dim \tilde{u}_k^0 = 2^n l^n$.*

¹This action on \mathbf{U}' can be easily extended to \mathbf{U} .

Proof. At the integral level, $U_{\mathcal{A}}^0(n)$ is isomorphic to $U_{\mathcal{A}}^0(n+1)$, the integral 0-part of the quantum \mathfrak{sl}_{n+1} . Thus, there is an isomorphism $f : U_k^0(n+1) \xrightarrow{\sim} U_k^0(n)$ which sends the basis for the infinitesimal 0-part described in [12, 5.8] to the set \mathcal{X} . Therefore, \mathcal{X} is a basis for \tilde{u}_k^0 . \square

The elements $K_1^l - 1, \dots, K_n^l - 1$ are central in U_k (and in \tilde{u}_k). They generate an ideal $\langle K_1^l - 1, \dots, K_n^l - 1 \rangle$ of \tilde{u}_k . Let $u_k = \tilde{u}_k / \langle K_1^l - 1, \dots, K_n^l - 1 \rangle$. We call u_k the *infinitesimal quantum group* of \mathfrak{gl}_n . Let u_k^+, u_k^0 and u_k^- respectively be the image of $\tilde{u}_k^+, \tilde{u}_k^0, \tilde{u}_k^-$. Again, by abuse of notation, we shall denote the images of E_α, F_α , etc. in u_k by the same letters.

Theorem 2.3. *The algebras u_k^+, u_k^-, u_k^0 and u_k have the following k -bases :*

$$u_k^+ : \quad \prod_{\alpha \in R^+} E_\alpha^{N_\alpha} \quad (0 \leq N_\alpha \leq l-1),$$

$$u_k^- : \quad \prod_{\alpha \in R^+} F_\alpha^{N_\alpha} \quad (0 \leq N_\alpha \leq l-1),$$

$$u_k^0 : \quad \prod_{i=1}^n K_i^{N_i} \quad (0 \leq N_i \leq l-1),$$

$$u_k : \quad \prod_{\alpha \in R^+} F_\alpha^{N_\alpha} \prod_{i=1}^n K_i^{N_i} \prod_{\alpha \in R^+} E_\alpha^{N'_\alpha} \quad (0 \leq N_\alpha \leq l-1, 0 \leq N'_\alpha \leq l-1, 0 \leq N_i \leq l-1).$$

In particular, the dimension of u_k is l^{n^2} .

Proof. We first observe that, as a k -space

$$u_k \cong \tilde{u}_k^- \otimes (\tilde{u}_k^0/J) \otimes \tilde{u}_k^+.$$

Here J is the ideal of \tilde{u}_k^0 generated by $K_1^l - 1, \dots, K_n^l - 1$. So $u_k^\pm = \tilde{u}_k^\pm$. The bases for \tilde{u}_k^\pm are given in [12, 5.8]. The claim for u_k^0 follows from (2.1.2) and Lemma 2.2. The rest of the proof is obvious. \square

There is an alternative way to construct the algebra \tilde{u}_k and u_k . Let \mathcal{U}_k be the algebra over k with generators E_i, F_i and $K_j^{\pm 1}$ ($1 \leq i \leq n-1, 1 \leq j \leq n$) and the same relations 2.1(a)-(g) for \mathbf{U} but with v replaced by ε (noting our assumption on l). This algebra is a \mathfrak{gl}_n version of the k -algebra U_2 considered in [1, 1.3], attributed to De Concini and Kac [3].

Clearly, there is a algebra homomorphism $g : \mathcal{U}_k \rightarrow U_k$ mapping the generators of \mathcal{U}_k to their counterparts in U_k . The image of g is the algebra \tilde{u}_k .

Similar to \mathbf{U} , we can define root vectors E_α, F_α , where $\alpha \in R^+$, and triangular decomposition for \mathcal{U}_k . We have the following PBW basis which can be proved in the same way as in [3] over \mathbb{C} (cf. the basis in [1, p.15(1)]).

Lemma 2.4. *The elements $\prod_{\alpha \in R^+} F_\alpha^{N_\alpha} \prod_{i=1}^n K_i^{N_i} \prod_{\alpha \in R^+} E_\alpha^{N'_\alpha}$ ($0 \leq N_\alpha, N'_\alpha, (N_1, N_2, \dots, N_n) \in \mathbb{Z}^n$) form a k -basis for \mathcal{U}_k .*

Let I be the (two-sided) ideal of \mathcal{U}_k generated by $E_i^l, F_i^l, K_j^l - 1, 1 \leq i \leq n-1, 1 \leq j \leq n$.

Theorem 2.5. *There is an algebra isomorphism*

$$\mathcal{U}_k/I \xrightarrow{\sim} u_k.$$

Hence u_k is the k -algebra defined by generators

$$E_i, F_i, K_j \quad (1 \leq i \leq n-1, 1 \leq j \leq n)$$

and relations (a)–(g), which are the same as 2.1(a)–(g) with v replaced by ε , together with the relation:

$$(h) \quad K_j^l = 1, \quad E_i^l = 0, \quad F_i^l = 0;$$

Proof. By definition, the algebra epimorphism $g : \mathcal{U}_k \rightarrow \tilde{u}_k$ extends to an epimorphism $\bar{g} : \mathcal{U}_k \rightarrow u_k$. Observe from 2.4 and 2.3 that $\dim(\mathcal{U}_k/I) \leq l^{n^2} = \dim u_k$. So we must have an equality of dimensions and $\ker \bar{g} = I$, proving the statement. \square

Remark 2.6. The presentation for infinitesimal quantum groups associated to semisimple Lie algebras given in [12, 5.7] involves generators E_α, F_α for every positive root $\alpha \in R^+$. The presentation obtained above is a Drinfeld–Jimbo type presentation with generators involving only simple roots, and is stated for the type A case. Clearly, one can use the same approach to obtain similar presentations in general.

3. THE BLM CONSTRUCTION OF \mathbf{U} AND q -SCHUR ALGEBRAS

Before we turn to other constructions of u_k , we first briefly review the geometric settings for \mathbf{U} and q -Schur algebras in [2].

Let $\tilde{\Xi}$ be the set of all $n \times n$ matrices over \mathbb{Z} with all off diagonal entries in \mathbb{N} , and let $\Xi = M_n(\mathbb{N})$ be the subset of $\tilde{\Xi}$ consisting of matrices with entries all in \mathbb{N} . Let $\sigma : \Xi \rightarrow \mathbb{N}$ be the map sending a matrix to the sum of its entries. Then, for $r \in \mathbb{N}$, the inverse image $\Xi_r := \sigma^{-1}(r)$ is the set of $n \times n$ matrices in Ξ whose entries sum to r . We also write $\sigma(\mathbf{j}) = j_1 + \cdots + j_n$ for $\mathbf{j} \in \mathbb{N}^n$. For $1 \leq i, j \leq n$, let $E_{i,j} \in \Xi$ be the matrix $(a_{k,l})$ with $a_{k,l} = \delta_{i,k}\delta_{j,l}$.

Let $U_{\mathcal{A}}(n, r)$ be the algebra over \mathcal{A} introduced in [2, 1.2]. It has a normalized \mathcal{A} -basis $\{[A]\}_{A \in \Xi_r}$. In particular, if $\lambda \in \mathbb{N}^n$ with $D = \text{diag}(\lambda) \in \Xi_r$, then (cf. [2, 1.2, 1.3])

$$(3.0.1) \quad [D][A] = \begin{cases} [A] & \text{if } \lambda = ro(A) \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad [A][D] = \begin{cases} [A] & \text{if } \lambda = co(A) \\ 0 & \text{otherwise,} \end{cases}$$

where $ro(A) = (\sum_j a_{1,j}, \dots, \sum_j a_{n,j})$ and $co(A) = (\sum_i a_{i,1}, \dots, \sum_i a_{i,n})$ are the sequences of row and column sums of $A = (a_{i,j})$. We put $\mathbf{U}(n, r) = U_{\mathcal{A}}(n, r) \otimes_{\mathcal{A}} \mathbb{Q}(v)$.

In [8, 1.4], the algebra $U_{\mathcal{A}}(n, r)$ is shown to be naturally isomorphic to the q -Schur algebra introduced in [5]. In the sequel, we shall call $U_{\mathcal{A}}(n, r)$ and $\mathbf{U}(n, r)$ q -Schur algebras.

Let \mathbf{K} be the \mathcal{A} -algebra (without 1), defined in [2, §4], with basis $\{[A]\}_{A \in \tilde{\Xi}}$. The multiplication in \mathbf{K} is defined in [2, 4.4] by specializing v' to 1 from another algebra over $\mathbb{Q}(v)[v', v'^{-1}]$ whose multiplication is induced from the stabilization property of the multiplication of q -Schur algebras. As in [2, 5.1], let $\widehat{\mathbf{K}}$ be the vector space of all formal (possibly infinite) $\mathbb{Q}(v)$ -linear combinations $\sum_{A \in \tilde{\Xi}} \beta_A [A]$ satisfying

(F): for any $\mathbf{x} \in \mathbb{Z}^n$, the sets $\begin{matrix} \{A \in \tilde{\Xi} \mid \beta_A \neq 0, ro(A) = \mathbf{x}\} \\ \{A \in \tilde{\Xi} \mid \beta_A \neq 0, co(A) = \mathbf{x}\} \end{matrix}$ are finite.

We shall regard \mathbf{K} naturally as a subset of $\widehat{\mathbf{K}}$. We can define the product of two elements $\sum_{A \in \tilde{\Xi}} \beta_A [A]$, $\sum_{B \in \tilde{\Xi}} \gamma_B [B]$ in $\widehat{\mathbf{K}}$ to be $\sum_{A, B} \beta_A \gamma_B [A] \cdot [B]$ where $[A] \cdot [B]$ is the product in \mathbf{K} . This defines an associative algebra structure on $\widehat{\mathbf{K}}$. This algebra has a unit element: the sum of all $[D]$ with D a diagonal matrix in $\tilde{\Xi}$.

Let Ξ^\pm be the set of all $A \in \Xi$ whose diagonal entries are zero. Given $r > 0$, $A \in \Xi^\pm$ and $\mathbf{j} = (j_1, j_2, \dots, j_n) \in \mathbb{Z}^n$, we define

$$(3.0.2) \quad \begin{aligned} A(\mathbf{j}, r) &= \sum_{\substack{D \in \Xi^0 \\ \sigma(A+D)=r}} v^{\sum_i d_i j_i} [A + D] \in \mathbf{U}(n, r), \\ A(\mathbf{j}) &= A(\mathbf{j}, \infty) = \sum_{D \in \tilde{\Xi}^0} v^{\sum_i d_i j_i} [A + D] \in \widehat{\mathbf{K}}. \end{aligned}$$

where Ξ^0 (resp., $\tilde{\Xi}^0$) denotes the subset of diagonal matrices in Ξ (resp., $\tilde{\Xi}$) and d_i are diagonal entries of D .

Let \mathbf{V} be the subspace of $\widehat{\mathbf{K}}$ spanned by

$$\mathfrak{B} = \{A(\mathbf{j}) \mid A \in \Xi^\pm, \mathbf{j} \in \mathbb{Z}^n\}.$$

The next result is proved in [2, 5.5, 5.7].

Theorem 3.1. (1) \mathbf{V} is a subalgebra of $\widehat{\mathbf{K}}$ with $\mathbb{Q}(v)$ -basis \mathfrak{B} . It is generated by $E_{h, h+1}(\mathbf{0})$, $E_{h+1, h}(\mathbf{0})$ and $0(\mathbf{j})$ for all $1 \leq h < n$ and $\mathbf{j} \in \mathbb{Z}^n$.

(2) For any positive integer r , the q -Schur algebra $\mathbf{U}(n, r)$ is generated by the elements

$$E_{h, h+1}(\mathbf{0}, r), E_{h+1, h}(\mathbf{0}, r), \text{ and } 0(\mathbf{j}, r)$$

for all $1 \leq h < n$ and $\mathbf{j} \in \mathbb{N}^n$.

(3) There is an algebra isomorphism $\mathbf{U} \xrightarrow{\sim} \mathbf{V}$ satisfying

$$E_h \mapsto E_{h, h+1}(\mathbf{0}), K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n} \mapsto 0(\mathbf{j}), F_h \mapsto E_{h+1, h}(\mathbf{0})$$

and an algebra epimorphism $\zeta_r : \mathbf{U}(n) \rightarrow \mathbf{U}(n, r)$ satisfying

$$E_h \mapsto E_{h, h+1}(\mathbf{0}, r), K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n} \mapsto 0(\mathbf{j}, r), F_h \mapsto E_{h+1, h}(\mathbf{0}, r).$$

(4) $\zeta_r(A(\mathbf{j})) = A(\mathbf{j}, r)$, for any $A \in \Xi^\pm$, $\mathbf{j} \in \mathbb{Z}^n$.

We shall identify \mathbf{U} with \mathbf{V} and hence identify E_h with $E_{h,h+1}(\mathbf{0})$ etc., in the sequel. Thus,

$$E_h^{(m)} = (mE_{h,h+1})(\mathbf{0}) \quad \text{and} \quad F_h^{(m)} = (mE_{h+1,h})(\mathbf{0})$$

(see [2, p.673]).

Let Ξ^+ (resp., Ξ^-) be the subset of Ξ consisting of those matrices $(a_{i,j})$ with $a_{i,j} = 0$ for all $i \geq j$ (resp., $i \leq j$). For $A \in \Xi$, write $A = A^\pm + A^0 = A^+ + A^0 + A^-$ with $A^+ \in \Xi^+$, $A^0 \in \Xi^0$, $A^- \in \Xi^-$ and $A^\pm \in \Xi^\pm$.

For $A \in \Xi^\pm$ and $\mathbf{j} \in \mathbb{Z}^n$ let

$$(3.1.1) \quad E^{(A^+)} = \prod_{1 \leq i \leq h < j \leq n} E_h^{(a_{i,j})} \quad \text{and} \quad F^{(A^-)} = \prod_{1 \leq j \leq h < i \leq n} F_h^{(a_{i,j})}$$

The orders in which the products $E^{(A^+)}$ and $F^{(A^-)}$ are taken are defined as follows. Put

$$M_j = E_{j-1}^{(a_{j-1,j})} (E_{j-2}^{(a_{j-2,j})} E_{j-1}^{(a_{j-2,j-1})}) \cdots (E_1^{(a_{1,j})} E_2^{(a_{1,j})}) \cdots E_{j-1}^{(a_{1,j-1})}$$

Similarly, put

$$M'_j = (F_{j-1}^{(a_{j,1})} \cdots F_2^{(a_{j,1})} F_1^{(a_{j,1})}) \cdots (F_{j-1}^{(a_{j,j-2})} F_{j-2}^{(a_{j,j-2})}) F_{j-1}^{(a_{j,j-1})}$$

Then $E^{(A^+)} = M_n M_{n-1} \cdots M_2$ and $F^{(A^-)} = M'_2 M'_3 \cdots M'_n$. We obtain another basis (see [2]).

Proposition 3.2. *The set*

$$\{M^{(A,\mathbf{j})} := E^{(A^+)} \mathbf{0}(\mathbf{j}) F^{(A^-)} \mid A \in \Xi^\pm, \mathbf{j} \in \mathbb{Z}^n\}$$

is a $\mathbb{Q}(v)$ -basis for \mathbf{U} . This basis will be called the monomial basis of \mathbf{U} .

4. THE ALGEBRA \mathcal{W}

Recall that k is a field and ε is a primitive l -th root of unity where $l > 1$ is an odd integer. Let $\mathbf{K}_k = \mathbf{K} \otimes_{\mathcal{A}} k$; this inherits from \mathbf{K} a basis and a multiplication which will be denoted as in \mathbf{K} . Mimicking the construction of $\widehat{\mathbf{K}}$, we define $\widehat{\mathbf{K}}_k$ to be the k -vector space of all formal (possibly infinite) k -linear combinations $\sum_{A \in \Xi} \beta_A [A]$ satisfying the property (F) with a similar multiplication. This is an associative algebra with an identity: the sum of all $[D]$ with D a diagonal matrix in $\widetilde{\Xi}$. The elements $A(\mathbf{j})$ defined in (3.0.2) become

$$A(\mathbf{j}) = \sum_{\mathbf{z} \in \mathbb{Z}^n} \varepsilon^{\mathbf{j} \cdot \mathbf{z}} [A + \text{diag}(\mathbf{z})] \in \widehat{\mathbf{K}}_k \quad (\mathbf{j} \cdot \mathbf{z} = \sum_i j_i z_i).$$

Clearly, $A(\mathbf{j}) = A(\mathbf{j}')$ whenever $\bar{\mathbf{j}} = \bar{\mathbf{j}'}$. Here $\bar{\cdot} : \mathbb{Z}^n \rightarrow (\mathbb{Z}_l)^n$ ($\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$) is the map defined by $\overline{(j_1, j_2, \dots, j_n)} = (\bar{j}_1, \bar{j}_2, \dots, \bar{j}_n)$. Thus, we shall write $A(\bar{\mathbf{j}}) := A(\mathbf{j})$. Similarly, we shall use $A(\bar{\mathbf{j}}, r) := A(\mathbf{j}, r)$ to denote the element defined in (3.0.2) with v replaced by ε for the q -Schur algebra $U_k(n, r) = U_{\mathcal{A}}(n, r) \otimes_{\mathcal{A}} k$ over k .

