QUANTUM AFFINE $\mathfrak{gl}_n$ VIA HECKE ALGEBRAS

JIE DU AND QIANG FU†

Abstract. The quantum loop algebra of $\mathfrak{gl}_n$ is the affine analogue of quantum $\mathfrak{gl}_n$. In the seminal work [1], Beilinson–Lusztig–MacPherson gave a beautiful realisation for quantum $\mathfrak{gl}_n$ via a geometric setting of quantum Schur algebras. Since then, generalising this work to the affine case and other cases (see, e.g., [9]) attracted much attention. For example, in [13, 21, 15, 20], affine quantum Schur algebras and their connection to the quantum loop algebras have been geometrically constructed and thoroughly investigated; while [8, 3] provides an algebraic treatment of the theory. As a continuation of our algebraic approach, we will present in this paper a full generalisation of BLM’s realisation to the affine case.

1. Introduction

The quantum enveloping algebra $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ of the loop algebra of $\mathfrak{gl}_n$, or simply quantum affine $\mathfrak{gl}_n$, has two usual definitions, the $R$-matrix one and the Drinfeld one, known as Drinfeld’s new realisation. Both are presented by generators and relations (see, e.g., [10, §2.3] and the references therein). In [3, 2.3.1,2.5.3], a third presentation is given via the double Ringel–Hall algebra. In this presentation, the Ringel–Hall algebra of the cyclic quiver and its opposite algebra become the $\pm$-part of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$. Thus, with this construction, one may consider semisimple or indecomposable generators for $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$, defined by the semisimple or indecomposable representations of the quiver; see [3, §1.4]. In particular, one sees easily the fact that the subalgebra generated by simple generators is a proper subalgebra. This subalgebra is isomorphic to the quantum loop algebra of $\mathfrak{sl}_n$.

The double Ringel–Hall algebra construction of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ is an affine generalisation of a similar construction for a quantum enveloping algebra $\mathcal{U}$ of a finite type quiver via a Ringel–Hall algebra which, as the positive or negative part of $\mathcal{U}$, is spanned by the basis of isoclasses of representations of the quiver and whose multiplication is defined by Hall polynomials, see [16, 17, 22].

However, there is another construction for quantum $\mathfrak{gl}_n$ by Beilinson, Lusztig and MacPherson [1, 5.7], which directly displays a basis for the entire quantum enveloping algebra $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ and displays the multiplication rules by explicit formulas of basis elements by generators. This construction is geometric in nature and has been partially generalised to the case of the quantum loop algebra of $\mathfrak{sl}_n$ in [13, 15]. More precisely, they construct an explicit algebra homomorphism.

† Corresponding author.

Supported by the National Natural Science Foundation of China, the Program NCET, Fok Ying Tung Education Foundation, the Fundamental Research Funds for the Central Universities, and the Australian Research Council DP120101436.
multiplication rules are given by the formulas in Proposition 4.2(1)–(3). Though the map \( \zeta_r : U(\hat{\mathfrak{sl}}_n) \to S_\alpha(n, r) \) is not surjective in the case of \( n \leq r \), the natural algebra homomorphism from \( U(\hat{\mathfrak{gl}}_n) \) to \( S_\alpha(n, r) \) is still surjective for any \( n, r \) (see [21, 2.4], [3, 3.6.3, 3.8.1]).

BLM’s geometric approach uses the definition of quantum Schur algebras as the convolution algebras of (partial) flag varieties over finite fields and then, by a process of “quantumization”, to get a construction over the polynomials ring. Progress on generalising BLM’s work to the affine case via an algebraic approach has been made in the works [8, 3, 11]. In particular, a realisation conjecture [8, 5.5(2)] for \( U(\hat{\mathfrak{gl}}_n) \) was formulated and was proved in the classical \((v = 1)\) case in [3, Ch. 6]. We will prove this conjecture in this paper.

We will use directly the definition of affine quantum Schur algebras as endomorphism algebras of certain \( \epsilon \)-permutation modules over the affine Hecke algebra which has a basis indexed by certain double cosets of the affine symmetric group. The double cosets associated with semisimple representations will play a key role in the establishment of multiplication rules of BLM type basis elements by semisimple generators.

It should be pointed out that a recent work by Bridgeland constructs quantum enveloping algebras via Hall algebras of complexes and a complete realisation [2, Th. 4.9] is obtained for simply-laced finite types; see [23] for the general (finite type) case. A complete realisation for algebras via Hall algebras of complexes and a complete realisation [2, Th. 4.9] is obtained for the affine quantum Schur algebra \( U(\hat{\mathfrak{gl}}_n) \) over the polynomials ring. Progress on generalising BLM’s work to the affine case via an algebraic approach has been made in the works [8, 3, 11]. In particular, a realisation conjecture [8, 5.5(2)] for \( U(\hat{\mathfrak{gl}}_n) \) was formulated and was proved in the classical \((v = 1)\) case in [3, Ch. 6]. We will prove this conjecture in this paper.

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We now describe the main result of the paper. For a positive integer \( n \), let \( \hat{\mathfrak{gl}}_n := M_{\alpha,n}(\mathbb{C}) \) be the loop algebra of \( \mathfrak{gl}_n(\mathbb{C}) \) consisting of all matrices \( A = (a_{i,j})_{i,j\in\mathbb{Z}} \) with \( a_{i,j} \in \mathbb{C} \) such that

(a) \( a_{i,j} = a_{i+n,j+n} \) for \( i, j \in \mathbb{Z} \);
(b) for every \( i \in \mathbb{Z} \), both sets \( \{ j \in \mathbb{Z} \mid a_{i,j} \neq 0 \} \) and \( \{ j \in \mathbb{Z} \mid a_{j,i} \neq 0 \} \) are finite.

A basis for \( \hat{\mathfrak{gl}}_n \) can be described as \( \{ E_{i,j}^\alpha \mid i, j \in \mathbb{Z} \} \), where the matrix \( E_{i,j}^\alpha = (e_{k,l}^{i,j})_{k,l \in \mathbb{Z}} \) is defined by

\[
e_{k,l}^{i,j} = \begin{cases} 1 & \text{if } k = i + sn, l = j + sn \text{ for some } s \in \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( \Theta_\alpha(n) = M_{\alpha,n}(\mathbb{N}) \) be the \( \mathbb{N} \)-span of the basis. Then \( \Theta_\alpha(n) \) serves as the index set of the PBW basis of the universal enveloping algebra \( U(\hat{\mathfrak{gl}}_n) \). Let \( U(\hat{\mathfrak{gl}}_n) \) be the quantum enveloping algebra of \( \hat{\mathfrak{gl}}_n \) over \( \mathbb{Q}(v) \) and let

\[
\Theta_\alpha^\pm(n) = \{ A \in \Theta_\alpha(n) \mid a_{i,i} = 0 \text{ for all } i \} \text{ and } \mathbb{Z}_\alpha^n = \{ (\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z} \}.
\]

Then \( \Theta_\alpha^\pm(n) \times \mathbb{Z}_\alpha^n \) serves as an index set of a PBW type basis for \( U(\hat{\mathfrak{gl}}_n) \) (see, e.g., [3, 1.4.6]). We will construct a new basis \( \{ A(j) \mid A \in \Theta_\alpha^\pm(n), j \in \mathbb{Z}_\alpha^n \} \) and prove the following main result.

**Main Theorem 1.1.** The quantum enveloping algebra \( U(\hat{\mathfrak{gl}}_n) \) is the \( \mathbb{Q}(v) \)-algebra which is spanned by the basis \( \{ A(j) \mid A \in \Theta_\alpha^\pm(n), j \in \mathbb{Z}_\alpha^n \} \) and generated by \( 0(j), S_\alpha(0) \) and \( t^\alpha S_\alpha(0) \) for all \( j \in \mathbb{Z}_\alpha^n \) and \( \alpha \in \mathbb{N}^n \), where \( S_\alpha = \sum_{1 \leq i < n} \alpha_i E_{i,i+1}^\alpha \) and \( t^\alpha S_\alpha \) is the transpose of \( S_\alpha \), and whose multiplication rules are given by the formulas in Proposition 4.2(1)–(3).
We organise this paper as follows. We first recall in §2 some preliminary results on the double Ringel–Hall algebra of a cyclic quiver and affine quantum Schur algebras. In particular, we display a PBW type basis for the former and a basis defined by certain double cosets for the latter. In §3, we derive in the affine quantum Schur algebra some multiplication formulas (Theorem 3.6) of the basis elements by those associated with semisimple representations of the cyclic quiver. We then prove the main result via Theorem 4.4 in §4. As an application of the work, we obtain certain multiplication formulas in the Ringel–Hall algebra which are not directly seen from the Hall algebra multiplication. In the Appendix, we give a proof for the length formula of the shortest representative of a double coset defined by a matrix.

Further Notations 1.2. We need the following index sets for bases of the triangular parts of \( U(\hat{gl}_n) \). Let

\[
\Theta^\pm(n) := \{ A \in \Theta(n) \mid a_{i,j} = 0 \text{ for } i \geq j \} \quad \text{and} \quad \Theta(n) := \{ A \in \Theta(n) \mid a_{i,j} = 0 \text{ for } i \leq j \}.
\]

For \( A \in \Theta(n) \), we write

\[
A = A^+ + A^0 = A^+ + A^0 + A^- \tag{1.2.1}
\]

where \( A^\pm \in \Theta^\pm(n) \), \( A^+ \in \Theta^+(n) \) and \( A^- \in \Theta^-(n) \). We also need the following notation in 2.1(2)(e)

\[
\lambda^1, \ldots, \lambda^m \in \mathbb{N}_0^n \quad \text{with} \quad \lambda = \lambda^1 + \cdots + \lambda^m.
\]

2. Preliminary results

In this section, we briefly discuss the affine symmetric group and its associated Hecke algebra, the affine \( v \)-Schur algebra, the double Hall algebra interpretation of affine \( gl_n \) and the connections between them.