Let Γ be the set of all $A = (a_{i,j}) \in \widetilde{\Xi}$ such that $a_{i,j} < l$ for all $i \neq j$. Let Γ^\pm be the set of all $A \in \Gamma$ whose diagonal entries are zero, and let Γ^+ (resp., Γ^- , Γ^0) denote the subset of Γ consisting of those matrices $(a_{i,j})$ with $a_{i,j} = 0$ for all $i \geq j$ (resp. $i \leq j$, $i \neq j$).

Let \mathcal{W} be the subspace of $\widehat{\mathbf{K}}_k$ spanned by

$$\mathfrak{B}_k = \{A(\bar{\mathbf{j}}) \mid A \in \Gamma^\pm, \bar{\mathbf{j}} \in (\mathbb{Z}_l)^n\}.$$

This is the finite version of \mathbf{V} over k . We have clearly $\dim \mathcal{W} \leq l^{n^2}$. As for \mathbf{V} , we want to prove that \mathcal{W} is actually a subalgebra of $\widehat{\mathbf{K}}_k$.

Following the notation used in [2], we also denote by $\bar{\cdot} : \mathcal{A} \rightarrow \mathcal{A}$ the ring homomorphism sending \bar{v} to v^{-1} . It should not be confused with the notation $\bar{\mathbf{j}}$ on n -tuples of integers. Put

$$\mathbf{e}_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0) \in \mathbb{Z}^n,$$

and $\llbracket a \rrbracket_\varepsilon = \frac{\varepsilon^{2a} - 1}{\varepsilon^2 - 1}$ for $a \geq 1$. The following result is a version of [2, 5.3] over k .

Lemma 4.1. *The subspace \mathcal{W} is stabilized under the left multiplication by elements of the form $0(\bar{\mathbf{j}}), E_{h,h+1}(\bar{\mathbf{0}}), E_{h+1,h}(\bar{\mathbf{0}})$. More precisely, for $1 \leq h, i \leq n$ ($h \neq n$), $\bar{\mathbf{j}}, \bar{\mathbf{j}}' \in (\mathbb{Z}_l)^n$ and $A \in \Gamma^\pm$, if we put $\alpha_h = \mathbf{e}_h - \mathbf{e}_{h+1}$, $\beta_h = -\mathbf{e}_h - \mathbf{e}_{h+1}$, $f(i) = f(i, A) = \sum_{j \geq i} a_{h,j} - \sum_{j > i} a_{h+1,j}$ and $f'(i) = f'(i, A) = \sum_{j < i} a_{h,j} - \sum_{j \leq i} a_{h+1,j}$, then the following multiplication identities hold in $\widehat{\mathbf{K}}_k$:*

$$(4.1.1) \quad \begin{aligned} 0(\bar{\mathbf{j}})A(\bar{\mathbf{j}}') &= \varepsilon^{\sum_{i,k} j_i a_{i,k}} A(\bar{\mathbf{j}} + \bar{\mathbf{j}}') \\ A(\bar{\mathbf{j}}')0(\bar{\mathbf{j}}) &= \varepsilon^{\sum_{i,k} j_i a_{k,i}} A(\bar{\mathbf{j}} + \bar{\mathbf{j}}') \end{aligned}$$

where 0 stands for the zero matrix.

$$(4.1.2) \quad \begin{aligned} E_{h,h+1}(\bar{\mathbf{0}})A(\bar{\mathbf{j}}) &= \sum_{\substack{i < h; a_{h+1,i} \geq 1 \\ a_{h,i} + 1 < l}} \varepsilon^{f(i)} \overline{\llbracket a_{h,i} + 1 \rrbracket_\varepsilon} (A + E_{h,i} - E_{h+1,i})(\bar{\mathbf{j}} + \bar{\alpha}_h) \\ &+ \sum_{\substack{i > h+1; a_{h+1,i} \geq 1 \\ a_{h,i} + 1 < l}} \varepsilon^{f(i)} \overline{\llbracket a_{h,i} + 1 \rrbracket_\varepsilon} (A + E_{h,i} - E_{h+1,i})(\bar{\mathbf{j}}) \\ &+ \alpha \varepsilon^{f(h) - j_h - 1} \frac{(A - E_{h+1,h})(\bar{\mathbf{j}} + \bar{\alpha}_h) - (A - E_{h+1,h})(\bar{\mathbf{j}} + \bar{\beta}_h)}{1 - \varepsilon^{-2}} \\ &+ \varepsilon^{f(h+1) + j_{h+1}} \overline{\llbracket a_{h,h+1} + 1 \rrbracket_\varepsilon} (A + E_{h,h+1})(\bar{\mathbf{j}}) \end{aligned}$$

where α is 1 if $a_{h+1,h} \geq 1$ and is 0 otherwise.

$$\begin{aligned}
 E_{h+1,h}(\bar{\mathbf{0}})A(\bar{\mathbf{j}}) &= \sum_{\substack{i < h; a_{h,i} \geq 1 \\ a_{h+1,i+1} < l}} \varepsilon^{f'(i)} \overline{[a_{h+1,i} + 1]}_\varepsilon (A - E_{h,i} + E_{h+1,i})(\bar{\mathbf{j}}) \\
 &+ \sum_{\substack{i > h+1; a_{h,i} \geq 1 \\ a_{h+1,i+1} < l}} \varepsilon^{f'(i)} \overline{[a_{h+1,i} + 1]}_\varepsilon (A - E_{h,i} + E_{h+1,i})(\bar{\mathbf{j}} - \bar{\alpha}_h) \\
 (4.1.3) \quad &+ \alpha' \varepsilon^{f'(h+1) - j_{h+1} - 1} \frac{(A - E_{h,h+1})(\bar{\mathbf{j}} - \bar{\alpha}_h) - (A - E_{h,h+1})(\bar{\mathbf{j}} + \bar{\beta}_h)}{1 - \varepsilon^{-2}} \\
 &+ \varepsilon^{f'(h) + j_h} \overline{[a_{h+1,h} + 1]}_\varepsilon (A + E_{h+1,h})(\bar{\mathbf{j}})
 \end{aligned}$$

where α' is 1 if $a_{h,h+1} \geq 1$ and is 0 otherwise.

The same formulas hold in the q -Schur algebra $U_k(n, r) = U_{\mathcal{A}}(n, r) \otimes_{\mathcal{A}} k$ over k with $A(\bar{\mathbf{j}})$ replaced by $A(\bar{\mathbf{j}}, r)$ for $\bar{\mathbf{j}} \in \mathbb{N}^n$ (and $v = \varepsilon$ in (3.0.2)).

Proof. Note that the multiplication formulas given in [2, 5.3] hold² over the localization of \mathcal{A} at $1 - v^2$. Now specializing v to ε gives formulas over k . Since $[[l]]_\varepsilon = 0$ and $a_{ij} < l$, those terms with $a_{h,i} + 1 = l$ in (4.1.2) and with $a_{h+1,i} + 1 = l$ in (4.1.3) disappeared. This means that the matrices appeared on the right hand sides are all in Γ^\pm . Therefore, \mathcal{W} is closed under multiplication by elements $0(\bar{\mathbf{j}})$, $E_{h,h+1}(\bar{\mathbf{0}})$ and $E_{h+1,h}(\bar{\mathbf{0}})$. \square

Corollary 4.2. For any $m \geq 0$, we have the following formula in $\widehat{\mathbf{K}}_k$:

$$(4.2.1) \quad E_{h,h+1}(\bar{\mathbf{0}})^m = \prod_{i=1}^m \frac{\varepsilon^i - \varepsilon^{-i}}{\varepsilon - \varepsilon^{-1}} (mE_{h,h+1})(\bar{\mathbf{0}})$$

$$(4.2.2) \quad E_{h+1,h}(\bar{\mathbf{0}})^m = \prod_{i=1}^m \frac{\varepsilon^i - \varepsilon^{-i}}{\varepsilon - \varepsilon^{-1}} (mE_{h+1,h})(\bar{\mathbf{0}})$$

$$(4.2.3) \quad 0(\bar{e}_j)^l = 1$$

In particular $E_{h,h+1}(\bar{\mathbf{0}})^l = 0$, $E_{h+1,h}(\bar{\mathbf{0}})^l = 0$.

Proof. Using (4.1.2) for $A = E_{h,h+1}$, we get (4.2.1). Using (4.1.3) for $A = E_{h+1,h}$, we get (4.2.2). By (4.1.1), we see that $0(\bar{e}_j)^l = 0(\bar{l}e_j) = 0(\bar{\mathbf{0}}) = 1$. \square

For $A = (a_{s,t}) \in \tilde{\Xi}$ and $i < j$, let $\sigma_{i,j}(A) = \sum_{s \leq i; t \geq j} a_{s,t}$ and $\sigma_{j,i}(A) = \sum_{s \leq i; t \geq j} a_{t,s}$. Define $A' \preceq A$ iff $\sigma_{i,j}(A') \leq \sigma_{i,j}(A)$ and $\sigma_{j,i}(A') \leq \sigma_{j,i}(A)$ for all $1 \leq i < j \leq n$. Put $A' \prec A$ if $A' \preceq A$ and, for some pair (i, j) with $i < j$, either $\sigma_{i,j}(A') < \sigma_{i,j}(A)$ or $\sigma_{j,i}(A') < \sigma_{j,i}(A)$.

Proposition 4.3. Let $A = (a_{i,j}) \in \Gamma^\pm$ and assume that the orders in which the products below are taken are the same as in (3.1.1).

²This is because the formulas [2, 4.6(a),(b)] used in the proof of [2, 5.3] hold in the \mathcal{A} -algebra \mathbf{K} .

(1) We have the following formula in the q -Schur algebra $U_k(n, r) = U_{\mathcal{A}}(n, r) \otimes_{\mathcal{A}} k$:

$$(4.3.1) \quad \prod_{1 \leq i \leq h < j \leq n} (a_{i,j} E_{h,h+1})(\bar{\mathbf{0}}, r) \cdot \prod_{1 \leq j \leq h < i \leq n} (a_{i,j} E_{h+1,h})(\bar{\mathbf{0}}, r) = A(\bar{\mathbf{0}}, r) + f$$

where f stands for a k -linear combination of elements $A'(\bar{\mathbf{j}}, r)$ with $A' \in \Gamma^{\pm}$ and $A' \prec A$.

(2) We have the following formula in $\widehat{\mathbf{K}}_k$:

$$(4.3.2) \quad \prod_{1 \leq i \leq h < j \leq n} (a_{i,j} E_{h,h+1})(\bar{\mathbf{0}}) \cdot \prod_{1 \leq j \leq h < i \leq n} (a_{i,j} E_{h+1,h})(\bar{\mathbf{0}}) = A(\bar{\mathbf{0}}) + f$$

where f stands for a k -linear combination of elements $A'(\bar{\mathbf{j}})$ with $A' \in \Gamma^{\pm}$ and $A' \prec A$.

Proof. The statement (1) follows directly from the result [2, 5.5] for q -Schur algebras together with the formulas in 4.1. We now prove (2). By [2, 5.5(c)], we have

$$\prod_{1 \leq i \leq h < j \leq n} (a_{i,j} E_{h,h+1})(\bar{\mathbf{0}}) \cdot \prod_{1 \leq j \leq h < i \leq n} (a_{i,j} E_{h+1,h})(\bar{\mathbf{0}}) = A(\bar{\mathbf{0}}) + f$$

where f is a formal k -linear combination of elements $[A']$ with $A' \in \tilde{\Xi}$ such that $A' \prec A$. Since $A \in \Gamma^{\pm}$, $[a_{ij}]_{\varepsilon}! \neq 0$ for any $i \neq j$, we have by 4.2 $(a_{i,j} E_{h,h+1})(\bar{\mathbf{0}}) = E_{h,h+1}(\bar{\mathbf{0}})^{a_{ij}} / [a_{ij}]_{\varepsilon}!$. So we can use 4.1 to obtain that f is also a (finite) k -linear combination of $B(\bar{\mathbf{j}})$ with $B \in \Gamma^{\pm}$ and $B \prec A$. Hence (2) follows. \square

As in [2, p.668], let for $A \in \Gamma^{\pm}$,

$$\|A\| = \sum_{r < s} \frac{(s-r)(s-r+1)}{2} a_{rs} + \sum_{r > s} \frac{(r-s)(r-s+1)}{2} a_{rs} \in \mathbb{N}.$$

Theorem 4.4. (1) \mathcal{W} is a subalgebra of $\widehat{\mathbf{K}}_k$.

(2) The elements $E_{h,h+1}(\bar{\mathbf{0}})$, $E_{h+1,h}(\bar{\mathbf{0}})$, $0(\bar{e}_i)$ (for $h \in [1, n]$ and $i \in [1, n]$) generate \mathcal{W} as an algebra.

Proof. The proof is almost the same as the proof of [2, 5.5]. Let \mathcal{W}_1 be the subalgebra of $\widehat{\mathbf{K}}_k$ generated by the elements indicated in (2). From 4.1, we see that \mathcal{W} is stable under left multiplication by elements of the form $E_{h,h+1}(\bar{\mathbf{0}})$, $E_{h+1,h}(\bar{\mathbf{0}})$, $0(\bar{e}_i)$; it follows that $\mathcal{W}_1 \subseteq \mathcal{W}$. So it is enough to prove $A(\bar{\mathbf{j}}) \in \mathcal{W}_1$ for any $\bar{\mathbf{j}} \in (\mathbb{Z}_l)^n$ and $A \in \Gamma^{\pm}$. We shall prove this by induction on $\|A\|$. When $\|A\| = 0$, then $A = 0$ and $0(\bar{\mathbf{j}}) = \prod_{i=1}^n 0(e_i)^{j_i} \in \mathcal{W}_1$ by the definition. Hence we may assume that $\|A\| > 0$ and our statement is already known for A' with $\|A'\| < \|A\|$. By (4.3.2), if $A = (a_{ij}) \in \Gamma^{\pm}$, we have

$$\prod_{1 \leq i \leq h < j \leq n} (a_{i,j} E_{h,h+1})(\bar{\mathbf{0}}) \cdot \prod_{1 \leq j \leq h < i \leq n} (a_{i,j} E_{h+1,h})(\bar{\mathbf{0}}) = A(\bar{\mathbf{0}}) + f$$

where f stands for a k -linear combination of elements $A'(\bar{\mathbf{j}})$ with $A' \in \Gamma^{\pm}$ such that $A' \prec A$ (hence $\|A'\| < \|A\|$). By induction, we see that $f \in \mathcal{W}_1$. Since the product on the left hand side is also in \mathcal{W}_1 by 4.2, we have $A(\bar{\mathbf{0}}) \in \mathcal{W}_1$.

For general $\bar{\mathbf{j}} \in (\mathbb{Z}_l)^n$, we have $A(\bar{\mathbf{j}}) = \varepsilon^a 0(\bar{\mathbf{j}})A(\bar{\mathbf{0}})$ for some $a \in \mathbb{Z}$ by (4.1.1). Hence $A(\bar{\mathbf{j}}) \in \mathcal{W}_1$. Thus $\mathcal{W}_1 = \mathcal{W}$. \square

5. REALIZATIONS OF u_k

Recall that Γ is the set of all $A = (a_{i,j}) \in \tilde{\Xi}$ such that $a_{i,j} < l$ for all $i \neq j$. Following [2, §6], we will denote by \mathcal{K} the k -subspace of \mathbf{K}_k spanned by the elements $[A]$ with $A \in \Gamma$. Clearly, \mathcal{K} is infinite dimensional.

Lemma 5.1. ([2, 6.2]) (1) \mathcal{K} is a subalgebra of \mathbf{K}_k .

(2) Let D be any diagonal matrix in $\tilde{\Xi}$. The map $\tau_D : \mathcal{K} \rightarrow \mathcal{K}$ given by $[A] \rightarrow [A + lD]$ is an algebra homomorphism.

Let Γ' be the set of all $n \times n$ matrices $A = (a_{i,j})$ with $a_{i,j} \in \mathbb{N}$, $a_{i,j} < l$ for all $i \neq j$ and $a_{i,i} \in \mathbb{Z}_l$ for all i . We have an obvious map $pr : \Gamma \rightarrow \Gamma'$ defined by reducing the diagonal entries modulo l .