Let \( S_{\Delta,r} \) be the affine symmetric group consisting of all permutations \( w : \mathbb{Z} \to \mathbb{Z} \) satisfying \( w(i+r) = w(i) + r \) for \( i \in \mathbb{Z} \). Let \( W_r \) be the subgroup of \( S_{\Delta,r} \), the Weyl group of affine type A,
generated by \( S = \{ s_i \}_{1 \leq i \leq r} \), where \( s_i \) is defined by \( s_i(j) = j \) for \( j \neq i, i + 1 \mod r \), \( s_i(j) = j - 1 \) for \( j \equiv i + 1 \mod r \), and \( s_i(j) = j + 1 \) for \( j \equiv i \mod r \). Let \( \rho \) be the permutation of \( \mathbb{Z} \) sending \( j \) to \( j + 1 \) for all \( j \in \mathbb{Z} \). We extend the length function \( \ell \) on \( W_r \) to \( \mathfrak{S}_{\Delta, r} \) by setting \( \ell(\rho^m w) = \ell(w) \) for all \( m \in \mathbb{Z}, w \in W_r \).

The (extended) affine Hecke algebra \( H_\delta(r) \) over \( \mathbb{Z} \) associated to \( \mathfrak{S}_{\Delta, r} \) is the \( \mathbb{Z} \)-algebra which is spanned by (basis) \( \{ T_w \}_{w \in \mathfrak{S}_{\Delta, r}} \) and generated by \( T_\rho, T_{\rho^\pm 1}, T_s, s \in S \), and whose multiplication rules are given by the formulas, for all \( s \in S \) and \( w \in \mathfrak{S}_{\Delta, r} \),

\[
T_s T_w = \begin{cases} (v^2 - 1)T_w + v^2 T_{sw}, & \text{if } \ell(sw) < \ell(w); \\ T_{sw}, & \text{if } \ell(sw) = \ell(w) + 1, \\
\end{cases}
\]

\[
T_\rho T_w = T_{\rho w}.
\]

Let \( H_\delta(r) = H_\delta(r) \otimes_{\mathbb{Z}} \mathbb{Q}(v) \). We will discover a similar description for \( \mathfrak{U}(\tilde{g}_l) \).

For \( \lambda \in \Lambda_\delta(n, r) \), let \( \mathfrak{S}_\lambda := \mathfrak{S}_{(\lambda_1, \ldots, \lambda_n)} \) be the corresponding standard Young subgroup of the symmetric group \( \mathfrak{S}_r \). For a finite subset \( X \subseteq \mathfrak{S}_{\Delta, r} \), let

\[
T_X = \sum_{x \in X} T_x \in H_\delta(r) \quad \text{and} \quad x_\lambda = T_{\mathfrak{S}_\lambda}.
\]

The endomorphism algebras over \( \mathbb{Z} \) or \( \mathbb{Q}(v) \)

\[
\mathcal{S}_\delta(n, r) := \text{End}_{H_\delta(r)} \left( \bigoplus_{\lambda \in \Lambda_\delta(n, r)} x_\lambda H_\delta(r) \right) \quad \text{and} \quad \mathcal{S}_\delta(n, r) := \text{End}_{H_\delta(r)} \left( \bigoplus_{\lambda \in \Lambda_\delta(n, r)} x_\lambda H_\delta(r) \right)
\]

are called affine quantum Schur algebras or, more specifically, affine \( v \)-Schur algebras (cf. [13, 14, 15]). Note that \( \mathcal{S}_\delta(n, r) \cong \mathcal{S}_\delta(n, r) \otimes_{\mathbb{Z}} \mathbb{Q}(v) \).

For \( \lambda \in \Lambda_\delta(n, r) \), denote the set of shortest representatives of right cosets of \( \mathfrak{S}_\lambda \) in \( \mathcal{S}_\delta(n, r) \) by

\[
\mathcal{D}_\lambda = \{ d \mid d \in \mathfrak{S}_{\Delta, r}, \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathfrak{S}_\lambda \}.
\]

Note that elements in \( \mathcal{D}_\lambda \) can be characterised as follows:

\[
d^{-1} \in \mathcal{D}_\lambda \iff d(\lambda_0, i - 1 + 1) < d(\lambda_0, i - 1 + 2) < \cdots < d(\lambda_0, i - 1 + \lambda_i), \forall 1 \leq i \leq n, \quad (2.0.2)
\]

where \( \lambda_0, i - 1 := \sum_{1 \leq t \leq i - 1} \lambda_t \). Moreover, \( \mathcal{D}_{\lambda, \mu} := \mathcal{D}_\lambda \cap \mathcal{D}_\mu \) is the set of shortest representatives of \( (\mathfrak{S}_\lambda, \mathfrak{S}_\mu) \) double cosets.

For \( \lambda, \mu \in \Lambda_\delta(n, r) \) and \( d \in \mathcal{D}_{\lambda, \mu} \), define \( \phi^d_{\lambda, \mu} \in \mathcal{S}_\delta(n, r) \) by

\[
\phi^d_{\lambda, \mu}(x_{\nu} h) = \delta_{\mu \nu} \sum_{w \in \mathfrak{S}_\lambda \cap \mathfrak{S}_\mu} T_w h
\]

where \( \nu \in \Lambda_\delta(n, r) \) and \( h \in H_\delta(r) \). Then by [14] the set \( \{ \phi^d_{\lambda, \mu} \mid \lambda, \mu \in \Lambda_\delta(n, r), d \in \mathcal{D}_{\lambda, \mu} \} \) forms a \( \mathbb{Z} \)-basis for \( \mathcal{S}_\delta(n, r) \).

For \( 1 \leq i \leq n, k \in \mathbb{Z} \) and \( \lambda \in \Lambda_\delta(n, r) \) let \( \lambda_{k, i - 1} := kr + \sum_{1 \leq t \leq i - 1} \lambda_t \) and

\[
R_{k+i}^\lambda = \{ \lambda_{k, i - 1} + 1, \lambda_{k, i - 1} + 2, \ldots, \lambda_{k, i - 1} + \lambda_i = \lambda_{k, i} \},
\]

By [20, 7.4] (see also [8, 9.2]), there is a bijective map

\[
\mathfrak{b}_d : \{ (\lambda, d, \mu) \mid d \in \mathcal{D}_{\lambda, \mu}, \lambda, \mu \in \Lambda_\delta(n, r) \} \longrightarrow \Theta_\delta(n, r)
\]
sending \((\lambda, w, \mu)\) to \(A = (a_{k,l})\), where \(a_{k,l} = |R_k^\lambda \cap wR_l^\mu|\) for all \(k, l \in \mathbb{Z}\). For \(A \in \Theta_\Delta(n, r)\) let \(e_A = \phi_{\lambda, \mu}^d\) where \(A = \phi_{\lambda, \mu}(\lambda, d, \mu)\). Furthermore, let
\[
[A] = v^{-d_A}e_A, \quad \text{where} \quad d_A = \sum_{1 \leq i < n \atop i \geq k, j < l} a_{i,j}a_{k,l}.
\] (2.0.3)

Later, in 3.5 and 3.6, we will consider basis elements associated with matrices of the form \(M = A + T - \tilde{T}\) for some \(T, \tilde{T} \in \Theta_\Delta(n)\). We will automatically set \(e_M = 0 = [M]\) if one of the entry of \(M\) is zero.

For \(A \in \Theta_\Delta^+(n)\) and \(j \in \mathbb{Z}_n^n\), define elements in \(\mathcal{S}_\Delta(n, r)\):
\[
A(j, r) = \sum_{\mu \in \mathcal{A}_\Delta(n, r - \sigma(A))} v^{\mu j}[A + \text{diag}(\mu)]; \quad \text{(cf. [1])}
\] (2.0.4)
where \(\mu \cdot j = \sum_{1 \leq i \leq n} \mu_i j_i\). The set \(\{A(j, r)\}_{j \in \mathbb{Z}_n^n, \, \mu \in \mathcal{A}_\Delta(n, r)}\) spans \(\mathcal{S}_\Delta(n, r)\).

Let \(\Delta(n)\) \((n \geq 2)\) be the cyclic quiver with vertex set \(I = \mathbb{Z}/n\mathbb{Z} = \{1, 2, \ldots, n\}\) and arrow set \(\{i \to i + 1 \mid i \in I\}\). Let \(\mathbb{F}\) be a field. For \(i \in I\), let \(S_i\) be the irreducible nilpotent representation of \(\Delta(n)\) over \(\mathbb{F}\) with \((S_i)_i = \mathbb{F}\) and \((S_i)_j = 0\) for \(i \neq j\). For any \(A = (a_{i,j}) \in \Theta_\Delta^+(n)\), let
\[
M(A) = M_\mathbb{F}(A) = \bigoplus_{1 \leq i < j \leq n} a_{i,j} M^{i,j},
\]
where \(M^{i,j} = M(E_{i,j})\) is the unique indecomposable nilpotent representation for \(\Delta(n)\) of length \(j - i\) with top \(S_i\). Thus, the set \(\{M(A)\}_{A \in \Theta_\Delta^+(n)}\) is a complete set of representatives of isomorphism classes of finite dimensional nilpotent representations of \(\Delta(n)\).