Let \mathcal{K}' be the free k -module with basis elements $[A]$ in bijection with the elements $A \in \Gamma'$. Following [2, 6.3], there is an algebra structure on \mathcal{K}' given, for $A, A' \in \Gamma'$, by

$$[A] \cdot [A'] = \begin{cases} 0, & \text{if } co(A) \neq ro(A') \text{ in } \mathbb{Z}_l, \\ \sum \rho_{\tilde{A}''} [pr(\tilde{A}'')], & \text{otherwise,} \end{cases}$$

where $\rho_{\tilde{A}''}$ and \tilde{A}'' are determined by a product in \mathcal{K} : $[\tilde{A}] \cdot [\tilde{A}'] = \sum \rho_{\tilde{A}''} [\tilde{A}'']$ for any $\tilde{A}, \tilde{A}' \in \Gamma$ satisfying $co(\tilde{A}) = ro(\tilde{A}')$ (in \mathbb{Z}), $pr(\tilde{A}) = A$ and $pr(\tilde{A}') = A'$. Unlike \mathbf{K} or \mathcal{K} , the algebra \mathcal{K}' has unit element: the sum all $[A]$ with $A \in \Gamma'$ diagonal.

It was remarked at the end of [2] that \mathcal{K}' is “essentially” the algebra defined in [12, §5]. The rest of the section is devoted to prove this remark. More precisely, we shall prove the k -algebras \mathcal{K}' and u_k are isomorphic. We shall achieve this by proving that u_k is isomorphic to the algebra \mathcal{W} introduced in the previous section and that \mathcal{W} is isomorphic to \mathcal{K}' .

Recall that the map $\bar{\cdot} : \mathbb{Z}^n \rightarrow (\mathbb{Z}_l)^n$ is defined by $\overline{(j_1, j_2, \dots, j_n)} = (\bar{j}_1, \bar{j}_2, \dots, \bar{j}_n)$. Given $A \in \Gamma^\pm$ and $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n$, we rewrite

$$\begin{aligned} A(\bar{\mathbf{j}}) &:= \sum_{\mathbf{z} \in \mathbb{Z}^n} \varepsilon^{\mathbf{j} \cdot \mathbf{z}} [A + \text{diag}(\mathbf{z})] \in \widehat{\mathbf{K}}_k \\ (5.1.1) \quad &= \sum_{\bar{\mathbf{z}} \in (\mathbb{Z}_l)^n} \varepsilon^{\mathbf{j} \cdot \bar{\mathbf{z}}} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ \bar{\mathbf{x}} = \bar{\mathbf{z}}} [A + \text{diag}(\mathbf{x})] \\ &= \sum_{\bar{\mathbf{z}} \in (\mathbb{Z}_l)^n} \varepsilon^{\mathbf{j} \cdot \bar{\mathbf{z}}} [A + \text{diag}(\bar{\mathbf{z}})] \end{aligned}$$

where $\mathbf{j} \cdot \mathbf{z} = \sum_{i=1}^n j_i z_i$, $\varepsilon^{\mathbf{j} \cdot \mathbf{z}} = \varepsilon^{\mathbf{j} \cdot \mathbf{z}}$ and

$$[[A + \text{diag}(\bar{\mathbf{z}})]] = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ \bar{\mathbf{x}} = \bar{\mathbf{z}}} } [A + \text{diag}(\mathbf{x})] \in \widehat{\mathbf{K}}_k.$$

Note that $A + \text{diag}(\mathbf{x}) \in \Gamma'$ and the elements $[[A]]$, $A \in \Gamma'$, are linearly independent.³

Lemma 5.2. (BLM bases) *Each of the following sets forms a k -basis for \mathcal{W} :*

- (1) $\mathfrak{B}_k = \{A(\bar{\mathbf{j}}) \mid A \in \Gamma^\pm, \bar{\mathbf{j}} \in (\mathbb{Z}_l)^n\}$;
- (2) $\mathfrak{B}'_k = \{[[A]] \mid A \in \Gamma'\}$.

In particular, $\dim \mathcal{W} = l^{n^2}$.

Proof. Fix $A \in \Gamma^\pm$. By definition, for $\bar{\mathbf{j}} \in (\mathbb{Z}_l)^n$,

$$(5.2.1) \quad A(\bar{\mathbf{j}}) = \sum_{\bar{\mathbf{j}}' \in (\mathbb{Z}_l)^n} \varepsilon^{\mathbf{j} \cdot \bar{\mathbf{j}}'} [[A + \bar{\mathbf{j}}']].$$

Note that the coefficient matrix of (5.2.1) is $B_n = (\varepsilon^{\mathbf{j} \cdot \bar{\mathbf{j}}'})_{\bar{\mathbf{j}}, \bar{\mathbf{j}}' \in (\mathbb{Z}_l)^n}$. If we order the index set $(\mathbb{Z}_l)^n \times (\mathbb{Z}_l)^n$ lexicographically, the matrix B_n with $B_0 = 1$ is of the form

$$B_n = \begin{pmatrix} B_{n-1} & B_{n-1} & \cdots & B_{n-1} \\ B_{n-1} & \varepsilon B_{n-1} & \cdots & \varepsilon^{l-1} B_{n-1} \\ \dots & \dots & \dots & \dots \\ B_{n-1} & \varepsilon^{l-1} B_{n-1} & \cdots & (\varepsilon^{l-1})^{l-1} B_{n-1} \end{pmatrix}.$$

Clearly, as a generalization of a Vandermonde determinant, we have

$$\det(B_n) = \det(B_{n-1})^l \prod_{0 \leq j < i \leq l-1} (\varepsilon^i - \varepsilon^j)^s$$

with $s = l^{n-1}$. Since ε is a l -primitive root of unity, we have $\det(B_n) \neq 0$ by induction. Thus, the linear independence of \mathfrak{B}_k follows from the independence of \mathfrak{B}'_k . Hence, \mathfrak{B}_k is a basis, proving (1). Now, the invertibility of the matrix B_n also implies that \mathfrak{B}'_k is contained in \mathcal{W} . So (2) follows from (1) since both sets have the same cardinality. \square

There is a third basis for \mathcal{W} , a monomial basis (cf. 3.2). For any $A = (a_{ij}) \in \Gamma^\pm, \bar{\mathbf{j}} \in (\mathbb{Z}_l)^n$, let

$$M^{(A, \bar{\mathbf{j}})} = \prod_{1 \leq i \leq h < j \leq n} (a_{i,j} E_{h,h+1})(\bar{\mathbf{0}}) \cdot 0(\bar{\mathbf{j}}) \cdot \prod_{1 \leq j \leq h < i \leq n} (a_{i,j} E_{h+1,h})(\bar{\mathbf{0}}),$$

where the ordering of the products is the same as in (3.1.1).

Corollary 5.3. *The set $\{M^{(A, \bar{\mathbf{j}})} \mid A \in \Gamma^\pm, \bar{\mathbf{j}} \in (\mathbb{Z}_l)^n\}$ forms a k -basis for \mathcal{W} .*

³Since \mathcal{W} is naturally a subspace of the direct product $\prod_{A \in \Gamma} k[A]$, the linear independence of the elements $[[A]]$ follows easily from the identity $\sum_{A \in \Gamma'} \rho_A [[A]] = \sum_{\substack{A \in \Gamma^\pm \\ \bar{\mathbf{x}} \in \mathbb{Z}^n}} \rho_{A + \text{diag}(\bar{\mathbf{x}})} [A + \text{diag}(\mathbf{x})]$.

Proof. By (4.1.1) and (4.3.2), we see that

$$(5.3.1) \quad M^{(A, \bar{\mathbf{j}})} = \varepsilon^a A(\bar{\mathbf{j}}) + \text{lower terms}$$

and the assertion follows from 5.2(1). \square

Lemma 5.4. *With E_i, F_i, K_j being replaced by $E_{i, i+1}(\bar{\mathbf{0}})$, $E_{i+1, i}(\bar{\mathbf{0}})$, $0(\bar{e}_j)$, the relations (a)-(h) given in 2.5 continue to hold in $\widehat{\mathbf{K}}_k$.*

Proof. The relation 2.5(a) is obvious and 2.5(b) and (c) follow from (4.1.1). The other relations can also be proved by 4.1 in a way similar to the proof of [2, 5.6]. \square

We now are ready to prove the main result of this section.

Theorem 5.5. (1) *There is an algebra isomorphism $\varphi : u_k \xrightarrow{\sim} \mathcal{W}$ satisfying $E_h \mapsto E_{h, h+1}(\bar{\mathbf{0}})$, $F_h \mapsto E_{h+1, h}(\bar{\mathbf{0}})$, $K_j \mapsto 0(\bar{e}_j)$.*

(2) *There is an algebra isomorphism $\psi : \mathcal{W} \xrightarrow{\sim} \mathcal{K}'$ satisfying $\llbracket A \rrbracket \mapsto [A]$ for $A \in \Gamma'$.*

Proof. (1) By 2.5, 5.4 and 4.4, there is a surjective algebra homomorphism φ from u_k to \mathcal{W} sending E_h (resp., F_h and K_j) to $E_{h, h+1}(\bar{\mathbf{0}})$ (resp., $E_{h+1, h}(\bar{\mathbf{0}})$ and $0(\bar{e}_j)$). By 2.3 and 5.2, we have $\dim u_k = l^{n^2} = \dim \mathcal{W}$. It follows that φ is an isomorphism.

(2) By 5.2, ψ is a linear isomorphism. So it is enough to prove ψ is an algebra homomorphism. Let $\bar{\mathbf{j}}, \bar{\mathbf{j}}' \in (\mathbb{Z}_l)^n$, $A, A' \in \Gamma^\pm$. Fix $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}^n$ satisfying $co(A + \text{diag}(\mathbf{x})) = ro(A' + \text{diag}(\mathbf{x}'))$ and $\bar{\mathbf{x}} = \bar{\mathbf{j}}, \bar{\mathbf{x}}' = \bar{\mathbf{j}}'$. By 5.1(1), we may assume in \mathcal{K}

$$[A + \text{diag}(\mathbf{x})] \cdot [A' + \text{diag}(\mathbf{x}')] = \sum_{C \in \Gamma^\pm, \mathbf{z} \in \mathbb{Z}^n} \rho_{C, \mathbf{z}} [C + \text{diag}(\mathbf{z})] \quad (\rho_{C, \mathbf{z}} \in k).$$

By the definition of the product in \mathcal{K}' , we have in \mathcal{K}'

$$\llbracket A + \text{diag}(\bar{\mathbf{j}}) \rrbracket \cdot \llbracket A' + \text{diag}(\bar{\mathbf{j}}') \rrbracket = \sum_{C \in \Gamma^\pm, \mathbf{z} \in \mathbb{Z}^n} \rho_{C, \mathbf{z}} [C + \text{diag}(\bar{\mathbf{z}})].$$

On the other hand, we have in $\widehat{\mathbf{K}}_k$

$$\llbracket A + \text{diag}(\bar{\mathbf{j}}) \rrbracket \cdot \llbracket A' + \text{diag}(\bar{\mathbf{j}}') \rrbracket = \sum_{\substack{\mathbf{y} \in \mathbb{Z}^n, \bar{\mathbf{y}} = \bar{\mathbf{j}} \\ \mathbf{y}' \in \mathbb{Z}^n, \bar{\mathbf{y}}' = \bar{\mathbf{j}}'}} [A + \text{diag}(\mathbf{y})] \cdot [A + \text{diag}(\mathbf{y}')].$$

If $[A + \text{diag}(\mathbf{y})] \cdot [A + \text{diag}(\mathbf{y}')] \neq 0$, then $co(A + \text{diag}(\mathbf{y})) = ro(A' + \text{diag}(\mathbf{y}'))$. Since the fixed pair $(\mathbf{x}, \mathbf{x}')$ above satisfies the same conditions, it follows easily that there exists $\lambda \in \mathbb{Z}^n$ such that $\mathbf{y} = \mathbf{x} + l\lambda$ and $\mathbf{y}' = \mathbf{x}' + l\lambda$. Moreover, the map $(\mathbf{y}, \mathbf{y}') \mapsto \lambda$ is a bijection from the set of

all such pairs to the set \mathbb{Z}^n . Thus, we obtain

$$\begin{aligned}
\llbracket A + \text{diag}(\bar{\mathbf{j}}) \rrbracket \cdot \llbracket A' + \text{diag}(\bar{\mathbf{j}}') \rrbracket &= \sum_{\lambda \in \mathbb{Z}^n} [A + \text{diag}(\mathbf{x}) + l\text{diag}(\lambda)] \cdot [A' + \text{diag}(\mathbf{x}') + l\text{diag}(\lambda)] \\
&= \sum_{\lambda \in \mathbb{Z}^n} \sum_{C \in \Gamma^\pm, \mathbf{z} \in \mathbb{Z}^n} \rho_{C, \mathbf{z}} [C + \text{diag}(\mathbf{z}) + l\text{diag}(\lambda)] \quad (\text{by 5.1(2)}) \\
&= \sum_{C \in \Gamma^\pm, \mathbf{z} \in \mathbb{Z}^n} \rho_{C, \mathbf{z}} \sum_{\lambda \in \mathbb{Z}^n} [C + \text{diag}(\mathbf{z}) + l\text{diag}(\lambda)] \\
&= \sum_{C \in \Gamma^\pm, \mathbf{z} \in \mathbb{Z}^n} \rho_{C, \mathbf{z}} \llbracket C + \text{diag}(\bar{\mathbf{z}}) \rrbracket.
\end{aligned}$$

Therefore,

$$\psi(\llbracket A + \text{diag}(\bar{\mathbf{j}}) \rrbracket) \cdot \psi(\llbracket A' + \text{diag}(\bar{\mathbf{j}}') \rrbracket) = \psi(\llbracket A + \text{diag}(\bar{\mathbf{j}}) \rrbracket) \cdot \psi(\llbracket A' + \text{diag}(\bar{\mathbf{j}}') \rrbracket),$$

and hence, it is an isomorphism of algebras. \square

With this theorem, we shall identify u_k with \mathcal{W} and \mathcal{K}' in the remaining sections. In particular, the identity $1 = \sum_{\lambda \in \mathbb{Z}_1^n} \llbracket \text{diag}(\lambda) \rrbracket$ in u_k .

6. MONOMIAL BASES FOR q -SCHUR ALGEBRAS: A REVISIT

In the remaining sections, we will investigate little q -Schur algebras arising from the quantum group u_k . For later use, we first collect and improve in this section some results from [9] and then construct some bases conjectured in [6, 3.2] for q -Schur algebras $\mathbf{U}(n, r)$ over $\mathbb{Q}(v)$.

Recall the map ζ_r defined in 3.1(3). Let $\mathbf{e}_i = \zeta_r(E_i)$, $\mathbf{f}_i = \zeta_r(F_i)$, $\mathbf{k}_j = \zeta_r(K_j)$ for $1 \leq i \leq n-1$, $1 \leq j \leq n$. Let X be an indeterminate which is independent of v . For $t \in \mathbb{N}$, put

$$[X; t]! = (X-1)(X-v)\cdots(X-v^{t-1})$$

and $[X; 0]! = 1$. The following theorem is given in [9] (cf. [6]).

Theorem 6.1. *The q -Schur algebra $\mathbf{U}(n, r)$ over $\mathbb{Q}(v)$ is generated by the elements*

$$\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_i \quad (1 \leq i \leq n-1)$$

subject to the relations:

- (a) $\mathbf{k}_i \mathbf{k}_j = \mathbf{k}_j \mathbf{k}_i$;
- (b) $[\mathbf{k}_1; t_1]! [\mathbf{k}_2; t_2]! \cdots [\mathbf{k}_{n-1}; t_{n-1}]! = 0$ for $t_i \in \mathbb{N}$ such that $t_1 + \cdots + t_{n-1} = r + 1$;
- (c) $\mathbf{k}_i \mathbf{e}_j = v^{\epsilon(i,j)} \mathbf{e}_j \mathbf{k}_i$, $\mathbf{k}_j \mathbf{f}_i = v^{-\epsilon(i,j)} \mathbf{f}_i \mathbf{k}_j$ with $\epsilon(i, j)$ as in 2.1(b);
- (d) $\mathbf{e}_i \mathbf{e}_j = \mathbf{e}_j \mathbf{e}_i$, $\mathbf{f}_i \mathbf{f}_j = \mathbf{f}_j \mathbf{f}_i$ when $|i - j| > 1$;
- (e) $\mathbf{e}_i^2 \mathbf{e}_j - (v + v^{-1}) \mathbf{e}_i \mathbf{e}_j \mathbf{e}_i + \mathbf{e}_j \mathbf{e}_i^2 = 0$ when $|i - j| = 1$;
- (f) $\mathbf{f}_i^2 \mathbf{f}_j - (v + v^{-1}) \mathbf{f}_i \mathbf{f}_j \mathbf{f}_i + \mathbf{f}_j \mathbf{f}_i^2 = 0$ when $|i - j| = 1$;
- (g) $\mathbf{e}_i \mathbf{f}_j - \mathbf{f}_j \mathbf{e}_i = \delta_{ij} \frac{\tilde{\mathbf{k}}_i - \tilde{\mathbf{k}}_i^{-1}}{v - v^{-1}}$, where $\tilde{\mathbf{k}}_i = \mathbf{k}_i \mathbf{k}_{i+1}^{-1}$, with $\mathbf{k}_n = v^r \mathbf{k}_1^{-1} \cdots \mathbf{k}_{n-1}^{-1}$.