The Euler form associated with the cyclic quiver \(\Delta(n)\) is the bilinear form \(\langle -,-\rangle\): \(\mathbb{Z}_n^n \times \mathbb{Z}_n^n \rightarrow \mathbb{Z}\) defined by \(\langle \lambda, \mu \rangle = \sum_{1 \leq i \leq n} \lambda_i \mu_i - \sum_{1 \leq i \leq n} \lambda_i \mu_{i+1}\) for \(\lambda, \mu \in \mathbb{Z}_n^n\).

By [18], for \(A, B, C \in \Theta_\Delta^+(n)\), let \(\varphi^C_{A,B} \in \mathbb{Z}[v^2]\) be the Hall polynomials such that, for any finite field \(\mathbb{F}_q\), \(\varphi^C_{A,B}|_{v^2=q}\) is equal to the number of submodules \(N\) of \(M_{\mathbb{F}_q}(C)\) satisfying \(N \cong M_{\mathbb{F}_q}(B)\) and \(M_{\mathbb{F}_q}(C)/N \cong M_{\mathbb{F}_q}(A)\).

By definition, the (generic) twisted Ringel–Hall algebra \(\mathcal{J}_\Delta(n)\) of \(\Delta(n)\) is the \(\mathbb{Q}(v)\)-algebra spanned by basis \(\{u_A \mid A \in \Theta_\Delta^+(n)\}\) whose multiplication is defined by, for all \(A, B \in \Theta_\Delta^+(n)\),
\[
u_A u_B = v^{(\text{d}(A), \text{d}(B))} \sum_{C \in \Theta_\Delta^+(n)} \varphi^C_{A,B} u_C,
\]
where \(\text{d}(A) \in NI\) is the dimension vector of \(M(A)\).

By extending \(\mathcal{J}_\Delta(n)\) to Hopf algebras (see 2.1(2)(b) for multiplication)
\[
\mathcal{J}_\Delta(n)^{\geq 0} = \mathcal{J}_\Delta(n) \otimes \mathbb{Q}(v)[K_1^{\pm 1}, \ldots, K_n^{\pm 1}] \quad \text{and} \quad \mathcal{J}_\Delta(n)^{\leq 0} = \mathbb{Q}(v)[K_1^{\pm 1}, \ldots, K_n^{\pm 1}] \otimes \mathcal{J}_\Delta(n)^{op},
\]
we define the double Ringel–Hall algebra \(\mathcal{D}_\Delta(n)\) (cf. [22] and [3, (2.1.3.2)]) to be a quotient algebra of the free product \(\mathcal{J}_\Delta(n)^{\geq 0} \ast \mathcal{J}_\Delta(n)^{\leq 0}\) via a certain skew Hopf paring \(\psi: \mathcal{J}_\Delta(n)^{\geq 0} \times \mathcal{J}_\Delta(n)^{\leq 0} \rightarrow \mathbb{Q}(v)\). In particular, there is a triangular decomposition
\[
\mathcal{D}_\Delta(n) = \mathcal{D}_\Delta^+(n) \otimes \mathcal{D}_\Delta^0(n) \otimes \mathcal{D}_\Delta^-(n),
\]
where $D^+(\alpha) = S_\alpha(n)$, $D_0^+(\alpha) = Q(\nu)[K_1^{\pm1}, \ldots, K_n^{\pm1}]$ and $D^-(\alpha) = S_\alpha(n)^{op}$.

For $\alpha \in \mathbb{N}_0^n$, let
\[
S_\alpha = \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\pm \in \Theta^+(\alpha).
\] (2.0.5)
Then $M(S_\alpha) = \oplus_{1 \leq i \leq n} \alpha_i S_i$ is a semisimple representation of $\triangle(n)$. Let $u_\alpha = u_{S_\alpha}$.

For $\alpha, \beta \in \mathbb{Z}_0^n$, define a partial order on $\mathbb{Z}_0^n$ by setting
\[
\alpha \leq \beta \iff \alpha_i \leq \beta_i \text{ for all } i \in \mathbb{Z}.
\] (2.0.6)
We now collects some of the results we need later, see [3, Th. 2.5.3] for part (1) and [3, 2.6.7] for part (2)(e).

**Theorem 2.1.**
(1) Let $U(\mathfrak{gl}_n)$ be the quantum enveloping algebra of the loop algebra of $\mathfrak{gl}_n$ defined in [6] or [3, §2.5]. Then there is a Hopf algebra isomorphism $D_\alpha(n) \cong U(\mathfrak{gl}_n)$.

(2) The algebra $D_\alpha(n)$ is the algebra over $Q(\nu)$ which is spanned by basis
\[
\{ u_A^+ K_1^j u_A^- | A \in \Theta^+(\alpha), j \in \mathbb{Z}_0^n \}, \text{ where } K_j = K_1^j \cdots K_n^j,
\]
and generated by $u_A^+, K_1^{\pm1}$, $u_A^-$ ($\alpha, \beta \in \Theta^+(\alpha)$, $1 \leq i \leq n$), and whose multiplication is given by the following relations:

(a) $K_i K_j = K_j K_i, K_i K_i^{-1} = 1$;
(b) $K_j^i u_A^+ = v(\nu, d(\alpha)), j \geq i$, $u_A^+ K_j = v(\nu, d(\alpha)) K_j u_A^-$;
(c) $u_A^+ u_B^+ = \sum_{C \in \Theta^+(\alpha)} v(\nu, d(\alpha)) \delta_{S_A, S_B} u_C^+$;
(d) $u_A^- u_B^- = \sum_{C \in \Theta^+(\alpha)} v(\nu, d(\alpha)) \delta_{S_B, S_A} u_C^-$;
(e) For $\lambda, \mu \in \mathbb{N}_0^n, u_\lambda^+ u_\mu^- = \sum_{0 \leq \gamma \leq \alpha} \sum_{\alpha \leq \lambda, \alpha \leq \mu} x_{\alpha, \gamma} K^{2\gamma - \alpha \nu} u_\alpha^- u_\mu^-$, where
\[
x_{\alpha, \gamma} = v(\alpha, \lambda - \alpha) + (\mu, 2\gamma - \alpha) + 2(\gamma, \alpha - \gamma - \lambda) + 2(\alpha, \gamma) \left[ \gamma \alpha, \lambda - \alpha, \gamma \mu \right] \left[ \alpha, \gamma, \mu \right] a_{\alpha} a_{\lambda - \alpha} a_{\mu - \alpha} a_{\lambda} a_{\mu}^-.
\]

For $A \in \Theta^+(\alpha)$, let
\[
\overline{u}_A^\pm = v^{\dim \text{End}(M(A)) - \dim M(A)} u_A^\pm,
\]
and let $^t A$ be the transpose matrix of $A$. The relationship between $D_\alpha(n)$ and $S_\alpha(n,r)$ can be seen from the following (cf. [13, 15] and [20, Prop. 7.6]).

**Theorem 2.2** ([3, 3.6.3, 3.8.1]). For $r \in \mathbb{N}$, the map $\zeta_r : D_\alpha(n) \to S_\alpha(n,r)$ is a surjective algebra homomorphism such that, for all $j \in \mathbb{Z}_0^n$ and $A \in \Theta^+(\alpha)$,
\[
\zeta_r(K_j) = 0(j, r), \zeta_r(u_A^+) = A(0, r), \text{ and } \zeta_r(u_A^-) = (^t A)(0, r).
\]
3. SOME MULTIPLICATION FORMULAS IN THE AFFINE \( v \)-SCHUR ALGEBRA

We now derive certain useful multiplication formulas in the affine \( v \)-Schur algebra and, hence, in the quantum loop algebra of \( \mathfrak{gl}_n \). These formulas will be given in 3.6 and 4.2. They are the key to the realization of the quantum loop algebra of \( \mathfrak{gl}_n \).

We need some preparation before proving 3.6 and 4.2. The following result is given in [3, 3.2.3].

**Lemma 3.1.** Let \( \lambda, \mu \in \Lambda_\circ(n, r) \) and \( d \in \mathcal{D}_{\lambda, \mu}^0 \). Assume \( A = \mathcal{A}(\lambda, d, \mu) \). Then \( d^{-1} \mathcal{S}_\lambda d \cap \mathcal{S}_\mu = \mathcal{S}_\nu \), where \( \nu = (\nu^{(1)}, \ldots, \nu^{(n)}) \) and \( \nu^{(i)} = (a_{ki})_{k \in \mathbb{Z}} = (\ldots, a_{i1}, \ldots, a_{ni}, \ldots) \). In particular, we have

\[
x_\lambda T_d x_\mu = \prod_{1 \leq i \leq n, j \in \mathbb{Z}} [a_{ij}]_T^d T_{\mathcal{S}_\lambda} \mathcal{S}_\mu.
\]

Given \( A \in \Theta_\circ(n, r) \) with \( A = \mathcal{A}(\lambda, w, \mu) \), let \( y_A = w \) be the shortest representative of the double coset \( \mathcal{S}_\lambda \mathcal{S}_w \mathcal{S}_\mu \).

**Lemma 3.2.** (1) For \( \lambda \in \Lambda_\circ(2, r) \) and \( w \in \mathcal{D}_{\lambda, r} \cap \mathcal{S}_r \), we have \( \ell(w) = \sum_{1 \leq i \leq \lambda_1} (w^{-1}(i) - i) \).