Let, for $r \geq 1$,

$$\Lambda(n, r) = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 + \dots + \lambda_n = r\},$$

the set of all compositions of r into n parts. Let $U_{\mathcal{A}}^+(n, r)$ (resp., $U_{\mathcal{A}}^-(n, r)$, $U_{\mathcal{A}}^0(n, r)$) be the \mathcal{A} -subalgebras of $U_{\mathcal{A}}(n, r)$ generated by the $\mathbf{e}_i^{(m)}$ (resp., $\mathbf{f}_i^{(m)}$, \mathbf{k}_λ), where $\mathbf{k}_\lambda = \prod_{i=1}^n \begin{bmatrix} \mathbf{k}_i; 0 \\ \lambda_i \end{bmatrix}$. For $A \in \Xi = M_n(\mathbb{N})$, we further introduce monomials in $U_{\mathcal{A}}^+(n, r)$: $\mathbf{e}^{(A^+)} = \zeta_r(E^{(A^+)})$ and $\mathbf{f}^{(A^-)} = \zeta_r(F^{(A^-)})$ (cf. (3.1.1)).

For our convenience, we include the following useful results which are taken from [6] and [9].

Lemma 6.2. (1) *The set $\{\mathbf{k}_\lambda \mid \lambda \in \Lambda(n, r)\}$ is a complete set of orthogonal primitive idempotents (hence a basis) for $U_{\mathcal{A}}^0(n, r)$. In particular, $1 = \sum_{\lambda \in \Lambda(n, r)} \mathbf{k}_\lambda$.*

(2) *Let $\lambda \in \Lambda(n, r)$. Then $\mathbf{k}_i \mathbf{k}_\lambda = v^{\lambda_i} \mathbf{k}_\lambda$ for $1 \leq i \leq n$.*

(3) *Let $\lambda \in \Lambda(n, r)$. If $A \in \Xi^+$ (resp., $A \in \Xi^-$) and $\lambda_i < \sigma_i(A)$ for some i , then $\mathbf{e}^{(A)} \mathbf{k}_\lambda = 0$ (resp., $\mathbf{k}_\lambda \mathbf{f}^{(A)} = 0$).*

(4) *For a fixed i_0 , the elements $\mathbf{k}_1^{j_1} \cdots \mathbf{k}_n^{j_n}$ ($j_i \in \mathbb{N}, j_{i_0} = 0, j_1 + \dots + j_n \leq r$) form a basis for $\mathbf{U}(n, r)^0$.*

Following [9], for any $A \in \Xi^\pm$ and $\lambda \in \Lambda(n, r)$, let

$$\mathbf{m}^{(A, \lambda)} = \mathbf{e}^{(A^+)} \mathbf{k}_\lambda \mathbf{f}^{(A^-)},$$

and, if $\sigma(A) = \sum_{i,j} a_{ij} = r$, let

$$\mathbf{m}^{(A)} = \mathbf{e}^{(A^+)} \mathbf{k}_\lambda \mathbf{f}^{(A^-)},$$

where $\lambda = \lambda(A) = (\sigma_1(A), \dots, \sigma_n(A))$ with $\sigma_i(A) = a_{ii} + \sum_{1 \leq j < i} (a_{ij} + a_{ji})$.

The following result is proved in [9, 5.6] with the condition that all coefficients $f_{B,A} \in \mathbb{Q}(v)$. Since the basis is integral and $\mathbf{m}^{(A, \lambda)} \in U_{\mathcal{A}}(n, r)$, it is in fact an relation over \mathcal{A} . Recall the partial ordering \preceq defined above 4.3.

Lemma 6.3. *Suppose $\mathbf{m}^{(A, \lambda)} \neq 0$ for some $A \in \Xi^\pm$ and $\lambda \in \Lambda(n, r)$. If there exists $D \in \Xi^0$ such that $co(A + D) = \lambda + co(A^-) - ro(A^-)$, then $\mathbf{m}^{(A, \lambda)} = \mathbf{m}^{(A+D)}$. Otherwise*

$$\mathbf{m}^{(A, \lambda)} = \sum_{B \in \Xi_r, B \prec A} f_{B,A} \mathbf{m}^{(B)} \quad (f_{B,A} \in \mathcal{A}).$$

The following integral monomial basis for the q -Schur algebra $U_{\mathcal{A}}(n, r)$ is given in [9, 6.4].

Theorem 6.4. *The set*

$$\mathcal{M} = \{\mathbf{m}^{(A)} \mid A \in \Xi_r\} = \{\mathbf{e}^{(A^+)} \mathbf{k}_\lambda \mathbf{f}^{(A^-)} \mid A \in \Xi^\pm, \lambda \in \Lambda(n, r), \lambda_i \geq \sigma_i(A) \text{ for } 1 \leq i \leq n\}$$

forms an \mathcal{A} -basis for $U_{\mathcal{A}}(n, r)$.

Corollary 6.5. *Fix any i_0 with $1 \leq i_0 \leq n$ and, for $\lambda \in \Lambda(n, r)$, let*

$$\mathbf{k}_{\lambda, i_0} = \begin{bmatrix} \mathbf{k}_1; 0 \\ \lambda_1 \end{bmatrix} \cdots \begin{bmatrix} \mathbf{k}_{i_0-1}; 0 \\ \lambda_{i_0-1} \end{bmatrix} \begin{bmatrix} \mathbf{k}_{i_0+1}; 0 \\ \lambda_{i_0+1} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{k}_n; 0 \\ \lambda_n \end{bmatrix}.$$

Then the set

$$\{\mathbf{e}^{(A^+)} \mathbf{k}_{\lambda, i_0} \mathbf{f}^{(A^-)} \mid A \in \Xi^\pm, \lambda \in \Lambda(n, r), \lambda_i \geq \sigma_i(A) \forall i\}$$

forms an \mathcal{A} -basis for $U_{\mathcal{A}}(n, r)$.

Proof. By 6.2(1,2), we have

$$\mathbf{k}_{\lambda, i_0} = \mathbf{k}_\lambda + \sum_{\substack{\mu \in \Lambda(n, r) \\ \mu \neq \lambda}} \prod_{i \neq i_0} \begin{bmatrix} \mu_i \\ \lambda_i \end{bmatrix} \mathbf{k}_\mu.$$

The product $\prod_{i \neq i_0} \begin{bmatrix} \mu_i \\ \lambda_i \end{bmatrix}$ is zero unless $\mu_i \geq \lambda_i$ for all $i \neq i_0$. Assume $A \in \Xi^\pm$, $\lambda \in \Lambda(n, r)$, $\lambda_i \geq \sigma_i(A)$ for all i . Then, by 6.3,

$$\mathbf{e}^{(A^+)} \mathbf{k}_{\lambda, i_0} \mathbf{f}^{(A^-)} = \mathbf{e}^{(A^+)} \mathbf{k}_\lambda \mathbf{f}^{(A^-)} + \sum_{\substack{B \in \Xi_r \\ B \prec A}} g_B \mathbf{m}^{(B)} (g_B \in \mathcal{A}).$$

Now, by 6.4, the result follows. \square

Strictly speaking, if we consider these bases as bases for $\mathbf{U}(n, r)$ over $\mathbb{Q}(v)$, the middle part of each basis element above is *not* a monomial in the generators \mathbf{k}_i . So it is not clear from [9] how to get true monomial $\mathbb{Q}(v)$ -bases in the generators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_j$. We are going to answer this question below.

We are now in a position to construct new monomial bases for $\mathbf{U}(n, r)$. For $\mathbf{j} \in \mathbb{N}^n$, write $\mathbf{k}^{\mathbf{j}} = \mathbf{k}_1^{j_1} \cdots \mathbf{k}_n^{j_n}$ and recall $\sigma(\mathbf{j}) = j_1 + \cdots + j_n$.

Theorem 6.6. (The monomial and BLM bases for $\mathbf{U}(n, r)$)

Let i_0 be a fixed integer with $1 \leq i_0 \leq n$. Each of the following sets forms a $\mathbb{Q}(v)$ -basis for $\mathbf{U}(n, r)$:

- (1) $\mathcal{N}_{i_0} = \left\{ \mathbf{e}^{(A^+)} \mathbf{k}^{\mathbf{j}} \mathbf{f}^{(A^-)} \mid A \in \Xi^\pm, \mathbf{j} \in \mathbb{N}^n, j_{i_0} = 0, \sigma(\mathbf{j}) + \sigma(A) \leq r \right\};$
- (2) $\mathcal{B}_{i_0} = \left\{ A(\mathbf{j}, r) \mid A \in \Xi^\pm, \mathbf{j} \in \mathbb{N}^n, j_{i_0} = 0, \sigma(\mathbf{j}) + \sigma(A) \leq r \right\}.$

Proof. For notational simplicity, we prove the case $i_0 = n$; the general case can then be seen easily.

(1) Let

$$\mathcal{N} := \mathcal{N}_n = \left\{ \mathbf{n}^{(A, \mathbf{j})} := \mathbf{e}^{(A^+)} \mathbf{k}_1^{j_1} \mathbf{k}_2^{j_2} \cdots \mathbf{k}_{n-1}^{j_{n-1}} \mathbf{f}^{(A^-)} \mid A \in \Xi^\pm, j_i \in \mathbb{N} \forall i, \sum_{i=1}^{n-1} j_i + \sigma(A) \leq r \right\}$$

There is an obvious bijective map

$$\begin{aligned} & \{(j_1, \dots, j_{n-1}, A) \mid j_1 + \dots + j_{n-1} + \sigma(A) \leq r, A \in \Xi^\pm\} \longrightarrow \Xi_r \\ & (j_1, \dots, j_{n-1}, A) \longmapsto A + \text{diag}(j_1, \dots, j_{n-1}, r - \sum_{i=1}^{n-1} j_i + \sigma(A)). \end{aligned}$$

Hence it is enough to prove that the set \mathcal{N} is linearly independent since $\dim \mathbf{U}(n, r) = \#\Xi_r$. By 6.2(1-2), for $A \in \Xi^\pm$, we have, for any j_1, \dots, j_n ,

$$\mathbf{e}^{(A^+)} \mathbf{k}_1^{j_1} \dots \mathbf{k}_n^{j_n} \mathbf{f}^{(A^-)} = \sum_{\substack{\lambda \in \Lambda(n, r) \\ \forall j, \lambda_j \geq \sigma_j(A)}} v^{\sum_{i=1}^n \lambda_i j_i} \mathbf{e}^{(A^+)} \mathbf{k}_\lambda \mathbf{f}^{(A^-)} + \sum_{\substack{\mu \in \Lambda(n, r) \\ \exists j, \mu_j < \sigma_j(A)}} v^{\sum_{i=1}^n \mu_i j_i} \mathbf{e}^{(A^+)} \mathbf{k}_\mu \mathbf{f}^{(A^-)}.$$

Hence by 6.3, we have

$$(6.6.1) \quad \mathbf{e}^{(A^+)} \mathbf{k}_1^{j_1} \dots \mathbf{k}_n^{j_n} \mathbf{f}^{(A^-)} = \sum_{\substack{\lambda \in \Lambda(n, r) \\ \lambda_i \geq \sigma_i(A)}} v^{\sum_{i=1}^n \lambda_i j_i} \mathbf{e}^{(A^+)} \mathbf{k}_\lambda \mathbf{f}^{(A^-)} + \sum_{\substack{B \in \Xi_r \\ B < A}} f_{B, A} \mathbf{m}^{(B)}$$

where $f_{B, A} \in \mathbb{Q}(v)$. Note that the monomials appearing in the first sum are part of basis elements in 6.4.

For fixed $A \in \Xi^\pm$, let

$$\mathcal{N}_A = \{\mathbf{n}^{(A, \mathbf{j})} \mid j_1 + \dots + j_{n-1} + \sigma(A) \leq r, j_i \in \mathbb{N}, \forall i\}.$$

Then $\mathcal{N} = \cup_{A \in \Xi^\pm; \sigma(A) \leq r} \mathcal{N}_A$. Consider the sets $\Lambda_{n, r, A} = \{\lambda \mid \lambda \in \Lambda(n, r), \lambda_i \geq \sigma_i(A) \text{ for all } i\}$ and

$$\Lambda'_{n, r, A} = \{(j_1, \dots, j_{n-1}) \in \mathbb{N}^{n-1} \mid j_1 + \dots + j_{n-1} + \sigma(A) \leq r\}.$$

There is an obvious bijective map

$$\begin{aligned} & \Lambda'_{n, r, A} \longrightarrow \Lambda_{n, r, A} \\ & (j_1, \dots, j_{n-1}) \longmapsto (j_1 + \sigma_1(A), \dots, j_{n-1} + \sigma_{n-1}(A), r - \sum_{i=1}^{n-1} (j_i + \sigma_i(A))). \end{aligned}$$

Hence, from 6.4 and the calculation above, it is enough to prove that \mathcal{N}_A is linearly independent.

This is equivalent to show that

$$(6.6.2) \quad \det \left(v^{\sum_{i=1}^{n-1} \lambda_i j_i} \right)_{\lambda \in \Lambda_{n, r, A}, \mathbf{j} \in \Lambda'_{n, r, A}} \neq 0.$$

We apply induction on $\sigma(A)$. If $\sigma(A) = 0$, then the matrix is the transition matrix from the basis 6.2(4) to 6.2(1). So (6.6.2) is clearly true. Assume now $\sigma(A) > 0$ and the result is true for all B with $\sigma(B) = \sigma(A) - 1$. Now $\sigma(A) = \sum_{i \neq j} a_{ij} > 0$ implies that there exists some $a_{ij} > 0$ for $i \neq j$. Without loss of generality, we can assume $a_{12} > 0$. Let $B = A - E_{12}$, then $B \in \Xi^\pm$, $\sigma_2(B) = \sigma_2(A) - 1$ and $\sigma_i(B) = \sigma_i(A)$ for $i \neq 2$. Since there is an obvious bijective map:

$\Lambda_{n,r,A} \rightarrow \Lambda_{n,r-1,B}$ by sending $(\lambda_1, \dots, \lambda_n)$ to $(\lambda_1, \lambda_2 - 1, \lambda_3, \dots, \lambda_n)$ and $\Lambda'_{n,r,A} = \Lambda'_{n,r-1,B}$, we have

$$\left(v^{-j_2} v^{\sum_{i=1}^{n-1} \lambda_i j_i} \right)_{\lambda \in \Lambda_{n,r,A}, \mathbf{j} \in \Lambda'_{n,r,A}} = \left(v^{\sum_{i=1}^{n-1} \mu_i j_i} \right)_{\mu \in \Lambda_{n,r-1,B}, \mathbf{j} \in \Lambda'_{n,r-1,B}}.$$

Therefore,

$$\det \left(v^{\sum_{i=1}^{n-1} \lambda_i j_i} \right)_{\lambda \in \Lambda_{n,r,A}, \mathbf{j} \in \Lambda'_{n,r,A}} = v^s \det \left(v^{\sum_{i=1}^{n-1} \mu_i j_i} \right)_{\mu \in \Lambda_{n,r-1,B}, \mathbf{j} \in \Lambda'_{n,r-1,B}}$$

for some integer s , and so the result follows from induction.

By setting $\Lambda'_{n,r,A} = \{(j_1, \dots, j_{i_0-1}, j_{i_0+1}, \dots, j_n) \in \mathbb{N}^{n-1} \mid \sum_{i \neq i_0} j_i + \sigma(A) \leq r\}$, the above proof works for a general i_0 .

(2) If $i_0 = n$, then by definition (3.0.2),

$$\begin{aligned} A(\mathbf{j}, r) &= \sum_{\lambda \in \Lambda(n, r - \sigma(A))} v^{\lambda_1 j_1 + \dots + \lambda_{n-1} j_{n-1}} [A + \text{diag}(\lambda)] \\ (6.6.3) \quad &= \sum_{\lambda \in \Lambda'_{n,r,A}} v^{\lambda \cdot \mathbf{j}} [A + \text{diag}(\lambda)]. \end{aligned}$$

Thus, for a fixed $A \in \Xi^\pm$, the coefficient matrix is $(v^{\lambda \cdot \mathbf{j}})_{\lambda, \mathbf{j} \in \Lambda'_{n,r,A}}$. From the argument above, we see that

$$(v^{\lambda \cdot \mathbf{j}})_{\lambda, \mathbf{j} \in \Lambda'_{n,r,A}} = \left(v^{-\sum_{i=1}^{n-1} \sigma_i(A) j_i} v^{\lambda \cdot \mathbf{j}} \right)_{\lambda \in \Lambda_{n,r,A}, \mathbf{j} \in \Lambda'_{n,r,A}}.$$

Thus,

$$\det(v^{\lambda \cdot \mathbf{j}})_{\lambda, \mathbf{j} \in \Lambda'_{n,r,A}} = v^s \det(v^{\lambda \cdot \mathbf{j}})_{\lambda \in \Lambda_{n,r,A}, \mathbf{j} \in \Lambda'_{n,r,A}}$$

for some integer s . So this matrix is invertible by (6.6.2). Therefore, \mathcal{B}_n is linearly independent, and consequently, it forms a basis. The proof for any i_0 is entirely similar. \square

For any $A = (a_{ij}) \in \Xi_r$, let $\mathbf{n}^{(A)} = \mathbf{e}^{(A^+)} \mathbf{k}_1^{(a_{11})} \dots \mathbf{k}_{n-1}^{(a_{n-1, n-1})} \mathbf{f}^{(A^-)}$. Then we have

$$\mathcal{N} = \{\mathbf{n}^{(A)} \mid A \in \Xi_r\}.$$

Corollary 6.7. *Let $A \in \Xi^\pm$, $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{N}^n$. If $j_1 + j_2 + \dots + j_n + \sigma(A) > r$, then*

$$(6.7.1) \quad \mathbf{e}^{(A^+)} \mathbf{k}_1^{j_1} \dots \mathbf{k}_n^{j_n} \mathbf{f}^{(A^-)} = \sum_{B \in \Xi_r, B \preceq A} f_{B,A} \mathbf{n}^{(B)}$$

where $f_{B,A} \in \mathbb{Q}(v)$. The same is true for the basis \mathcal{N}_{i_0} .

Proof. Applying (6.6.1) to $\mathbf{n}^{(A)}$, we see that each $\mathbf{n}^{(A)}$ is a linear combination of $\mathbf{m}^{(B)}$ with $B \in \Xi_r, B \preceq A$. Thus, each $\mathbf{m}^{(A)}$ is a linear combination of $\mathbf{n}^{(B)}$ with $B \in \Xi_r, B \preceq A$. This together with (6.6.1) proves our result. \square

With the new monomial bases given in 6.6, we may follow the argument in [9, §6] to produce new PBW type bases for q -Schur algebras.

Fix the reduced expression

$$\mathbf{i} = (i_1, i_2, \dots, i_\nu) = (n-1, \dots, 2, 1, \dots, n-1, n-2, n-1)$$

of the longest word w_0 of the symmetric group \mathfrak{S}_n , that is,

$$w_0 = s_{i_1} s_{i_2} \cdots s_{i_\nu} = (s_{n-1} s_{n-2} \cdots s_1)(s_{n-1} s_{n-2} \cdots s_2)(s_{n-1} s_{n-2}) s_{n-1},$$

where $s_i = (i, i+1)$ are basic transpositions. For any $\mathbf{c} = (c_1, \dots, c_\nu) \in \mathbb{N}^\nu$, define monomials in root vectors $E_{\mathbf{i}}^{\mathbf{c}}$ and $F_{\mathbf{i}}^{\mathbf{c}}$ as in [13, 2.2], and let $A_{\mathbf{c}}^+ = (a_{ij})$ where the first $n-1$ components of \mathbf{c} become the n -th column reading upwards, and the next $n-2$ components become the $(n-1)$ -th column and so on, i.e.,

$$c_1 = a_{n-1, n}, \dots, c_{n-1} = a_{1n}, c_n = a_{n-2, n-1}, \dots$$

Define $A_{\mathbf{c}}^-$ symmetrically. Then $\{E_{\mathbf{i}}^{\mathbf{c}}\}_{\mathbf{c} \in \mathbb{N}^\nu}$ resp. $\{F_{\mathbf{i}}^{\mathbf{c}}\}_{\mathbf{c} \in \mathbb{N}^\nu}$ is a \mathcal{A} -basis for $U_{\mathcal{A}}^+$ resp. $U_{\mathcal{A}}^-$. Moreover, the transition matrix between this basis and the monomial basis is unipotent upper triangular. More precisely, we have by [13, 7.8(b)] and [9, 6.2(2)].

Lemma 6.8. *Let $\mathbf{i} = (n-1, \dots, 2, 1, \dots, n-1, n-2, n-1)$ and let $\mathbf{c} \in \mathbb{N}^\nu$. For any $\mathbf{c}' \in \mathbb{N}^\nu$, there exists $h_{\mathbf{c}, \mathbf{c}'} \in \mathcal{A}$ such that*

$$E^{(A_{\mathbf{c}}^+)} = E_{\mathbf{i}}^{\mathbf{c}} + \sum_{A_{\mathbf{c}'}^+ \prec A_{\mathbf{c}}^+} h_{\mathbf{c}, \mathbf{c}'} E_{\mathbf{i}}^{\mathbf{c}'}, \text{ and } F^{(A_{\mathbf{c}}^-)} = F_{\mathbf{i}}^{\mathbf{c}} + \sum_{A_{\mathbf{c}'}^- \prec A_{\mathbf{c}}^-} h_{\mathbf{c}, \mathbf{c}'} F_{\mathbf{i}}^{\mathbf{c}'}$$

Let $\mathbf{e}_{\mathbf{i}}^{\mathbf{c}} = \zeta_r(E_{\mathbf{i}}^{\mathbf{c}})$ and $\mathbf{f}_{\mathbf{i}}^{\mathbf{c}} = \zeta_r(F_{\mathbf{i}}^{\mathbf{c}})$. For any $A \in \Xi^\pm$, let $\mathbf{c}(A^+) \in \mathbb{N}^\nu$ (resp., $\mathbf{c}(A^-) \in \mathbb{N}^\nu$) correspond to A^+ (resp., A^-) under the bijection $\mathbf{c} \mapsto A_{\mathbf{c}}^+$ (resp., $\mathbf{c} \mapsto A_{\mathbf{c}}^-$) above. We now derive a new PBW-basis for $\mathbf{U}(n, r)$.

Theorem 6.9. (The PBW bases for $\mathbf{U}(n, r)$)

For any integer i_0 with $1 \leq i_0 \leq n$, the set

$$\mathcal{P}_{i_0} = \{ \mathbf{e}_{\mathbf{i}}^{\mathbf{c}(A^+)} \mathbf{k}^{\mathbf{j}} \mathbf{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} \mid A \in \Xi^\pm, \mathbf{j} \in \mathbb{N}^n, j_{i_0} = 0, \sigma(\mathbf{j}) + \sigma(A) \leq r \}$$

forms a $\mathbb{Q}(v)$ -basis for $\mathbf{U}(n, r)$.

Proof. We again assume $i_0 = n$ for simplicity. Using 6.8, and noting that if $A \in \Xi^\pm$ and $\sigma(A^+) > r$ (resp., $\sigma(A^-) > r$), then $\mathbf{e}_{\mathbf{i}}^{\mathbf{c}(A^+)} = 0$ (resp., $\mathbf{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} = 0$), we may write $\mathbf{e}_{\mathbf{i}}^{\mathbf{c}(A^+)} = \mathbf{e}_{\mathbf{i}}^{\mathbf{c}(A^+)} +$ lower terms and $\mathbf{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} = \mathbf{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} +$ lower terms. Here the lower terms are relative to \preccurlyeq . So (6.7.1) gives

$$\mathbf{e}_{\mathbf{i}}^{\mathbf{c}(A^+)} \mathbf{k}_1^{j_1} \cdots \mathbf{k}_{n-1}^{j_{n-1}} \mathbf{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} = \mathbf{e}^{(A^+)} \mathbf{k}_1^{j_1} \cdots \mathbf{k}_{n-1}^{j_{n-1}} \mathbf{f}^{(A^-)} + \text{lower terms (relative to } \preccurlyeq),$$

where $A \in \Xi^\pm$, $j_i \in \mathbb{N}$, $j_1 + \cdots + j_{n-1} + \sigma(A) \leq r$. Now the assertion follows from 6.6. \square

Remark 6.10. (1) The new monomial bases for $\mathbf{U}(n, r)$ constructed in 6.6 are the “little” version of the monomial basis for $\mathbf{U}(n)$ in [2, 5.7]; while the bases in 6.9 are the q -Schur algebra version of the PBW-basis given in [12, 1.13].

(2) Note that the bases given in Theorem 6.9 are the conjectured bases in [6, 3.2].

7. LITTLE q -SCHUR ALGEBRAS

We now define little q -Schur algebras.

By restricting the map $\zeta_r : \mathbf{U}(n) \twoheadrightarrow \mathbf{U}(n, r)$ to the \mathcal{A} -form $U_{\mathcal{A}}(n)$, we obtain by [8, 3.4] a surjective map $\zeta_r : U_{\mathcal{A}}(n) \twoheadrightarrow U_{\mathcal{A}}(n, r)$. Thus, base change induces a surjective homomorphism $\zeta_{r,k} := \zeta_r \otimes 1 : U_k(n) \twoheadrightarrow U_k(n, r)$, and hence a map $\zeta_{r,k} : \tilde{u}_k(n) \rightarrow U_k(n, r)$ by restriction. The image $\zeta_{r,k}(\tilde{u}_k(n))$, denoted $u_k(n, r)$, is called a *little q -Schur algebra*. Since, by 6.2(1)-(2),

$$\mathbf{k}_i^l = \sum_{\lambda \in \Lambda(n, r)} \mathbf{k}_i^l \mathbf{k}_\lambda = \sum_{\lambda \in \Lambda(n, r)} \varepsilon^{\lambda_i l} \mathbf{k}_\lambda = 1,$$

it follows that $\zeta_{r,k}(K_i^l - 1) = 0$ for all i . Hence $\zeta_{r,k}$ induces a surjective map, the version of ζ_r over k ,

$$\zeta_{r,k} : u_k(n) \twoheadrightarrow u_k(n, r).$$

Let $u_k(n, r)^+ = \zeta_{r,k}(u_k(n)^+)$, $u_k(n, r)^- = \zeta_{r,k}(u_k(n)^-)$ and $u_k(n, r)^0 = \zeta_{r,k}(u_k(n)^0)$. By abuse of notation, we shall continue to denote the images of the generators E_i, F_i, K_j for $u_k(n)$ by the same letters $\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_j$ used for $\mathbf{U}(n, r)$.

Lemma 7.1. *There is a k -algebra anti-automorphism τ on $u_k(n, r)$ satisfying*

$$\tau(\mathbf{e}_i) = \mathbf{f}_i, \quad \tau(\mathbf{f}_i) = \mathbf{e}_i, \quad \tau(\mathbf{k}_i) = \mathbf{k}_i.$$

In particular, $\tau(u_k(n, r)^+) = u_k(n, r)^-$.

Proof. By [9, 4.2], there is a unique $\mathbb{Q}(v)$ -algebra anti-automorphism τ on $\mathbf{U}(n, r)$ satisfying $\tau(\mathbf{e}_i) = \mathbf{f}_i$, $\tau(\mathbf{f}_i) = \mathbf{e}_i$, $\tau(\mathbf{k}_i) = \mathbf{k}_i$. It is clear $\tau(U_{\mathcal{A}}(n, r)) = U_{\mathcal{A}}(n, r)$. Hence τ induces a anti-automorphism $\tau \otimes id$ on $U_k(n, r)$. Since $(\tau \otimes id)(u_k(n, r)) = u_k(n, r)$, the result follows. \square

Our first aim to develop a relation between q -Schur algebras and little ones, analogous to the relation between U_k and u_k .

Lemma 7.2. *Let $\lambda, \mu \in \Lambda(n, r)$. The following are equivalent.*

- (1) $\bar{\lambda} = \bar{\mu} \in (\mathbb{Z}_l)^n$;
- (2) $\overline{\lambda_1 j_1 + \cdots + \lambda_{n-1} j_{n-1}} = \overline{\mu_1 j_1 + \cdots + \mu_{n-1} j_{n-1}}$ for any $0 \leq j_1, \dots, j_{n-1} < l$;
- (3) *the coefficients of \mathbf{k}_λ and \mathbf{k}_μ in the expression of $\mathbf{k}_1^{j_1} \cdots \mathbf{k}_{n-1}^{j_{n-1}}$ as a linear combination of \mathbf{k}_λ ($\lambda \in \Lambda(n, r)$) are equal for any $0 \leq j_1, \dots, j_{n-1} < l$;*

Proof. The first equivalence is clear. To see the second, we use the formula

$$\mathbf{k}_1^{j_1} \cdots \mathbf{k}_{n-1}^{j_{n-1}} = \sum_{\nu \in \Lambda(n,r)} \varepsilon^{j_1 \nu_1 + \cdots + j_{n-1} \nu_{n-1}} \mathbf{k}_\nu \in U_k(n,r),$$

which can be derived from 6.2(1)-(2). \square

Let $\overline{\Lambda(n,r)} = \{\bar{\lambda} \in (\mathbb{Z}_l)^n \mid \lambda \in \Lambda(n,r)\}$, and let $\mathbb{N}_l = \{0, 1, \dots, l-1\} \subseteq \mathbb{Z}$. Note that the set \mathbb{N}_l is a particular set of representations for elements of \mathbb{Z}_l . For $\bar{\lambda} \in (\mathbb{Z}_l)^n$, define

$$\mathbf{p}_{\bar{\lambda}} = \begin{cases} \sum_{\mu \in \Lambda(n,r), \bar{\mu} = \bar{\lambda}} \mathbf{k}_\mu & \text{if } \bar{\lambda} \in \overline{\Lambda(n,r)} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7.3. *Fix any i_0 with $1 \leq i_0 \leq n$. Each of the following sets form a k -basis for $u_k(n,r)^0$.*

- (1) $\mathcal{X}_{i_0} = \{\mathbf{k}^{\mathbf{j}} = \mathbf{k}_1^{j_1} \cdots \mathbf{k}_n^{j_n} \mid \mathbf{j} \in \mathbb{N}_l^n, j_{i_0} = 0, \sigma(\mathbf{j}) \leq r\}$;
- (2) $\mathcal{Y}_{i_0} = \{\mathbf{k}_{\lambda, i_0} \mid \lambda \in \Lambda(n,r), \lambda_i < l, i \neq i_0\}$ (see 6.5);
- (3) $\mathcal{Z} = \{\mathbf{p}_{\bar{\lambda}} \mid \bar{\lambda} \in \overline{\Lambda(n,r)}\}$.

In particular, we have $\dim u_k(n,r)^0 = \#\overline{\Lambda(n,r)}$.

Proof. By 2.3 and 6.1, $u_k(n,r)^0$ is spanned by \mathcal{X}_{i_0} . So $\dim u_k(n,r)^0 \leq \#\mathcal{I}_1$, where

$$\mathcal{I}_1 := \{\mathbf{j} = (j_1, j_2, \dots, j_n) \mid \mathbf{j} \in \mathbb{N}_l^n, j_{i_0} = 0, \sigma(\mathbf{j}) \leq r\}.$$

The condition $\lambda_i < l$ in (2) guarantees that \mathcal{Y}_{i_0} is in the span of \mathcal{X}_{i_0} and hence is contained in $u_k(n,r)^0$. Thus, $\#\mathcal{I}_2 \leq \dim u_k(n,r)^0 \leq \#\mathcal{I}_1$ where

$$\mathcal{I}_2 = \{\lambda \in \Lambda(n,r) \mid \lambda_i < l, i \neq i_0\}.$$

Since \mathcal{Y}_{i_0} is linearly independent by 6.5 and $\#\mathcal{I}_1 = \#\mathcal{I}_2$, it follows that both \mathcal{X}_{i_0} and \mathcal{Y}_{i_0} are bases, proving (1) and (2). To prove (3), we use 7.2 to write a basis element $\mathbf{k}_1^{j_1} \cdots \mathbf{k}_{n-1}^{j_{n-1}}$ in \mathcal{X}_n ($i_0 = n$) as

$$(7.3.1) \quad \mathbf{k}_1^{j_1} \cdots \mathbf{k}_{n-1}^{j_{n-1}} = \sum_{\bar{\lambda} \in (\mathbb{Z}_l)^n} \varepsilon^{\lambda_1 j_1 + \cdots + \lambda_{n-1} j_{n-1}} \mathbf{p}_{\bar{\lambda}}.$$

Now both \mathcal{X}_n and \mathcal{Z} are linear independent sets in $U_k(n,r)$ and $\#\overline{\Lambda(n,r)} = \#\mathcal{I}_1$. Thus, (7.3.1) implies that all $\mathbf{p}_{\bar{\lambda}} \in u_k(n,r)^0$ and \mathcal{Z} forms a basis for $u_k(n,r)^0$, proving (3). \square

Remark 7.4. By 6.2(1), the elements $\mathbf{p}_{\bar{\lambda}}$ ($\bar{\lambda} \in \overline{\Lambda(n,r)}$) is a complete set of orthogonal primitive idempotents in $u_k(n,r)^0$.