(2) For any \( A \in \Theta_\circ(n, r) \), \( \ell(y_A) = \sum_{1 \leq i \leq n, l \leq k \leq j \leq r} a_{ij} a_{kl} \).

**Proof.** Since \( w \in \mathcal{S}_r \), \( w^{-1}(1) < \cdots < w^{-1}(\lambda_1) \) and \( w^{-1}(\lambda_1 + 1) < \cdots < w^{-1}(r) \), it follows that

\[
\ell(w) = |\{(i, j) \mid 1 \leq i < j \leq r, w^{-1}(i) > w^{-1}(j)\}|
\]

\[
= |\{(i, j) \mid 1 \leq i \leq \lambda_1, \lambda_1 + 1 \leq j \leq r, w^{-1}(i) > w^{-1}(j)\}|.
\]

On the other hand, for every \( 1 \leq i \leq \lambda_1, w^{-1}(i) - i \) of the numbers \( 1, 2, \ldots, w^{-1}(i) \) must lie in \( \{w^{-1}(\lambda_1 + 1), \ldots, w^{-1}(r)\} \) which contribute \( w^{-1}(i) - i \) inversions. Hence, \( \ell(w) = \ell(w^{-1}) = (w^{-1}(1) - 1) + (w^{-1}(2) - 2) + \cdots + (w^{-1}(\lambda_1) - \lambda_1) \), proving part (1).

Part (2) is probably known. Since we couldn’t find a proof in the literature, a proof is given in the Appendix. \( \square \)

**Lemma 3.3.** For \( a \geq 0, r \geq 1 \) and \( 0 \leq t \leq r \) we have

\[
\sum_{X \subseteq \{a+1, \ldots, a+r\}} v^2 \sum_{x \in X} x = v^{2at} + t(t+1) \left[ \begin{array}{c} r \\ t \end{array} \right].
\]

**Proof.** We proceed by induction on \( r \). The case \( r = 1 \) is trivial. Assume now that \( r > 1 \). Then, by induction hypothesis,

\[
\sum_{X \subseteq \{a+1, \ldots, a+r\}} v^2 \sum_{x \in X} x = \sum_{X \subseteq \{a+1, \ldots, a+r-1\}} v^2 \sum_{x \in X} x + \sum_{Y \subseteq \{a+1, \ldots, a+r-1\}} v^2 \sum_{y \in Y} y
\]

\[
= v^{2at} + t(t+1) \left[ \begin{array}{c} r-1 \\ t \end{array} \right] + v^{2(a+r)} t^{2at} t(t+1) \left[ \begin{array}{c} r-1 \\ t-1 \end{array} \right]
\]

\[
= v^{2at} + t(t+1) \left( \left[ \begin{array}{c} r-1 \\ t \end{array} \right] + v^{2(r-t)} \left[ \begin{array}{c} r-1 \\ t-1 \end{array} \right] \right)
\]

\[
= v^{2at} + t(t+1) \left[ \begin{array}{c} r \\ t \end{array} \right].
\]
as desired. \[\square\]

For \(i \in \mathbb{Z}\) let \(e_i^\phi \in \mathbb{N}_0^n\) be such that

\[
(e_i^\phi)_j = \begin{cases} 
1 & \text{if } j \equiv i \mod n \\
0 & \text{otherwise.}
\end{cases}
\]

**Lemma 3.4** ([11, 5.2]). Let \(\mu \in \Lambda_\Delta(n, r), \beta \in \mathbb{N}_0^n\) and assume \(\mu \succeq \beta\).

(1) If \(\alpha = \sum_{1 \leq i \leq n} (\mu_i - \beta_i) e_i^\phi, \delta = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \beta_n)\) and \(\gamma = \{(Y_0, Y_1, \cdots, Y_{n-1}) \mid Y_i \subseteq R_{i+1}^{\mu}, |Y_i| = \alpha_i, \text{ for } 0 \leq i \leq n-1\}\), then there is a bijective map

\[g : \mathcal{D}_\Delta \cap \mathfrak{S}_\mu \to \gamma\]

defined by sending \(w\) to \((w^{-1}X_0, w^{-1}X_1, \cdots, w^{-1}X_{n-1})\) where \(X_i = \{\mu_0,i + 1, \mu_0,i + 2, \cdots, \mu_0,i + \alpha_i\}\), with \(\mu_{0,i} = \sum_{1 \leq s \leq i} \mu_s\) and \(\mu_{0,0} = 0\).

(2) If \(\gamma = \mu - \beta, \theta = (\beta_1, \gamma_1, \beta_2, \gamma_2, \cdots, \beta_n, \gamma_n)\) and \(\gamma' = \{(Y'_0, Y'_1, \cdots, Y'_{n-1}) \mid Y'_i \subseteq R_{i}^{\mu}, |Y'_i| = \gamma_i, \text{ for } 1 \leq i \leq n\}\), then there is a bijective map

\[g' : \mathcal{D}_\Delta \cap \mathfrak{S}_\mu \to \gamma'\]

defined by sending \(w\) to \((w^{-1}X'_1, w^{-1}X'_2, \cdots, w^{-1}X'_{n})\) where \(X'_i = \{\mu_{0,i-1} + \beta_1 + 1, \mu_{0,i-1} + \beta_i + 2, \cdots, \mu_{0,i}\}\).

The injection can be seen easily by noting that

\[
(\alpha_0 + \beta_1, \alpha_1 + \beta_2, \cdots, \alpha_{n-1} + \beta_n) = (\mu_1, \mu_2, \cdots, \mu_n) = (\beta_1 + \gamma_1, \beta_2 + \gamma_2, \cdots, \beta_n + \gamma_n)
\]

and \(X_i\) (resp., \(X'_{i+1}\)) consists of the first \(\alpha_i\) (resp., the last \(\gamma_{i+1}\)) numbers in \(R_{i+1}^{\mu}\) for all \(0 \leq i \leq n\), while the subjection is to define \(w = g^{-1}(Y_0, Y_1, \cdots, Y_{n-1})\) by \(w^{-1}(\mu_{0,i} + s) = k_{i,s}\) for all \(0 \leq i \leq n-1\) and \(1 \leq s \leq \mu_{i+1}\), where \(Y_i = \{k_{i,1}, \cdots, k_{i,\alpha_i}\}, R_{i+1}^{\mu} \setminus Y_i = \{k_{i,\alpha_i+1}, k_{i,\alpha_i+2}, \cdots, k_{i,\mu_{i+1}}\}\), and both are strictly increasing.

For \(A \in M_\Delta(n)\) with \(\sigma(A) = r\), we denote \(e_A = [A] = 0 \in \mathcal{S}_\Delta(n, r)\) if \(a_{i,j} < 0\) for some \(i, j \in \mathbb{Z}\). There is a natural map

\[\sim : \Theta_\Delta(n) \to \Theta_\Delta(n) \quad A = (a_{i,j}) \mapsto \tilde{A} = (\tilde{a}_{i,j}), \quad (3.4.1)\]

where \(\tilde{a}_{i,j} = a_{i-1,j}\) for all \(i, j \in \mathbb{Z}\). For \(A \in \Theta_\Delta(n)\) let \(\text{ro}(A) = (\sum_{j \in \mathbb{Z}} a_{i,j})_{i \in \mathbb{Z}}\) and \(\text{co}(A) = (\sum_{i \in \mathbb{Z}} a_{i,j})_{j \in \mathbb{Z}}\).

We are now ready to establish multiplication formulas of an arbitrary basis elements \(e_A\) by certain basis elements \(e_B\) in the affine Schur algebra \(S_\Delta(n, r)\) over \(\mathbb{Z}\), where \(B^+\) or \(^t(B^-)\) defines a semisimple representation of the cyclic quiver. The significance of these formulas is the generalisation of [15, 3.5] (cf. [1, 3.1]) from real roots to all roots including all imaginary roots.
Proposition 3.5. Let $A \in \mathfrak{O}_q(n,r)$ and $\mu = \text{ro}(A)$. Assume $\beta \in \mathbb{N}_0^n$ and $\beta \leq \mu$. Let $\alpha = w = \sum_{1 \leq i \leq n} (\mu_i - \beta_i) e_i^{\lambda-1}$ and $\gamma = \sum_{1 \leq i \leq n} (\mu_i - \beta_i) e_i^{\lambda} = \mu - \beta$, $B = \sum_{1 \leq i \leq n} \alpha_i E_{i, i+1}^{\lambda} + \text{diag}(\beta)$, and $C = \sum_{1 \leq i \leq n} \gamma_i E_{i, i+1}^{\lambda} + \text{diag}(\beta)$. Then the following identities hold in $\mathcal{S}_\lambda(n, r)$.

1. $e_{BE_A} = \sum_{T \in \mathcal{O}_q(n)_{\text{ro}(T) = \alpha}} v^2 \sum_{1 \leq i \leq n, j \neq i} (a_{i,j} - t_{i,j}) t_{i,j} \prod_{1 \leq i \leq n, j \neq i} \prod_{1 \leq i \leq n, j \neq i} a_{i,j} - t_{i,j} \prod_{1 \leq i \leq n, j \neq i} e_{A + T - \bar{T}}$.