Recall from §5 the identity $1 = \sum_{\lambda \in \mathbb{Z}_l^n} [\text{diag}(\lambda)]$.

Corollary 7.5. *For $\bar{\lambda} \in (\mathbb{Z}_l)^n$, we have $\zeta_{r,k}([\text{diag}(\bar{\lambda})]) = \mathbf{p}_{\bar{\lambda}}$.*

Proof. Let $\mathbf{s}_{\bar{\lambda}} = \zeta_{r,k}(\llbracket \text{diag}(\bar{\lambda}) \rrbracket)$. In $u_k(n)$, we have

$$K_i \cdot \llbracket \text{diag}(\bar{\lambda}) \rrbracket = 0(\bar{\mathbf{e}}_i) \cdot \llbracket \text{diag}(\bar{\lambda}) \rrbracket = \sum_{\bar{\mathbf{j}} \in (\mathbb{Z}/l)^n} \varepsilon^{\mathbf{j} \cdot \mathbf{e}_i} \llbracket \text{diag}(\bar{\mathbf{j}}) \rrbracket \cdot \llbracket \text{diag}(\bar{\lambda}) \rrbracket = \varepsilon^{\lambda_i} \llbracket \text{diag}(\bar{\lambda}) \rrbracket.$$

Thus, $\mathbf{k}_i \mathbf{s}_{\bar{\lambda}} = \varepsilon^{\lambda_i} \mathbf{s}_{\bar{\lambda}}$ and, by 6.2, $\mathbf{k}_i \mathbf{p}_{\bar{\lambda}} = \varepsilon^{\lambda_i} \mathbf{p}_{\bar{\lambda}}$ for $1 \leq i \leq n$. So both sets $\mathbf{k} \mathbf{s}_{\bar{\lambda}}$ and $\mathbf{k} \mathbf{p}_{\bar{\lambda}}$ are two sided ideals of $u_k(n, r)^0$, and we obtain two central primitive idempotent decompositions:

$$1 = \sum_{\bar{\mu} \in (\mathbb{Z}_l)^n, \mathbf{s}_{\bar{\mu}} \neq 0} \mathbf{s}_{\bar{\mu}} = \sum_{\bar{\lambda} \in \Lambda(n, r)} \mathbf{p}_{\bar{\lambda}}.$$

If $\bar{\lambda} \in \overline{\Lambda(n, r)}$, then $\mathbf{p}_{\bar{\lambda}} = \mathbf{p}_{\bar{\lambda}} \mathbf{s}_{\bar{\mu}} = \mathbf{s}_{\bar{\mu}}$ for some $\bar{\mu}$. Multiplying the equation by \mathbf{k}_i gives $\varepsilon^{\lambda_i} \mathbf{p}_{\bar{\lambda}} = \varepsilon^{\mu_i} \mathbf{s}_{\bar{\mu}}$ which implies $\bar{\lambda} = \bar{\mu}$ and so $\mathbf{s}_{\bar{\lambda}} = \mathbf{p}_{\bar{\lambda}}$. \square

The result below is the little version of [9, 8.3]. Recall the sets $\Gamma, \Gamma^\pm, \Gamma^+, \Gamma^-$ defined in §4.

Proposition 7.6. *The set $\{\mathbf{e}^{(A)} \mid A \in \Gamma^+, \sigma(A) \leq r\}$ (resp., $\{\mathbf{f}^{(A)} \mid A \in \Gamma^-, \sigma(A) \leq r\}$) forms a k -basis of $u_k(n, r)^+$ (resp., $u_k(n, r)^-$).*

Proof. By 5.3 and 5.5(1), we can easily deduce a monomial basis of the form $\{E^{(A)}\}_{A \in \Gamma^+}$ for $u_k(n)$ (see (3.1.1) for the notation $E^{(A)}$). Thus, we have $u_k(n, r)^+ = \text{span}\{\mathbf{e}^{(A)} \mid A \in \Gamma^+\}$. For $A \in \Gamma^+$, if $\sigma(A) = \sum_i \sigma_i(A) > r$, then $\mathbf{e}^{(A)} = \sum_{\lambda \in \Lambda(n, r)} \mathbf{e}^{(A)} \mathbf{k}_\lambda = 0$ by 6.2(1),(3); if $\sigma(A) \leq r$, then $\mathbf{e}^{(A)} = \sum_{\lambda \in \Lambda(n, r), \lambda_i \geq \sigma_i(A)} \mathbf{e}^{(A)} \mathbf{k}_\lambda \neq 0$ by 6.2(3). By [9, 8.3], we see that the set $\{\mathbf{e}^{(A)} \mid A \in \Gamma^+, \sigma(A) \leq r\}$ is linearly independent. Therefore, the result is proven. \square

We derive some commutator relations between the generators of $u_k(n, r)$. The following result follows from [6, 2.4] or [9, 4.8, 4.12(1)]. Recall from 4.1 $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$.

Proposition 7.7. *Let $\bar{\lambda} \in \overline{\Lambda(n, r)}$. Then in the k -algebra $u_k(n, r)$ we have the following.*

- (1) *If there exist some $\mu \in \Lambda(n, r)$ such that $\bar{\mu} = \bar{\lambda}$ and $\mu_{i+1} \geq 1$, then $\mathbf{e}_i \mathbf{p}_{\bar{\lambda}} = \mathbf{p}_{\bar{\lambda} + \bar{\alpha}_i} \mathbf{e}_i$; otherwise $\mathbf{e}_i \mathbf{p}_{\bar{\lambda}} = 0$.*
- (2) *If there exist some $\mu \in \Lambda(n, r)$ such that $\bar{\mu} = \bar{\lambda}$ and $\mu_i \geq 1$, then $\mathbf{f}_i \mathbf{p}_{\bar{\lambda}} = \mathbf{p}_{\bar{\lambda} - \bar{\alpha}_i} \mathbf{f}_i$; otherwise $\mathbf{f}_i \mathbf{p}_{\bar{\lambda}} = 0$.*

Proof. The statement (2) follows by applying the anti-automorphism τ given in 7.1 to (1). We now prove (1). Recall from [9] that, in $U_{\mathcal{A}}(n, r)$,

- (a) If $\lambda_{i+1} \geq 1$, then $\mathbf{e}_i \mathbf{k}_\lambda = \mathbf{k}_{\lambda + \alpha_i} \mathbf{e}_i$; otherwise, $\mathbf{e}_i \mathbf{k}_\lambda = 0$. ([9, 4.8, 4.10])
- (b) If $\lambda_i = 0$ for some i with $1 \leq i \leq n - 1$, then $\mathbf{k}_\lambda \mathbf{e}_i = \mathbf{f}_i \mathbf{k}_\lambda = 0$. ([9, 4.12])

Thus, if there exists a $\mu \in \Lambda(n, r)$ such that $\bar{\mu} = \bar{\lambda}$ and $\mu_{i+1} \geq 1$, then

$$\mathbf{e}_i \mathbf{p}_{\bar{\lambda}} = \sum_{\substack{\mu \in \Lambda(n, r), \bar{\mu} = \bar{\lambda} \\ \mu_{i+1} \geq 1}} \mathbf{k}_{\mu + \alpha_i} \mathbf{e}_i \quad (\text{by (a)}), \quad \mathbf{p}_{\bar{\lambda} + \bar{\alpha}_i} \mathbf{e}_i = \sum_{\substack{\nu \in \Lambda(n, r), \bar{\nu} = \bar{\lambda} + \bar{\alpha}_i \\ \nu_i \geq 1}} \mathbf{k}_\nu \mathbf{e}_i \quad (\text{by (b)}).$$

It is clear that

$$\{\mu + \alpha_i \mid \mu \in \Lambda(n, r), \bar{\mu} = \bar{\lambda}, \mu_{i+1} \geq 1\} = \{\nu \mid \nu \in \Lambda(n, r), \bar{\nu} = \bar{\lambda} + \bar{\alpha}_i, \nu_i \geq 1\}.$$

Hence $\mathbf{e}_i \mathbf{p}_{\bar{\lambda}} = \mathbf{p}_{\bar{\lambda} + \bar{\alpha}_i} \mathbf{e}_i$. If $\{\mu \in \Lambda(n, r) \mid \bar{\mu} = \bar{\lambda}, \mu_{i+1} \geq 1\} = \emptyset$, then $\mathbf{e}_i \mathbf{p}_{\bar{\lambda}} = 0$ by (a). \square

Corollary 7.8 (cf. [9, 4.9]). *Let $A \in \Gamma^\pm$. Then we have the following result in the k -algebra $u_k(n, r)$.*

(1) *If there exist some $\mu \in \Lambda(n, r)$ such that $\bar{\mu} = \bar{\lambda}$ and $\mu_i \geq \sigma_i(A^+)$ for all i , then $\mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\lambda}} = \mathbf{p}_{\bar{\lambda}'} \mathbf{e}^{(A^+)}$ where $\bar{\lambda}' = \bar{\lambda} - \overline{co(A^+)} + \overline{ro(A^+)}$; otherwise, $\mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\lambda}} = 0$.*

(2) *If there exist some $\mu \in \Lambda(n, r)$ such that $\bar{\mu} = \bar{\lambda}$ and $\mu_i \geq \sigma_i(A^-)$ for all i , then $\mathbf{p}_{\bar{\lambda}'} \mathbf{f}^{(A^-)} = \mathbf{f}^{(A^-)} \mathbf{p}_{\bar{\lambda}''}$ where $\bar{\lambda}'' = \bar{\lambda} + \overline{co(A^+)} - \overline{ro(A^-)}$; otherwise, $\mathbf{p}_{\bar{\lambda}'} \mathbf{f}^{(A^-)} = 0$.*

Proof. Assume there exist some $\mu \in \Lambda(n, r)$ such that $\bar{\mu} = \bar{\lambda}$ and $\mu_i \geq \sigma_i(A^+)$ for all i . If $i < j$, $a \leq \mu_j$ and $0 \leq a \leq l-1$, by 7.7, $(\mathbf{e}_i^a \cdots \mathbf{e}_{j-1}^a) \mathbf{p}_{\bar{\lambda}} = \mathbf{p}_{\bar{\nu}} (\mathbf{e}_i^a \cdots \mathbf{e}_{j-1}^a)$ where $\bar{\nu} = \bar{\lambda} + a(\bar{\alpha}_i + \cdots + \bar{\alpha}_{j-1})$. Since $\mathbf{e}^{(A^+)} = \prod_{1 \leq i < j \leq n} \mathbf{e}_i^{(a_{ij})} \cdots \mathbf{e}_{j-1}^{(a_{ij})}$, we obtain $\mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\lambda}} = \mathbf{p}_{\bar{\lambda}'} \mathbf{e}^{(A^+)}$ where $\bar{\lambda}' = \bar{\lambda} - \overline{co(A^+)} + \overline{ro(A^+)}$. On the other hand, if $\{\mu \in \Lambda(n, r) \mid \bar{\mu} = \bar{\lambda}, \mu_i \geq \sigma_i(A^+)$ for all $i\} = \emptyset$, then by 6.2(3), $\mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\lambda}} = 0$. Hence the result in (1) follows. Applying τ in 7.1 to the result in (1) gives that in (2). \square

8. BASES FOR A LITTLE q -SCHUR ALGEBRA

Recall from §4 that for $A \in \Gamma^\pm$, $\mathbf{j} \in \mathbb{Z}^n$

$$A(\bar{\mathbf{j}}, r) = \sum_{\mathbf{j}' \in \Lambda(n, r - \sigma(A))} e^{\mathbf{j} \cdot \mathbf{j}'} [A + \text{diag}(\mathbf{j}')].$$

Note that by (3.0.2) $A(\bar{\mathbf{j}}, r) = A(\mathbf{j}, r)|_{v=\varepsilon}$.

It is known that from 3.1(4) the map $\zeta_r : \mathbf{U}(n) \rightarrow \mathbf{U}(n, r)$ satisfies that $\zeta_r(A(\mathbf{j})) = A(\bar{\mathbf{j}}, r)$. Since $A(\mathbf{j})$ is not necessarily in U_A , the first part of the following result at k level can not be deduced from the fact at $\mathbb{Q}(v)$ level. Recall from §5 the basis $\{A(\bar{\mathbf{j}})\}$ for $u_k(n)$.

Lemma 8.1. (1) *We have $\zeta_{r,k}(A(\bar{\mathbf{j}})) = A(\bar{\mathbf{j}}, r)$ for $A \in \Gamma^\pm$, $\bar{\mathbf{j}} \in (\mathbb{Z}_l)^n$.*

(2) *Fix any $A \in \Gamma^\pm$ with $\sigma(A) \leq r$ and let*

$$\mathcal{V}_A := \text{span}_k \{A(\bar{\mathbf{j}}, r) \mid \bar{\mathbf{j}} \in (\mathbb{Z}_l)^n\}.$$

Then $\dim_k \mathcal{V}_A \leq \#\overline{\Lambda(n, r - \sigma(A))}$.

Proof. (1) We apply induction on $\|A\|$. If $\|A\| = 0$, then $A = 0$. It is obvious that $\zeta_{r,k}(0(\bar{\mathbf{j}})) = 0(\bar{\mathbf{j}}, r)$ for $\bar{\mathbf{j}} \in (\mathbb{Z}_l)^n$. Assume $\|A\| > 0$. We apply $\zeta_{r,k}$ to (4.3.2) to obtain

$$\prod_{1 \leq i \leq h < j \leq n} (a_{ij} E_{h, h+1})(\bar{\mathbf{0}}, r) \prod_{1 \leq j \leq h < i \leq n} (a_{ij} E_{h+1, h})(\bar{\mathbf{0}}, r) = \zeta_{r,k}(A(\bar{\mathbf{0}})) + \zeta_{r,k}(f).$$

By induction, we see that $\zeta_{r,k}(f)$ is a k -linear combination of elements $A'(\bar{\mathbf{j}}, r) \in U_k(n, r)$ with $A' \in \Gamma^\pm$ and $A' \prec A$ (hence $\|A'\| < \|A\|$). Equating this with (4.3.1), we obtain $\zeta_{r,k}(A(\bar{\mathbf{0}})) = A(\bar{\mathbf{0}}, r)$.

In general, we have $A(\bar{\mathbf{j}}) = \varepsilon^{\sum_{i,k} j_i a_{k,i}} A(\bar{\mathbf{0}})0(\bar{\mathbf{j}})$ by (4.1.1). Hence, by (4.1.1) for the q -Schur algebra, we have $\zeta_{r,k}(A(\bar{\mathbf{j}})) = \varepsilon^{\sum_{i,k} j_i a_{k,i}} A(\bar{\mathbf{0}}, r)0(\bar{\mathbf{j}}, r) = A(\bar{\mathbf{j}}, r)$.

(2) For $\bar{\mathbf{j}} \in (\mathbb{Z}_l)^n$, we have (cf. (5.1.1))

$$\begin{aligned} A(\bar{\mathbf{j}}, r) &= \sum_{\lambda \in \Lambda(n, r - \sigma(A))} \varepsilon^{\lambda \cdot \bar{\mathbf{j}}} [A + \text{diag}(\lambda)] \\ &= \sum_{\bar{\lambda} \in \overline{\Lambda(n, r - \sigma(A))}} \varepsilon^{\lambda \cdot \bar{\mathbf{j}}} \sum_{\mu \in \Lambda(n, r - \sigma(A)), \bar{\mu} = \bar{\lambda}} [A + \text{diag}(\mu)] \\ &= \sum_{\bar{\lambda} \in \overline{\Lambda(n, r - \sigma(A))}} \varepsilon^{\lambda \cdot \bar{\mathbf{j}}} \llbracket A + \text{diag}(\bar{\lambda}), r \rrbracket, \end{aligned}$$

where

$$\llbracket A + \text{diag}(\bar{\lambda}), r \rrbracket = \sum_{\substack{\mu \in \Lambda(n, r - \sigma(A)) \\ \bar{\mu} = \bar{\lambda}}} [A + \text{diag}(\mu)] \in U_k(n, r).$$

Thus, \mathcal{V}_A is spanned by all $\llbracket A + \text{diag}(\bar{\lambda}), r \rrbracket$ with $\bar{\lambda} \in \overline{\Lambda(n, r - \sigma(A))}$. Hence the result follows. \square

We now derive the first two bases for $u_k(n, r)$ as the counterparts of the basis \mathcal{M} in 6.4 and the basis in 5.2.