2. $e_{CE_A} = \sum_{T \in \mathcal{O}_q(n)_{\text{ro}(T) = \alpha}} v^2 \sum_{1 \leq i \leq n, j \neq i} (a_{i,j} - t_{i,j}) t_{i,j} \prod_{1 \leq i \leq n, j \neq i} \prod_{1 \leq i \leq n, j \neq i} a_{i,j} - t_{i,j} \prod_{1 \leq i \leq n, j \neq i} e_{A - T + \bar{T}}$.

Proof. We only prove (1). The proof for (2) is entirely similar.

Let $\lambda = \text{ro}(B)$ and $\nu = \text{co}(A)$. Assume $d_1 \in \mathcal{D}_\lambda$ and $d_2 \in \mathcal{D}_\nu$ defined by $\mathcal{D}_\lambda(\lambda, d_1, \mu) = B$ and $\mathcal{D}_\lambda(\mu, d_2, \nu) = A$. Then $\lambda_i = \alpha_i + \beta_i$ and $\mu_i = \alpha_{i+1} + \beta_i$ for all $1 \leq i \leq n$. From 3.1 we see that

$$e_{BE_A}(x_{\nu}) = T_{\Theta, d_1, \Theta, \mu} \cdot T_{d_2, \Theta, \nu}$$

$$= \sum_{w \in \mathcal{S}_\mu} v^{2 \ell(w)} T_{\Theta, d_1, \Theta, \mu} \cdot T_{\Theta, d_2, \Theta, \nu}$$

$$= \prod_{1 \leq i \leq n} [a_{i,j}]^{-1} T_{\Theta, d_1, \Theta, \mu} \cdot T_{d_2, \Theta, \nu}$$

$$= \prod_{1 \leq i \leq n} [a_{i,j}]^{-1} T_{\Theta, d_1, \Theta, \mu} \cdot T_{d_2, \Theta, \nu}$$

where $\mathcal{S}_\mu = d_1^{-1} \mathcal{S}_\mu d_2 \cap \mathcal{S}_\nu$, $\mathcal{S}_\delta = d_1^{-1} \mathcal{S}_\lambda d_1 \cap \mathcal{S}_\mu$ with $\delta = (\alpha_0, \beta_1, \alpha_1, \beta_2, \ldots, \alpha_{n-1}, \beta_n)$. By (2.0.2), we have $\lambda_1 = \rho^{-\alpha_0}$ (so $\ell(d_1) = 0$). This together with the fact that $d_2 \in \mathcal{D}_\mu$ implies that $\ell(d_1 w d_2) = \ell(d_1) + \ell(w) + \ell(d_2) = \ell(w) + \ell(d_2)$ for $w \in \mathcal{D}_\mu \cap \mathcal{S}_\mu$. Thus, we have

$$e_{BE_A}(x_{\nu}) = \prod_{1 \leq i \leq n} [a_{i,j}]^{-1} \sum_{w \in \mathcal{S}_\mu \cap \mathcal{D}_\mu} T_{\Theta, d_1, \Theta, \mu} T_{d_2, \Theta, \nu}$$

(3.5.1)

For $w \in \mathcal{D}_\mu \cap \mathcal{S}_\mu$ let $C(w) = (c_{i,j}^{(w)}) \in \mathcal{O}_q(n,r)$, where $c_{i,j}^{(w)} = \lfloor R_i^\lambda \cap d_1 w d_2 R_j^\nu \rfloor$, and let $T(w) = (t_{i,j}^{(w)}) \in \mathcal{O}_q(n)$, where $t_{i,j}^{(w)} = \lfloor w^{-1} X_i \cap d_2 R_j^\nu \rfloor$ with $X_i = \{\mu_{0,i} + 1, \mu_{0,i} + 2, \ldots, \mu_{0,i} + \alpha_i\}$ for all $1 \leq i \leq n$ and $j \in \mathbb{Z}$. Then $\text{ro}(T(w)) = \alpha$ and $\text{co}(T(w)) = \nu$. Since $d_1^{-1} R_i^\lambda = \alpha_0 + R_i^\lambda = (R_i^\lambda \setminus X_{i-1} \cup X_i)$, we see that $c_{i,j}^{(w)} = R_i^\lambda \cap d_1 w d_2 R_j^\nu = \lfloor w^{-1} d_1^{-1} R_i^\lambda \cap d_2 R_j^\nu \rfloor = a_{i,j} - t_{i,j}^{(w)} + t_{a_{i,j}}^{(w)}$ (see the proof of [11, 5.3]). In other words, for all $w \in \mathcal{D}_\mu \cap \mathcal{S}_\mu$,

$$C(w) = A + T(w) - \bar{T}(w).$$

(3.5.2)

In particular, $y_{C(w)} \in \mathcal{S}_\lambda d_1 w d_2 \mathcal{S}_\nu \cap \mathcal{D}_\mu$. 


Putting $\mathcal{S}_{\alpha w} = y_{C(w)}^{-1} \mathcal{S}_{\alpha} y_{C(w)} \cap \mathcal{S}_{\nu}$, we have by 3.1
\[
\sum_{w \in \mathcal{S}_{\mu} \cap \mathcal{S}_{\beta}} T_{\mathcal{S}_{\alpha}} T_{d_{1} w_{0} d_{2}} T_{\mathcal{S}_{\nu}} = \sum_{w \in \mathcal{S}_{\mu} \cap \mathcal{S}_{\beta}, d_{1} w_{0} d_{2} = w' y_{C(w)} w''} T_{\mathcal{S}_{\alpha}} T_{w'} T_{y_{C(w)} T_{w''}} T_{\mathcal{S}_{\nu}} = \sum_{w \in \mathcal{S}_{\mu} \cap \mathcal{S}_{\beta}, d_{1} w_{0} d_{2} = w' y_{C(w)} w''} v^{2(\ell(w') + \ell(w''))} T_{\mathcal{S}_{\alpha}} T_{y_{C(w)} T_{w''}} T_{\mathcal{S}_{\nu}} = \sum_{w \in \mathcal{S}_{\mu} \cap \mathcal{S}_{\beta}, d_{1} w_{0} d_{2} = w' y_{C(w)} w''} v^{2(\ell(w) + \ell(d_{2}) - \ell(y_{C(w)}))} \prod_{1 \leq i, j \leq n, j \neq Z_{i,j}} \left[ c_{i,j}^{(w)} \right] ! e_{C(w)}(x_{\nu}).
\]

Now by (3.5.1) and (3.5.2) and noting $\text{ro}(T(w)) = \alpha$ for $w \in \mathcal{S}_{\beta} \cap \mathcal{S}_{\mu}$, we have
\[
e_{B_{e_{A}}} = \sum_{w \in \mathcal{S}_{\mu} \cap \mathcal{S}_{\beta}} v^{2(\ell(w) + \ell(d_{2}) - \ell(y_{C(w)}))} \prod_{1 \leq i, j \leq n, j \neq Z_{i,j}} \left[ c_{i,j}^{(w)} \right] ! e_{C(w)} = \sum_{T \in \Theta_{n}(n), \text{ro}(T) = \alpha} \prod_{1 \leq i, j \leq n, j \neq Z_{i,j}} \left[ a_{i,j} - t_{i-1,j} + t_{i,j} \right] ! v^{2(\ell(d_{2}) - \ell(y_{A+T-T}))} \left( \sum_{w \in \mathcal{S}_{\mu} \cap \mathcal{S}_{\beta}} v^{2\ell(w)} \right) e_{A+T-T}.
\]

Given $T \in \Theta_{n}(n)$ with $\text{ro}(T) = \alpha$ let
\[
\mathcal{Z}(T) = \{ Z = (Z_{i,j})_{0 \leq i \leq n-1, j \leq Z} \mid |Z_{i,j}| = t_{i,j}, Z_{i,j} \subseteq R_{i+1}^{e} \cap d_{2} R_{j}^{e}, \text{ for } 0 \leq i \leq n-1, j \in \mathbb{Z} \}.
\]

If $T = T(w)$ then the bijective map $g$ in 3.4 induces a bijective map
\[
h_{T} : \{ w \in \mathcal{S}_{\beta} \cap \mathcal{S}_{\mu} \mid T(w) = T \} \rightarrow \mathcal{Z}(T)
\]
defined by sending $w$ to $(w^{-1}(X_{i}) \cap d_{2} R_{j}^{e})_{0 \leq i \leq n-1, j \in \mathbb{Z}}$. Since for $0 \leq i \leq n-1$ and $j \in \mathbb{Z}$
\[
R_{i+1}^{e} \cap d_{2} R_{j}^{e} = \left\{ \mu_{0,i} + \sum_{s \leq j-1} a_{i+1,s} + 1, \mu_{0,i} + \sum_{s \leq j-1} a_{i+1,s} + 2, \ldots, \mu_{0,i} + \sum_{s \leq j} a_{i+1,s} \right\},
\]
it follows from 3.2 and the definition of $g^{-1}$ that, for $Z = (Z_{i,j})_{0 \leq i \leq n-1, j \in \mathbb{Z}} \in \mathcal{Z}(T)$,
\[
\ell(h_{T}^{-1}(Z)) = \sum_{0 \leq i \leq n-1} \left( \sum_{j \in \mathbb{Z}} k - \sum_{1 \leq j \leq a_{i}} (\mu_{0,i} + j) \right) = \sum_{0 \leq i \leq n-1} \sum_{j \in \mathbb{Z}, k \in 2_{i,j}} (\alpha_{i} \mu_{0,i} + \alpha_{i}(\alpha_{i} + 1)/2).
\]

This implies that
\[
\sum_{w \in \mathcal{S}_{\mu} \cap \mathcal{S}_{\beta} \mid T(w) = T} v^{2\ell(w)} = \sum_{Z \in \mathcal{Z}(T)} v^{2\ell(h_{T}^{-1}(Z))} = v^{-2} \sum_{0 \leq i \leq n-1} (\alpha_{i} \mu_{0,i} + \alpha_{i}(\alpha_{i} + 1)/2) \prod_{0 \leq i \leq n-1} \left( \sum_{Z_{i,j} \in R_{i+1}^{e} \cap d_{2} R_{j}^{e}} v^{2k} \sum_{k \in 2_{i,j}} k \right).
Consequently, by (3.5.3), we have
\[
\sum_{w \in \Theta_{n}(n)} v^{2\ell(w)} = v^{2\alpha_{T}} \prod_{0 \leq i < n-1} \left[ \frac{a_{i+1,j}}{t_{i,j}} \right] = v^{2\alpha_{T}} \prod_{1 \leq i \leq n} \left[ \frac{a_{i,j}}{t_{i-1,j}} \right]
\]  
(3.5.4)
where
\[
a_{T} = \sum_{0 \leq i \leq n-1} \left( t_{i,j} (\mu_{0,i} + \frac{t_{i,j} (t_{i,j} + 1)}{2}) - \sum_{j \leq 1} \left( \alpha_{i} \mu_{0,i} + \frac{\alpha_{i} (\alpha_{i} + 1)}{2} \right) \right).
\]

Since \( \text{ro}(T) = \alpha \) we have \( \alpha_{i} = \sum_{j \in \mathbb{Z}} t_{i,j} \) and \( \alpha_{i}^{2} = \sum_{j \in \mathbb{Z}} t_{i,j}^{2} + 2 \sum_{j < l} t_{i,j} t_{i,l} \) for \( 0 \leq i \leq n - 1 \). This implies that
\[
a_{T} = \sum_{0 \leq i \leq n-1} a_{i+1,s} t_{i,j} - \sum_{j \leq 1} t_{i,j} t_{i,l}.
\]

Since \( d_{2} = y_{A} \) is the shortest representative in the double coset associated with \( A \), by 3.2(2),
\[
\ell(d_{2}) - \ell(y_{A+T-T}) = \sum_{1 \leq i \leq n} a_{i,j} t_{i,l} - \sum_{1 \leq i \leq n} a_{i+1,l} t_{i,j} + \sum_{1 \leq i \leq n} t_{i,j} t_{i,l} - \sum_{1 \leq i \leq n} t_{i-1,j} t_{i,l}.
\]

It follows that
\[
a_{T} + \ell(d_{2}) - \ell(y_{A+T-T}) = \sum_{1 \leq i \leq n} a_{i,j} t_{i,l} - \sum_{1 \leq i \leq n} t_{i-1,j} t_{i,l}.
\]

Consequently, by (3.5.3), (3.5.4) and noting \( \prod_{1 \leq i \leq n, j \in \mathbb{Z}} [t_{i,j}]! = \prod_{1 \leq i \leq n, j \in \mathbb{Z}} [t_{i,j}-1]! \) we have
\[
e_{B} e_{A} = \sum_{T \in \Theta_{n}(n) \text{ro}(T) = \alpha} v^{2(\ell(d_{2}) - \ell(y_{A+T-T}))} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} [a_{i,j} - t_{i-1,j} + t_{i,j}]! \prod_{1 \leq i \leq n, j \in \mathbb{Z}} [a_{i,j} - t_{i,j} - t_{i,j}]! e_{A+T-T} \]
\[
= \sum_{T \in \Theta_{n}(n) \text{ro}(T) = \alpha} v^{2 \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i-1,j}) t_{i,l}} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} [a_{i,j} + t_{i,j} - t_{i,j}] \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ \frac{a_{i,j} - t_{i,j} - t_{i,j}}{t_{i,j}} \right] e_{A+T-T},
\]
proving (1). \( \Box \)

Let \( \tilde{e} : \mathcal{Z} \to \mathcal{Z} \) be the ring homomorphism defined by \( \tilde{v} = v^{-1} \). We now use 3.5 to derive the corresponding formulas for the normalised basis \( \{ [A] \}_{A \in \Theta_{n}(n)} \) defined in (2.0.3).

**Proposition 3.6.** Let \( A \in \Theta_{n}(n, r) \) and \( \alpha, \gamma \in \mathbb{N}^{n} \).

1. For \( B \in \Theta_{n}(n, r) \), if \( B - \sum_{1 \leq i \leq n} \alpha_{i} E_{i,i+1} \) is a diagonal matrix and \( \text{co}(B) = \text{ro}(A) \), then
\[
[B][A] = \sum_{T \in \Theta_{n}(n) \text{ro}(T) = \alpha} v^{\beta(T,A)} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ \frac{a_{i,j} + t_{i,j} - t_{i-1,j}}{t_{i,j}} \right] [A + T - \tilde{T}],
\]
where \( \beta(T, A) = \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n, j > l} (a_{i+1,j} - t_{i,j}) t_{i,l} \).

2. For \( C \in \Theta_{n}(n, r) \), if \( C - \sum_{1 \leq i \leq n} \gamma_{i} E_{i,i+1} \) is a diagonal matrix and \( \text{co}(C) = \text{ro}(A) \), then
\[
[C][A] = \sum_{T \in \Theta_{n}(n) \text{ro}(T) = \gamma} v^{\beta'(T,A)} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ \frac{a_{i,j} - t_{i,j} + t_{i-1,j}}{t_{i,j}} \right] [A - T + \tilde{T}],
\]
where \( \beta'(T, A) = \sum_{1 \leq i \leq n, j \leq l} (a_{i,j} - t_{i,j}) t_{i-1,l} - \sum_{1 \leq i \leq n, l < j} (a_{i,j} - t_{i,j}) t_{i,l} \).
Proof. We only prove (1). The proof for (2) is entirely similar. Note that we have
\[
\prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ a_{i,j} + t_{i,j} - t_{i-1,j} \right] = v^2 \sum_{1 \leq i \leq n, j \in \mathbb{Z}} (a_{i,j} - t_{i-1,j}) t_{i,j} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ a_{i,j} + t_{i,j} - t_{i-1,j} \right]
\]
Thus by 3.5(1) we have
\[
[B][A] = \sum_{T \in B(n)} v^{\beta(T,A)} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ a_{i,j} + t_{i,j} - t_{i-1,j} \right][A + T - \tilde{T}],
\]
where
\[
\beta(T,A) = 2 \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i-1,j}) t_{i,l} + d_{A+T-\tilde{T}} - d_A - d_B + 2 \sum_{1 \leq i \leq n, j \in \mathbb{Z}} (a_{i,j} - t_{i-1,j}) t_{i,j}
\]
Fix $T \in \Theta_\alpha(n)$ satisfying $ro(T) = \alpha$. Then by definition we have $d_B = \sum_{1 \leq i \leq n} b_{i,i} \alpha_i$ and
\[
d_{A+T-\tilde{T}} - d_A = \sum_{1 \leq i \leq n, 1 \leq j < l} a_{i,j} t_{k,l} - t_{k-1,l} + \sum_{1 \leq i \leq n, 1 \leq j < l} a_{k,l} (t_{i,j} - t_{i-1,j}) + \sum_{1 \leq i \leq n, 1 \leq j < l} (t_{i,j} - t_{i-1,j}) (t_{k,l} - t_{k-1,l})
\]
\[
= \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n, j > l} (a_{i+1,j} - t_{i,j}) t_{i,l}
\]
Furthermore, since $ro(T) = \alpha$ and $co(B) = ro(A)$ we have $b_{i,i} = \sum_{j \in \mathbb{Z}} (a_{i,j} - t_{i-1,j})$ and $\alpha_i = \sum_{l \in \mathbb{Z}} t_{i,l}$ for each $i$, and hence
\[
d_{A+T-\tilde{T}} - d_A - d_B = - \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n, j > l} (a_{i+1,j} - t_{i,j}) t_{i,l}
\]
Consequently, $\beta(T,A) = \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n, j > l} (a_{i+1,j} - t_{i,j}) t_{i,l}$. The proof is completed. \qed

4. Proof of the main theorem

We now construct explicitly a subalgebra of the algebra $\mathcal{S}_\alpha(n) := \prod_{r \geq 0} \mathcal{S}_\alpha(n, r)$ and prove that this subalgebra is isomorphic to $U(\widehat{\mathfrak{gl}}_n)$. Recall the elements $A(j, r)$ defined in (2.0.4) and let
\[
A(j) = (A(j, r))_{r \geq 0} \in \mathcal{S}_\alpha(n) \quad \text{and} \quad \mathcal{B} = \{A(j) \mid A \in \Theta_\alpha^+(n), j \in \mathbb{Z}_n^m\}.
\]
Then $\mathcal{B}$ is linearly independent by [8, Prop.4.1(2)]. Let $\mathcal{V}_\alpha(n)$ be the $\mathbb{Q}(v)$-subspace of $\mathcal{S}_\alpha(n)$ spanned by $\mathcal{B}$. We will prove that $\mathcal{V}_\alpha(n)$ is a subalgebra of $\mathcal{S}_\alpha(n)$ isomorphic to $U(\widehat{\mathfrak{gl}}_n)$ or $\mathcal{D}_\alpha(n)$ by 2.1(a). For this purpose, we need a larger spanning set containing $\mathcal{B}$:
\[
\mathcal{\hat{B}} = \{A(j, \lambda) \mid A \in \Theta_\alpha^+(n), j \in \mathbb{Z}_\alpha^m, \lambda \in \mathbb{N}_\alpha\}
\]
where $A(j, \lambda) = (A(j, \lambda, r))_{r \geq 0}$ with $A(j, \lambda, r)$ defined by
\[ A(j, \lambda, r) = \sum_{\mu \in \Lambda_\lambda(n, r - \sigma(A))} v^{\mu \lambda} \left[ \frac{\mu}{\lambda} \right] [A + \text{diag}(\mu)] \quad (\text{cf. [12, §2]}) \] (4.0.2)

Note that, for $\sigma(\lambda) \leq r$, $0(j, \lambda, r) = \sum_{\mu \in \Lambda_\lambda(n, r), \lambda \leq \mu} v^{\mu \lambda} \left[ \frac{\mu}{\lambda} \right] [\text{diag}(\mu)]$

**Lemma 4.1.** The space $\mathcal{V}_\lambda(n)$ is spanned by the set $\widehat{B}$. In other words, every $A(j, \lambda) \in \mathcal{V}_\lambda(n)$.