Theorem 8.2. *Each of the following sets forms a k -basis for $u_k(n, r)$.*

- (1) $\mathcal{M}_k = \left\{ \mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\lambda}} \mathbf{f}^{(A^-)} \mid A \in \Gamma^\pm, \lambda \in \Lambda(n, r), \lambda_i \geq \sigma_i(A) \forall i \right\}$.
- (2) $\mathcal{L}_k = \left\{ \llbracket A + \text{diag}(\bar{\lambda}), r \rrbracket \mid A \in \Gamma^\pm, \sigma(A) \leq r, \lambda \in \Lambda(n, r - \sigma(A)) \right\}$.

Proof. (1) By 7.3, we have $\mathbf{p}_{\bar{\lambda}} \in u_k(n, r)$. Hence $\mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\lambda}} \mathbf{f}^{(A^-)} \in u_k(n, r)$ for any $\bar{\lambda} \in (\mathbb{Z}_l^n)$, $A \in \Gamma^\pm$. By 6.4, the set \mathcal{M}_k is linearly independent. By 5.2 and 8.1(1), we have

$$u_k(n, r) = \text{span}_k \left\{ A(\bar{\mathbf{j}}, r) \mid A \in \Gamma^\pm, \bar{\mathbf{j}} \in (\mathbb{Z}_l)^n \right\}.$$

Note that we can easily show that

$$\#\{(\bar{\lambda}_1, \dots, \bar{\lambda}_n) \in (\mathbb{Z}_l)^n \mid (\lambda_1, \dots, \lambda_n) \in \Lambda(n, r), \lambda_i \geq \sigma_i(A) \text{ for all } i\} = \#\overline{\Lambda(n, r - \sigma(A))}.$$

By 8.1(2), we have $\dim_k u_k(n, r) \leq \sum_{\substack{A \in \Gamma^\pm \\ \sigma(A) \leq r}} \#\overline{\Lambda(n, r - \sigma(A))} = \#\mathcal{M}_k$. Hence the result follows.

(2) Assume $A \in \Gamma^\pm$ and $\sigma(A) \leq r$. By the proof above, we have $\dim_k \mathcal{V}_A = \#\overline{\Lambda(n, r - \sigma(A))}$. Let \mathcal{V}'_A be the subspace of $U_k(n, r)$ spanned by the elements $\llbracket A + \text{diag}(\bar{\lambda}), r \rrbracket$ for $\bar{\lambda} \in \overline{\Lambda(n, r - \sigma(A))}$. Then $\dim_k \mathcal{V}'_A = \#\overline{\Lambda(n, r - \sigma(A))} = \dim_k \mathcal{V}_A$. Since $\mathcal{V}_A \subseteq \mathcal{V}'_A$, we have $\mathcal{V}_A = \mathcal{V}'_A$. In particular, we have $\llbracket A + \text{diag}(\bar{\lambda}), r \rrbracket \in \mathcal{V}_A \subseteq u_k(n, r)$ for $A \in \Gamma^\pm$, $\bar{\lambda} \in \overline{\Lambda(n, r - \sigma(A))}$. Since the set \mathcal{L}_k is k -linearly independent and $\#\mathcal{L}_k = \dim_k u_k(n, r)$, the result follows. \square

Let $\bar{\cdot} : M_n(\mathbb{Z}) \rightarrow M_n(\mathbb{Z}_l)$ be the map defined by $\overline{(a_{ij})} = (\bar{a}_{ij})$.

Lemma 8.3. *Let $\bar{\Xi}_r = \{\bar{A} \mid A \in \Xi_r\}$. There is a bijective mapping from $\bar{\Xi}_r$ to the set*

$$\{(A, \bar{\lambda}) \mid A \in \Gamma^\pm, \lambda \in \Lambda(n, r), \lambda_i \geq \sigma_i(A) \forall i\}.$$

Proof. For $A \in \Xi_r$, let A^\pm be the matrix obtained by replacing the diagonal entries of A by zeros. Then the map

$$f : \Xi_r \rightarrow \{(B, \lambda) \mid B \in \Xi^\pm, \lambda \in \Lambda(n, r), \lambda_i \geq \sigma_i(A) \forall i\},$$

defined by $f(A) = (A^\pm, \lambda)$ for all $A \in \Xi_r$, where $\lambda = (\sigma_1(A), \dots, \sigma_n(A))$, is a bijection. Since, for each $A \in \Xi_r$, there exist a unique $B \in \Gamma^\pm$ such that $\bar{A^\pm} = \bar{B}$, it follows that f induces a bijection

$$\bar{f} : \bar{\Xi}_r \rightarrow \{(A, \bar{\lambda}) \mid A \in \Gamma^\pm, \lambda \in \Lambda(n, r), \lambda_i \geq \sigma_i(A) \forall i\}.$$

□

With this lemma, we may rewrite a basis element in \mathcal{M}_k as

$$\bar{\mathbf{m}}^{(A)} := \mathbf{e}^{(B^+)} \mathbf{p}_{\bar{\lambda}} \mathbf{f}^{(B^-)},$$

where $B \in \Gamma^\pm$, $\lambda \in \Lambda(n, r)$ with $\lambda_i \geq \sigma_i(B)$ for all i , and $\bar{f}(A) = (B, \bar{\lambda})$. The last assertion of the following is seen easily from 6.3.

Corollary 8.4. *We have $\mathcal{M}_k = \{\bar{\mathbf{m}}^{(A)} \mid A \in \bar{\Xi}_r\}$. In particular, $\dim_k u_k(n, r) = \#\bar{\Xi}_r$. Moreover, for $A \in \Gamma^\pm$ and $\bar{\mu} \in \bar{\Lambda}(n, r)$ satisfying that if $\lambda \in \Lambda(n, r)$ with $\bar{\lambda} = \bar{\mu}$ then $\lambda_i < \sigma_i(A)$ for some i , we have*

$$\mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\mu}} \mathbf{f}^{(A^-)} = \sum_{B \in \bar{\Xi}_r, B^\pm \prec A} g_{B,A} \bar{\mathbf{m}}^{(B)} \quad (g_{B,A} \in k),$$

where B^\pm is the matrix obtained by replacing the diagonal entries by zeros and is regarded a matrix in Γ^\pm .

We have also the following monomial, BLM and PBW bases for $u_k(n, r)$ which are analogous to the bases in 6.6 and 6.9. Recall, for $\mathbf{j} \in \mathbb{N}^n$, $\mathbf{k}^{\mathbf{j}} = \mathbf{k}_1^{j_1} \cdots \mathbf{k}_n^{j_n}$ and $\sigma(\mathbf{j}) = j_1 + \cdots + j_n$, and $\mathbb{N}_l = \{0, 1, \dots, l-1\}$.

Theorem 8.5. *Fix any integer i_0 with $1 \leq i_0 \leq n$. Each of the following set forms a basis for $u_k(n, r)$.*

- (1) $\mathcal{N}_{i_0, k} = \{\mathbf{e}^{(A^+)} \mathbf{k}^{\mathbf{j}} \mathbf{f}^{(A^-)} \mid A \in \Gamma^\pm, \mathbf{j} \in \mathbb{N}_l^n, j_{i_0} = 0, \sigma(\mathbf{j}) + \sigma(A) \leq r\};$
- (2) $\mathcal{B}_{i_0, k} = \{A(\mathbf{j}, r) \mid A \in \Gamma^\pm, \mathbf{j} \in \mathbb{N}_l^n, j_{i_0} = 0, \sigma(\mathbf{j}) + \sigma(A) \leq r\};$
- (3) $\mathcal{P}_{i_0, k} = \{\mathbf{e}_{\mathbf{i}}^{\mathbf{c}(A^+)} \mathbf{k}^{\mathbf{j}} \mathbf{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} \mid A \in \Gamma^\pm, \mathbf{j} \in \mathbb{N}_l^n, j_{i_0} = 0, \sigma(\mathbf{j}) + \sigma(A) \leq r\}$, where \mathbf{i} is as in 6.8.

Proof. The proofs of these results are similar to those of 6.6 and 6.9. We prove (1) for example. As for 6.6(1), we prove the case $i_0 = n$. Let $\mathcal{N}_k = \mathcal{N}_{n,k}$. Clearly, $\#\mathcal{N}_k = \#\mathcal{M}_k$. So it is enough to prove the set \mathcal{N}_k is linearly independent.

By 6.2(1-2) and the definition of $\mathbf{p}_{\bar{\lambda}}$, we have

$$\mathbf{k}_1^{j_1} \cdots \mathbf{k}_n^{j_n} = \sum_{\bar{\lambda} \in \overline{\Lambda(n,r)}} \varepsilon^{\sum_{i=1}^n \lambda_i j_i} \mathbf{p}_{\bar{\lambda}} = \sum_{\bar{\lambda} \in \overline{\Lambda(n,r)}} \varepsilon^{\lambda \cdot \mathbf{j}} \mathbf{p}_{\bar{\lambda}},$$

It follows that for $A \in \Gamma^\pm$,

$$\mathbf{e}^{(A^+)} \mathbf{k}_1^{j_1} \cdots \mathbf{k}_n^{j_n} \mathbf{f}^{(A^-)} = \sum_{\substack{\bar{\lambda}: \lambda \in \overline{\Lambda(n,r)} \\ \lambda_i \geq \sigma_i(A)}} \varepsilon^{\lambda \cdot \mathbf{j}} \mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\lambda}} \mathbf{f}^{(A^-)} + \sum_{\substack{\bar{\mu}: \mu \in \overline{\Lambda(n,r)} \\ \exists i, \mu_i < \sigma_i(A)}} \varepsilon^{\mu \cdot \mathbf{j}} \mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\mu}} \mathbf{f}^{(A^-)}.$$

Hence by 8.4, we obtain (cf. (6.6.1))

$$\mathbf{e}^{(A^+)} \mathbf{k}_1^{j_1} \cdots \mathbf{k}_n^{j_n} \mathbf{f}^{(A^-)} = \sum_{\substack{\bar{\lambda}: \lambda \in \overline{\Lambda(n,r)} \\ \lambda_i \geq \sigma_i(A)}} \varepsilon^{\lambda \cdot \mathbf{j}} \bar{\mathbf{m}}^{(A_\lambda)} + \sum_{\substack{B \in \bar{\Xi}_r \\ B^\pm \prec A}} f_{B,A} \bar{\mathbf{m}}^{(B)}$$

where $f_{B,A} \in k$, and $A_\lambda = A + \text{diag}(\overline{\lambda_1 - \sigma_1(A)}, \dots, \overline{\lambda_n - \sigma_n(A)})$. Now, the linear independence of \mathcal{N}_k follows from the invertibility of the coefficient matrix $(\varepsilon^{\lambda \cdot \mathbf{j}})_{\lambda \in \overline{\Lambda_{n,r,A}}, \mu \in \overline{\Lambda'_{n,r,A}}}$ where

$$\overline{\Lambda_{n,r,A}} = \{\bar{\lambda} \mid \lambda \in \overline{\Lambda(n,r)}, \lambda_i \geq \sigma_i(A) \forall i\} \text{ and } \overline{\Lambda'_{n,r,A}} = \{\mathbf{j} \in \mathbb{N}_l^n \mid j_n = 0, \sigma(\mathbf{j}) + \sigma(A) \leq r\}.$$

The rest of the argument is entirely similar to the proof of (6.6.2) with $\Lambda_{n,r,A}$ and $\Lambda'_{n,r,A}$ replaced by $\overline{\Lambda_{n,r,A}}$ and $\overline{\Lambda'_{n,r,A}}$, and v by ε , of course. \square

Table 8.6. We display various bases discussed in previous sections in the following table.

Basis type (location)	$\mathbf{U}(n)$	$\mathbf{U}(n,r)$	$u_k(n)$	$u_k(n,r)$
Integral monomial (6.4; 8.2&8.4)	—	$\mathbf{m}^{(A)}, A \in \bar{\Xi}_r$	—	$\bar{\mathbf{m}}^{(A)}, A \in \bar{\Xi}_r$
Monomial (3.2; 6.6; 5.3; 8.5)	$M^{(A,\mathbf{j})}$	$\mathbf{e}^{(A^+)} \mathbf{k} \mathbf{j} \mathbf{f}^{(A^-)}$	$M^{(A,\bar{\mathbf{j}})}$	$\mathbf{e}^{(A^+)} \mathbf{k} \mathbf{j} \mathbf{f}^{(A^-)}$
BLM (3.1; 6.6; 5.2; 8.5)	$A(\mathbf{j})$	$A(\mathbf{j}, r)$	$A(\bar{\mathbf{j}})$	$A(\bar{\mathbf{j}}, r)$
PBW (6.9; 8.5)	—	$\mathbf{e}_i^{\mathbf{c}(A^+)} \mathbf{k} \mathbf{j} \mathbf{f}_i^{\mathbf{c}(A^-)}$	2.3	$\mathbf{e}_i^{\mathbf{c}(A^+)} \mathbf{k} \mathbf{j} \mathbf{f}_i^{\mathbf{c}(A^-)}$
BLM $_k$ (6.6; 8.2)	—	—	$\llbracket A \rrbracket, A \in \Gamma'$	$\llbracket A, r \rrbracket, \dots$
where for rows 2-4	$A \in \bar{\Xi}^\pm, \mathbf{j} \in \mathbb{Z}^n$	$A \in \bar{\Xi}^\pm, \mathbf{j} \in \mathbb{N}^n, j_{i_0} = 0$ $\sigma(\mathbf{j}) + \sigma(A) \leq r$	$A \in \Gamma^\pm, \mathbf{j} \in \mathbb{Z}_l^n$	$A \in \Gamma^\pm, \mathbf{j} \in \mathbb{N}_l^n, j_{i_0} = 0$ $\sigma(\mathbf{j}) + \sigma(A) \leq r$

9. DIMENSION FORMULAS

Let $X_{n,r,l} := \{\lambda \in \Lambda(n,r) \mid \lambda_2, \dots, \lambda_n < l\}$ and put $a_{n,r,l} = \#X_{n,r,l}$. It is well known that we have the following generating function:

$$(9.0.1) \quad \frac{1}{(1-x)^m} = \sum_{s \geq 0} \binom{s+m-1}{m-1} x^s,$$

where x is an indeterminant and m is a positive integer.