**Proof.** Let $\mathcal{V}_\lambda^n(n)$ be the $\mathbb{Q}(v)$-subalgebra of $\mathcal{S}_\lambda(n)$ generated by $0(\pm e^\lambda_i)$ for $1 \leq i \leq n$. Then the set $\{0(j) \mid j \in \mathbb{Z}_0^n\}$ forms a $\mathbb{Q}(v)$-basis for $\mathcal{V}_\lambda^n(n)$. Since
\[ 0(j, \lambda, r) = \prod_{1 \leq i \leq n} \left( 0(e^\lambda_i, r)^{j_i} \prod_{1 \leq s \leq \lambda_i} \frac{0(e^\lambda_i, r)v^{-s+1} - 0(-e^\lambda_i, r)v^{s-1}}{v^s - v^{-s}} \right), \] where $\sigma(\lambda) \leq r$,
we have
\[ 0(j, \lambda) = \prod_{1 \leq i \leq n} \left( 0(e^\lambda_i)^{j_i} \prod_{1 \leq s \leq \lambda_i} \frac{0(e^\lambda_i)v^{-s+1} - 0(-e^\lambda_i)v^{s-1}}{v^s - v^{-s}} \right) \in \mathcal{V}_\lambda(n). \] (4.1.1)

On the other hand, by the proof of [12, 3.4], we have
\[ 0(j, \lambda) A(0) = v^{\text{ro}(A)(j + \lambda)} A(j, \lambda) + \sum_{\mu \in \mathbb{N}^+, 0 < \mu \leq \lambda} v^{\text{ro}(A)(j + \lambda - \mu)} \left[ \frac{\text{ro}(A)}{\mu} \right] A(j - \mu, \lambda - \mu). \]

By induction, we see that $A(j, \lambda) \in \text{span}\{0(j, \lambda) A(0) \mid A \in \Theta\lambda^\pm(n), j \in \mathbb{Z}_0^n, \lambda \in \mathbb{N}_0^n\}$. Thus, by (4.1.1), $A(j, \lambda) \in \text{span}\{0(j) A(0) \mid A \in \Theta\lambda^\pm(n), j \in \mathbb{Z}_0^n\}$. This span equals $\mathcal{V}_\lambda(n)$ by [8, (4.2.1)] (i.e., 4.2(1) below).

For $T = (t_{i,j}) \in \Theta\lambda(n)$ let $\delta_T$ be the diagonal of $T$, i.e.,
\[ \delta_T = (t_{i,i})_{i \in \mathbb{Z}} \in \mathbb{N}_0^n. \]

We now use 3.6 to derive multiplication formulas of an arbitrary basis element by a “semisimple generators”, which is the key to solving the realisation problem. Recall the notation in (1.2.1).

**Proposition 4.2.** Let $j \in \mathbb{Z}_0^n$, $A \in \Theta\lambda^\pm(n)$, $\alpha \in \mathbb{N}_0^n$, and $S_\alpha = \sum_{1 \leq i \leq n} \alpha_i E^{\lambda}_{i,i+1}$. The following identities hold in $\mathcal{V}_\lambda(n)$:

1. $0(j') A(j) = v^{\text{ro}(A)} A(j' + j)$ and $A(j) 0(j') = v^{\text{co}(A)} A(j' + j)$ ([8, (4.2.1)]).

2. $S_\alpha A(j) = \sum_{T \in \Theta\lambda(n)} v^{f_{A,T}} \prod_{T(0(\alpha))} \left[ \frac{a_{i,j} + t_{i,j} - t_{i-1,j}}{t_{i,j}} \right] (A + T^\pm - \tilde{T}^\pm)(j_T, \delta_T),$
where $j_T = j + \sum_{1 \leq i \leq n} \left( \sum_{j \neq i} (t_{i,j} - t_{i-1,j}) \right) e_i^\lambda$ and
\[ f_{A,T} = \sum_{1 \leq i \leq n} a_{i,j} t_{i,j} - \sum_{1 \leq i,j \neq i} a_{i+1,j} t_{i,j} - \sum_{1 \leq i \leq n} t_{i-1,j} t_{i,j} + \sum_{1 \leq i \leq n} t_{i,j} t_{i,j} \]
\[ + \sum_{1 \leq i \leq n} t_{i,j} t_{i+1,j+1} + \sum_{1 \leq i \leq n} j_i (t_{i-1,j} - t_{i,j}). \]
(3) \( tS_\alpha(0)A(j) = \sum \psi_j T A, T \prod_{i \in \mathcal{E}_j, j \neq i} (a_{i,j} - t_{i,j} + t_{i-1,j}) \) 
\[ (A - T^\pm + \tilde{T}^\pm)(j, T, \delta_T), \]
where \( j_T = j + \sum_{1 \leq i \leq n} (t_{i-1,j} - t_{i,j}) e_i^\delta \) and 
\[ f_j T A = \sum_{1 \leq i \leq n} a_{i,j} t_{i-1,j} - \sum_{1 \leq i \leq n} a_{i,j} t_{i,j} - \sum_{1 \leq i \leq n} t_{i-1,j} t_{i,j} + \sum_{1 \leq i \leq n} t_{i,j} t_{i,j} \]
\[ + \sum_{1 \leq i \leq n} t_{i,j} t_{i-1,j} + \sum_{1 \leq i \leq n} j_i (t_{i,i} - t_{i-1,i}). \]

In particular, \( \mathcal{V}_\alpha(n) \) is closed under the multiplication by the “generators” \( 0(j), S_\alpha(0), tS_\alpha(0) \) for all \( j \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^n \).

**Proof.** We only prove (2). If \( r < \sigma(A) \), the \( r \)-th components of both sides are 0. Assume now \( r \geq \sigma(A) \). By 3.6 and noting the fact that, for \( X, Y \in \Theta_\alpha(n, r) \), \([X][Y] \neq 0 \implies \mathrm{co}(X) = \mathrm{ro}(Y) \) the \( r \)-th component of \( S_\alpha(0)A(j) \) becomes 
\[ S_\alpha(0, r)A(j, r) = \sum_{\gamma \in \mathcal{E}_\alpha(n, r - \sigma(A))} v^{\gamma j} \left[ S_\alpha + \text{diag} \left( \gamma + \text{ro}(A) - \sum_{1 \leq i \leq n} \alpha_i e_i^\delta \right) \right] [A + \text{diag}(\gamma)] \]
\[ = \sum_{\gamma \in \mathcal{E}_\alpha(n, r - \sigma(A))} \prod_{T \in \mathcal{E}_j, \ j \neq i} \left[ \frac{a_{i,j} + t_{i,j} - t_{i-1,j}}{t_{i,j}} \right] x_{T} \]
where 
\[ x_{T} = \sum_{\gamma \in \mathcal{E}_\alpha(n, r - \sigma(A))} v^{\gamma j} T A + \text{diag}(\gamma) \left[ \gamma + \delta_T - \delta_T^T \right] [A + \text{diag}(\gamma) + T - \tilde{T}]. \]
Let \( A + \text{diag}(\gamma) = (a_{i,j}^\gamma) \). Then \( a_{i,j}^\gamma = a_{i,j} \) for \( i \neq j \) and \( a_{i,i}^\gamma = \gamma_i \). Let \( \nu = \gamma + \delta_T - \delta_T^T \). Then 
\[ \beta(T, A + \text{diag}(\gamma)) = \sum_{1 \leq i \leq n} (a_{i,j}^\gamma - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n} (a_{i+1,j}^\gamma - t_{i,j}) t_{i,l} \]
\[ = \sum_{1 \leq i \leq n} (a_{i,j} - t_{i-1,j}) t_{i,l} + \sum_{1 \leq i \leq n} (\nu_i - t_{i,i}) t_{i,l} - \sum_{1 \leq i \leq n} (a_{i+1,j} - t_{i,j}) t_{i,l} \]
\[ - \sum_{1 \leq i \leq n} (\nu_{i+1} - t_{i+1,i+1}) t_{i,l} \]
\[ = \beta_{A,T} + \beta_{\nu,T}, \]
where \( \beta_{\nu,T} = \sum_{1 \leq i \leq n} \nu_i t_{i,l} - \sum_{1 \leq i \leq n} \nu_{i+1} t_{i,l} \) and 
\[ \beta_{A,T} = \sum_{1 \leq i \leq n} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n} a_{i+1,j} t_{i,l} + \sum_{1 \leq i \leq n} t_{i,j} t_{i,l} \]
\[ - \sum_{1 \leq i \leq n} t_{i,i}^2 + \sum_{1 \leq i \leq n} t_{i+1,i+1} t_{i,l}. \]
Clearly, we have \( \left[ \nu \right]_{\delta_T} = \nu^\delta_T (\delta_T - \nu) \left[ \nu \right]_{\delta_T} \). \( \beta_{A,T} + \delta_T \cdot \nu + j \cdot (\delta_T - \delta_T) = f_{A,T} \) and \( \beta_{\nu,T} + \nu \cdot (j - \delta_T) = \nu \cdot j_T \). This implies that

\[
x_T = \nu^{\beta_{A,T} + \delta_T \cdot \nu + j \cdot (\delta_T - \delta_T)} \sum_{\nu \in \Lambda_0(n,r \cdot \sigma(A + T^\pm - \bar{T}^\pm))} \nu^{\beta_{\nu,T} + \nu \cdot (j - \delta_T)} \left[ \nu \right]_{\delta_T} \times [A + T^\pm - \bar{T}^\pm + \text{diag}(\nu)]
\]

proving (2).