Lemma 9.1. *We have the following equality:*

$$a_{n,r,l} = \sum_{s \geq 0} (-1)^s \binom{n-1}{s} \binom{n+r-sl-1}{n-1}.$$

Proof. Let t_i ($1 \leq i \leq n$) be a collection of n commuting indeterminate and x be another indeterminate. Define functions $f_r(t_1, \dots, t_n)$ by

$$\frac{1}{1-t_1x} \cdot \prod_{i=2}^n \frac{1-(t_i x)^l}{1-t_i x} = \sum_{r \geq 0} f_r(t_1, \dots, t_n) x^r.$$

Then $f_r(t_1, \dots, t_n)$ is the sum of all monomials of degree r in $\{t_i\}$ such that for any $2 \leq i \leq n$, the exponent of t_i does not exceed l . Thus $a_{n,r,l}$ is obtained by evaluating $f_r(t_1, \dots, t_n)$ at $t_i = 1$ for all i ; that is,

$$\frac{1}{1-x} \cdot \frac{(1-x^l)^{n-1}}{(1-x)^{n-1}} = \sum_{r \geq 0} a_{n,r,l} x^r.$$

However, on the other hand,

$$\begin{aligned} \frac{1}{1-x} \cdot \frac{(1-x^l)^{n-1}}{(1-x)^{n-1}} &= (1-x^l)^{n-1} \cdot \frac{1}{(1-x)^n} \\ &= \sum_{s \geq 0} (-1)^s \binom{n-1}{s} x^{ls} \sum_{u \geq 0} \binom{n+u-1}{n-1} x^u && \text{(by (9.0.1))} \\ &= \sum_{r \geq 0} \sum_{r=u+ls} (-1)^s \binom{n-1}{s} \binom{n+u-1}{n-1} x^r \\ &= \sum_{r \geq 0} \sum_{s \geq 0} (-1)^s \binom{n-1}{s} \binom{n+r-sl-1}{n-1} x^r. \end{aligned}$$

The desired equality follows by equating coefficients. □

Theorem 9.2. *We have the following formula*

$$(9.2.1) \quad \#\overline{\Lambda(n,r)} = \sum_{s \geq 0} (-1)^s \binom{n-1}{s} \binom{n+r-sl-1}{n-1}.$$

In particular, we have

$$(1) \dim u_k(n, r)^0 = \sum_{s \geq 0} (-1)^s \binom{n-1}{s} \binom{n+r-sl-1}{n-1};$$

$$(2) \dim u_k(n, r)^+ = \sum_{s \geq 0} (-1)^s \binom{N-1}{s} \binom{N+r-sl-1}{N-1} \text{ with } N = \binom{n}{2} + 1;$$

$$(3) \dim_k u_k(n, r) = \sum_{s \geq 0} (-1)^s \binom{n^2-1}{s} \binom{n^2+r-sl-1}{n^2-1}.$$

Proof. By restricting the map $\bar{\cdot} : \mathbb{Z}^n \rightarrow (\mathbb{Z}_l)^n$ to the subset $X_{n,r,l}$, we obtain a map from $X_{n,r,l}$ to $\overline{\Lambda(n, r)}$. For any $\lambda \in \Lambda(n, r)$, we choose $0 \leq \mu_2, \dots, \mu_n < l$, such that $\overline{\mu_2} = \overline{\lambda_2}, \dots, \overline{\mu_n} = \overline{\lambda_n}$. Then $\sum_{i=2}^n \mu_i \leq \sum_{i=2}^n \lambda_i \leq r$. Let $\mu_1 = r - \sum_{i=2}^n \mu_i \geq 0$, then $\overline{\mu_1} = r - \sum_{i=2}^n \overline{\mu_i} = \overline{r} - \sum_{i=2}^n \overline{\lambda_i} = \overline{\lambda_1}$ and $(\overline{\mu_1}, \dots, \overline{\mu_n}) = \overline{\lambda}$. So the restriction is onto. On the other hand, if $\lambda, \mu \in X_{n,r,l}$ such that $\overline{\lambda} = \overline{\mu}$, then $\lambda_2 = \mu_2, \dots, \lambda_n = \mu_n$ by the definition of $X_{n,r,l}$. It follows that $\lambda_1 = \mu_1$ since $\lambda, \mu \in \Lambda(n, r)$. Hence the map from $X_{n,r,l}$ to $\overline{\Lambda(n, r)}$ is bijective. Thus, (1) follows immediately from 7.3, and (2) and (3) follow from 7.6 and 8.4, respectively, as we may identify the set $\{A \in \Gamma^+ \mid \sigma(A) \leq r\}$ with $X_{N,r,l}$, and $\overline{\Xi}_r$ with $\overline{\Lambda(n^2, r)}$. \square

Remark 9.3. From the dimension formulas, we see that the little q -Schur algebra $u_k(n, r)$ is different from the infinitesimal Schur algebras investigated in [7], [4]. We shall compare them in [10].

10. PRESENTING BOREL SUBALGEBRAS OF $u_k(n, r)$

Let $u_k(n, r)^{\geq 0}$ (resp., $u_k(n, r)^{\leq 0}$) be the k -subalgebra of $u_k(n, r)$ generated by \mathbf{e}_i (resp., \mathbf{f}_i) and \mathbf{k}_j ($1 \leq i \leq n-1$, $1 \leq j \leq n$). These algebras are called Borel subalgebras of $u_k(n, r)$ and will play an important role in the representation theory of $u_k(n, r)$.

The various bases for $u_k(n, r)$ give rise to bases for $u_k(n, r)^{\geq 0}$. As examples, we list some of them below.

Proposition 10.1. *Fix any i_0 with $1 \leq i_0 \leq n$. Each of the following sets forms a basis for $u_k(n, r)^{\geq 0}$.*

$$(1) \mathcal{M}_k^{\geq 0} := \{\mathbf{e}^{(A)} \mathbf{p}_{\overline{\lambda}} \mid A \in \Gamma^+, \lambda \in \Lambda(n, r), \lambda_i \geq \sigma_i(A) \text{ for all } i\};$$

$$(2) \mathcal{N}_{i_0, k}^{\geq 0} = \{\mathbf{e}^{(A)} \mathbf{k}^{\mathbf{j}} \mid A \in \Gamma^+, \mathbf{j} \in \mathbb{N}_l^n, j_{i_0} = 0, \sigma(\mathbf{j}) + \sigma(A) \leq r\};$$

$$(3) \mathcal{B}_{i_0, k}^{\geq 0} = \{A(\mathbf{j}, r) \mid A \in \Gamma^+, \mathbf{j} \in \mathbb{N}_l^n, j_{i_0} = 0, \sigma(\mathbf{j}) + \sigma(A) \leq r\};$$

$$(4) \mathcal{L}_k^{\geq 0} := \{[A + \text{diag}(\overline{\lambda}), r] \mid A \in \Gamma^+, \sigma(A) \leq r, \lambda \in \Lambda(n, r - \sigma(A))\}$$

In particular, $\dim u_k(n, r)^+ = \sum_{s \geq 0} (-1)^s \binom{N-1}{s} \binom{N+r-sl-1}{N-1}$ with $N = \binom{n+1}{2}$.

A similar result holds for $u_k(n, r)^{\leq 0}$.

Proof. By 7.1, it suffices to prove the case for $u_k(n, r)^{\geq 0}$. It is clear that $u_k(n, r)^{\geq 0} = u_k(n, r)^+ u_k(n, r)^0$. Hence by 7.3 and 7.6, $u_k(n, r)^{\geq 0}$ is spanned by the elements $\mathbf{e}^{(A)} \mathbf{p}_{\overline{\lambda}}$ with $A \in \Gamma^+$, $\overline{\lambda} \in (\mathbb{Z}_l)^n$. By 6.2(3), if for any $\mu \in \Lambda(n, r)$ with $\overline{\mu} = \overline{\lambda}$, there exist some i such that $\lambda_i < \sigma_i(A)$, then $\mathbf{e}^{(A)} \mathbf{p}_{\overline{\lambda}} = 0$. It follows that $u_k(n, r)^{\geq 0}$ is spanned by the elements $\mathbf{e}^{(A)} \mathbf{p}_{\overline{\lambda}}$ with

$A \in \Gamma^+$, $\lambda \in \Lambda(n, r)$ and $\lambda_i \geq \sigma_i(A)$ for all i . So (1) follows by 8.2. Clearly, (2) and (3) follows from 8.5 and (1). We now prove (4).

Assume $A \in \Gamma^+$. By (5.3.1) and 8.1, we have

$$\mathbf{e}^{(A)} \mathbf{k}^{\mathbf{j}} = \varepsilon^a A(\bar{\mathbf{j}}, r) + \sum_{\substack{B \prec A, B \in \Gamma^\pm \\ \mathbf{j}' \in \mathbb{Z}^n}} f_{B, \mathbf{j}'} B(\bar{\mathbf{j}}', r)$$

where $f_{B, \mathbf{j}'} \in k$. Note if $A \in \Gamma'^+$ and $B \prec A$, then $B \in \Gamma'^+$. It follows that

$$\mathbf{e}^{(A)} \mathbf{k}^{\mathbf{j}} = \varepsilon^a A(\bar{\mathbf{j}}, r) + \sum_{\substack{B \prec A, B \in \Gamma^+ \\ \mathbf{j}' \in \mathbb{Z}^n}} f_{B, \mathbf{j}'} B(\bar{\mathbf{j}}', r).$$

By definition, for $B \in \Gamma'^+$, we may write

$$B(\bar{\mathbf{j}}', r) = \sum_{\bar{\lambda} \in \overline{\Lambda(n, r - \sigma(B))}} \varepsilon^{\lambda \cdot \mathbf{j}'} \llbracket B + \text{diag}(\bar{\lambda}), r \rrbracket \in \text{span } \mathcal{L}_k^{\geq 0}.$$

Hence by (2), we have $u_k(n, r)^{\geq 0} \subseteq \text{span } \mathcal{L}_k^{\geq 0}$. Thus, we have

$$\dim u_k(n, r)^{\geq 0} \leq \#\mathcal{L}_k = \#\mathcal{M}_k^{\geq 0} = \dim u_k(n, r)^{\geq 0}.$$

Therefore, $\mathcal{L}_k^{\geq 0}$ is a basis. □

We turn to investigate presentations for Borel subalgebras.

Theorem 10.2. *The k -algebra $u_k(n, r)^{\geq 0}$ is generated by the elements*

$$\mathbf{e}_i \ (1 \leq i \leq n-1), \ \mathbf{p}_{\bar{\lambda}} \ (\bar{\lambda} \in \overline{\Lambda(n, r)})$$

subject to the following relations

$$(a) \ \mathbf{p}_{\bar{\lambda}} \mathbf{p}_{\bar{\mu}} = \delta_{\bar{\lambda}, \bar{\mu}} \mathbf{p}_{\bar{\lambda}}, \ 1 = \sum_{\bar{\lambda} \in \overline{\Lambda(n, r)}} \mathbf{p}_{\bar{\lambda}};$$

$$(b) \ \mathbf{e}_i^2 \mathbf{e}_j - (\varepsilon + \varepsilon^{-1}) \mathbf{e}_i \mathbf{e}_j \mathbf{e}_i + \mathbf{e}_j \mathbf{e}_i^2 = 0 \ \text{when } |i - j| = 1;$$

$$(c) \ \mathbf{e}_i \mathbf{e}_j = \mathbf{e}_j \mathbf{e}_i \ \text{when } |i - j| > 1;$$

$$(d) \ \mathbf{e}_i^l = 0;$$

$$(e) \ \mathbf{e}_i \mathbf{p}_{\bar{\lambda}} = \mathbf{p}_{\bar{\lambda} + \bar{\alpha}_i} \mathbf{e}_i \ \text{if } \mu_{i+1} \geq 1 \ \text{for some } \mu \in \Lambda(n, r) \ \text{with } \bar{\mu} = \bar{\lambda};$$

(f) $\mathbf{e}^{(A^+)} \mathbf{p}_{\bar{\lambda}} = 0$ for $A^+ \in \Gamma^+$ and $\bar{\lambda} \in \overline{\Lambda(n, r)}$ satisfying that if $\mu \in \Lambda(n, r)$ with $\bar{\lambda} = \bar{\mu}$ then $\mu_i < \sigma_i(A)$ for some i .

A similar result holds for $u_k(n, r)^{\leq 0}$.

Proof. Let $s_k^{\geq 0}$ be the algebra defined by generators \mathbf{e}_i ($1 \leq i \leq n-1$), $\mathbf{p}_{\bar{\lambda}}$ ($\bar{\lambda} \in \overline{\Lambda(n, r)}$) and relations (a)-(f). By 6.1, 7.5, 7.7 and 7.8, there is naturally an algebra epimorphism $s_k^{\geq 0} \rightarrow u_k(n, r)^{\geq 0}$. This in fact is an isomorphism as we now prove that $\dim s_k^{\geq 0} \leq \dim u_k(n, r)^{\geq 0}$.

Let s_k^+ (resp. s_k^0) be the subalgebra of $s_k^{\geq 0}$ generated by \mathbf{e}_i ($1 \leq i \leq n-1$) (resp. $\mathbf{p}_{\bar{\lambda}}$ ($\bar{\lambda} \in \overline{\Lambda(n, r)}$)). By (e), we have $s_k^{\geq 0} = s_k^+ s_k^0$. From 2.5, we may easily obtain a presentation for u_k^+ and

thus, an algebra epimorphism $u_k^+ \rightarrow s_k^+$. Hence, using the monomial basis for u_k in 5.3, we see that s_k^+ is spanned by the elements $\mathbf{e}^{(A)}$ with $A \in \Gamma^+$. Now the relation (e) guarantees that, for $A \in \Gamma^+$ and $\lambda \in \Lambda(n, r)$, if $\lambda_i \geq \sigma_i(A)$ for all i , then $\mathbf{e}^{(A)} \mathbf{p}_{\bar{\lambda}} = \mathbf{p}_{\bar{\lambda}'} \mathbf{e}^{(A)}$ where $\bar{\lambda}' = \bar{\lambda} - \overline{co(A)} + \overline{ro(A)}$ (see the proof of 7.8). This together with (f) implies that $s_k^{\geq 0}$ is spanned by the elements $\mathbf{e}^{(A)} \mathbf{p}_{\bar{\lambda}}$ with $A \in \Gamma^+$, $\lambda \in \Lambda(n, r)$ and $\lambda_i \geq \sigma_i(A)$ for all i . Thus, the dimension inequality follows from 10.1. \square

Remarks 10.3. (1) We remark that the relations involved in (f) are not simple at all. It is natural to expect that if (f) can be replaced by

$$(f') \quad \mathbf{e}_i \mathbf{p}_{\bar{\lambda}} = 0 \text{ if } \mu_{i+1} = 0 \text{ for all } \mu \in \Lambda(n, r) \text{ with } \bar{\mu} = \bar{\lambda}.$$

Unfortunately, this is not the case. We have constructed a counterexample.

(2) There is no nice and simple presentation for $u_k(n, r)$ either as given in 6.1 for the q -Schur algebra. In fact $\ker \zeta_{r,k}|_{u_k(n)}$ is more complex than $\ker \zeta_r$ since it can not be generated by certain elements in $u_k^0(n)$. For example, in the case assume $n = 2$ and $l = r = 3$, if $\ker \zeta_{r,k}|_{u_k(2)}$ was generated by certain elements in $u_k(2)^0$, then by 7.5 we would have

$$\ker \zeta_{r,k}|_{u_k(2)} = \langle \llbracket \text{diag}(\bar{\lambda}) \rrbracket \mid \bar{\lambda} \in (\mathbb{Z}_l)^n, \bar{\lambda} \notin \overline{\Lambda(2, 3)} \rangle.$$

Let $\beta = (1, -1) \in \mathbb{Z}^2$. It is clear for $\bar{\lambda} \notin \overline{\Lambda(2, 3)}$ we have $\bar{\lambda} \pm \bar{\beta} \notin \overline{\Lambda(2, 3)}$. We can easily see that $E_1 \llbracket \text{diag}(\bar{\lambda}) \rrbracket = \llbracket \text{diag}(\bar{\lambda} + \bar{\beta}) \rrbracket E_1$ and $F_1 \llbracket \text{diag}(\bar{\lambda}) \rrbracket = \llbracket \text{diag}(\bar{\lambda} - \bar{\beta}) \rrbracket F_1$, where $\llbracket \text{diag}(\bar{\lambda}) \rrbracket \in u_k(2)$. Hence we have

$$\ker \zeta_{r,k}|_{u_k(2)} = \text{span}\{E_1^a \llbracket \text{diag}(\bar{\lambda}) \rrbracket F_1^b \mid \bar{\lambda} \notin \overline{\Lambda(2, 3)}, 0 \leq a, b < 3\}.$$

It follows that $\dim \ker \zeta_{r,k}|_{u_k(2)} = 54$. But $\dim u_k(2) = 81$ and $\dim u_k(2, 3) = \overline{\Xi}_3 = 17$, so $\dim \ker \zeta_{r,k}|_{u_k(2)} < \dim u_k(2) - \dim u_k(2, 3) = 64$. This is a contradiction.

REFERENCES

- [1] H.H. Andersen, J.C. Jantzen and W. Soergel, *Representations of quantum groups at a p -th root of unity and of semisimple groups in characteristic p : independence of p* , Astérisque **220**, 1994.
- [2] A.A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of GL_n* , Duke Math.J. **61** (1990), 655–677.
- [3] C. De Concini, and V.G. Kac, *Representations of quantum groups at roots of 1*, in Operator algebras, Unitary Representations, Enveloping Algebras and Invariant Theory, A. Connes, M. Duflo, A. Joseph and R. Rentschler (eds) (1990), 471–506.
- [4] A.G. Cox, *On some applications of infinitesimal methods to quantum groups and related algebras*, Ph.D thesis, University of London 1997.
- [5] R. Dipper and G. James, *The q -Schur algebra*, Proc. London Math. Soc. **59** (1989), 23–50.
- [6] S. Doty and A. Giaquinto, *Presenting Schur algebras*, International Mathematics Research Notices IMRN, **36** (2002), 1907–1944.

- [7] S.R. Doty, D.K. Nakano and K.M. Peters, *On Infinitesimal Schur algebras*, Proc. London Math. Soc. **72** (1996), 588–612.
- [8] J. Du, *A note on the quantized Weyl reciprocity at roots of unity*, Alg. Colloq. **2** (1995), 363–372.
- [9] J. Du and B. Parshall, *Monomial bases for q -Schur algebras*, Trans. Amer. Math. Soc. **355** (2003), 1593–1620.
- [10] Q. Fu, *A comparison between little and infinitesimal q -Schur algebras*, Comm. Algebra, to appear.
- [11] G. Lusztig, *Modular representations and quantum groups*, Comtemp. Math. **82** (1989), 59–77.
- [12] G. Lusztig, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 257–296.
- [13] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.
- [14] M. Takeuchi, *Some topics on $Gl_q(n)$* , J. Algebra **147** (1992), 379–410.

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