For \( A, B \in \Theta^\pm_\Delta(n) \), define the ordering on \( \Theta^\pm_\Delta(n) \) by setting

\[
A \preceq B \iff \sum_{s \leq i, t \geq j} a_{s,t} \leq \sum_{s \leq i, t \geq j} b_{s,t}, \forall i < j, \text{ and } \sum_{s \geq i, t \leq j} a_{s,t} \leq \sum_{s \geq i, t \leq j} a_{s,t}, \forall i > j.
\] (4.2.1)

**Proposition 4.3.** With the notation in (1.2.1) we have, for any \( A \in \Theta^\pm_\Delta(n) \) and \( j \in \mathbb{N}_0^n \),

\[
A^+(0)0(j)A^-(0) = \nu^{\text{co}(A^+) + \text{ro}(A^-)} A(j) + \sum_{B \in \Theta^\pm_\Delta(n)} f_{A,B}^{B,j} B(j'),
\]

where \( f_{A,B}^{B,j} \in \mathbb{Q}(v) \).

**Proof.** Let \( D^+_\Delta(n) \) be the subspace of \( D_\Delta(n) \) spanned by the elements \( u_A^+ \) for \( A \in \Theta_\Delta^+(n) \). According to [5, 6.2], the algebra \( D^+_\Delta(n) \) is generated by the elements \( u_{S_{\alpha}}^+ \) for \( \alpha \in \mathbb{N}_0^n \), where \( S_{\alpha} \) is defined as in (2.0.5). This together with 2.2 implies that \( A^+(0) \) can be written as a linear combination of monomials in \( S_{\alpha}(0) \). Thus, by 4.2 and 4.1, we conclude that there exist \( f_{A,B}^{B,j} \in \mathbb{Q}(v) \) (independent of \( r \)) such that

\[
A^+(0)0(j)A^-(0) = \sum_{B \in \Theta^\pm_\Delta(n)} f_{A,B}^{B,j} B(j').
\] (4.3.1)

On the other hand, by the triangular relation given in [3, 3.7.3], we have

\[
A^+(0,r)0(j,r)A^-(0,r) = \nu^{\text{co}(A^+) + \text{ro}(A^-)} A(j,r) + f
\]

where \( f \) is a \( \mathbb{Q}(v) \)-combination of \( |B| \) with \( B \in \Theta_\Delta(n,r) \) and \( B \prec A \). Combining this with (4.3.1) proves the assertion. □

The maps \( \zeta_r \) given in 2.2 induce an algebra homomorphism

\[
\zeta = \prod_{r \geq 0} \zeta_r : D_\Delta(n) \rightarrow S_\Delta(n) = \prod_{r \geq 0} S_\Delta(n,r).
\]

We now prove the conjecture formulated in [8, 5.5(2)].

**Theorem 4.4.** The \( \mathbb{Q}(v) \)-space \( V_\Delta(n) \) is a subalgebra of \( S_\Delta(n) \) with \( \mathbb{Q}(v) \)-basis \( B \). Moreover, the map \( \zeta \) is injective and induces a \( \mathbb{Q}(v) \)-algebra isomorphism \( U(\widehat{gl}_n) = D_\Delta(n) \cong V_\Delta(n) \).
Proof. According to [8, 4.1], the set $\mathfrak{B}$ forms a $\mathbb{Q}(v)$-basis for $\mathcal{V}_\alpha(n)$. This together with 4.3 implies that the set $\{A^+(0)0(j)A^-(0) \mid A \in \Theta_\alpha^+(n), j \in \mathbb{Z}_n\}$ forms another basis for $\mathcal{V}_\alpha(n)$. Note that, by 2.1(2), the set $\{\tilde{u}_A^+, K^j\tilde{u}_A^- \mid A \in \Theta_\alpha^+(n), j \in \mathbb{Z}_n\}$ forms a $\mathbb{Q}(v)$-basis for $\mathcal{D}_\alpha(n)$. Furthermore, by 2.2,

$$
\zeta(\tilde{u}_A^+, K^j\tilde{u}_A^-) = A^+(0)0(j)A^-(0).
$$

Thus, $\zeta$ takes a basis for $\mathcal{D}_\alpha(n)$ onto the basis for $\mathcal{V}_\alpha(n)$. It follows that $\zeta$ is injective and $\zeta(\mathcal{D}_\alpha(n)) = \mathcal{V}_\alpha(n)$. The proof is completed. \hfill $\square$

Now, the Main Theorem 1.1 follows immediately.

We end the paper with an application to the (untwisted) Ringel–Hall algebra of a cyclic quiver.

Let $\mathcal{V}_\alpha^+(n)$ be the subalgebra of $\mathcal{V}_\alpha(n)$ spanned by $A(0)$ for all $A \in \Theta_\alpha^+(n)$. Then, by 2.2, the map sending $\tilde{u}_A$ to $A(0)$ is an algebra isomorphism from $\mathcal{D}_\alpha^+(n) = \mathcal{S}_\alpha(n)$ to $\mathcal{V}_\alpha^+(n)$. In particular, the formula 4.2(2) gives the following multiplication formula in the Ringel–Hall algebra $\mathfrak{H}(n)$ over $\mathbb{Z}$:

$$
\tilde{u}_a\tilde{u}_A = \sum_{T \in \Theta_\alpha^+(n) \atop \text{ro}(T) = \alpha} v^{f_{A,T}} \prod_{1 \leq i \leq n \atop j \notin \mathbb{Z}_n, j \neq \alpha} \begin{bmatrix} a_{i,j} + t_{i+1,j} - t_{i,j} & t_{i,j} \\ t_{i,j} & t_{i,j} + 1 \end{bmatrix} \tilde{u}_{A+T-\tilde{T}^+}
$$

where

$$
f_{A,T} = \sum_{1 \leq i \leq n \atop j \notin \mathbb{Z}_n, j \neq \alpha} a_{i,j}t_{i,j} - \sum_{1 \leq i \leq n \atop j \notin \mathbb{Z}_n, j \neq \alpha} a_{i+1,j}t_{i+1,j} - \sum_{1 \leq i \leq n \atop j \notin \mathbb{Z}_n, j \neq \alpha} t_{i,j+1}t_{i,j} + \sum_{1 \leq i \leq n \atop j \notin \mathbb{Z}_n, j \neq \alpha} t_{i,j}t_{i,j}.
$$

Untwisting the multiplication for $\mathfrak{H}(n)$ yields the following.

**Theorem 4.5.** The Ringel–Hall algebra $\mathfrak{H}(n)^\circ$ is the algebra over the polynomial ring $\mathbb{Z}[q] \quad (q = v^2)$ which is spanned by the basis $\{u_A \mid A \in \Theta_\alpha^+(n)\}$ and generated by $\{u_\alpha \mid \alpha \in \mathbb{N}_n\}$ and whose (untwisted) multiplication is given by the formulas: for any $A \in \Theta_\alpha^+(n)$ and $\alpha \in \mathbb{N}_n$,

$$
u_\alpha \circ u_A = \sum_{T \in \Theta_\alpha^+(n) \atop \text{ro}(T) = \alpha} q^{\sum_{1 \leq i \leq n, l < j} (a_{i,j}t_{i,j} - t_{i,j}t_{i+1,j})} \prod_{1 \leq i \leq n \atop j \notin \mathbb{Z}_n, j \neq \alpha} \begin{bmatrix} a_{i,j} + t_{i,j} - t_{i-1,j} & t_{i,j} \\ t_{i,j} & t_{i,j} + 1 \end{bmatrix} u_{A+T-\tilde{T}^+}
$$

In other words, if the Hall polynomial $\varphi_{B_S,A}$ is nonzero, then there exists $T = (t_{i,j}) \in \Theta_\alpha^+(n)$ with $\text{ro}(T) = \alpha$ such that $B = A + T - \tilde{T}^+$ and

$$
\varphi_{B_S,A}^{A+T-\tilde{T}^+} = q^{\sum_{1 \leq i \leq n, l < j} (a_{i,j}t_{i,j} - t_{i,j}t_{i+1,j})} \prod_{1 \leq i \leq n \atop j \notin \mathbb{Z}_n, j \neq \alpha} \begin{bmatrix} a_{i,j} + t_{i,j} - t_{i-1,j} & t_{i,j} \\ t_{i,j} & t_{i,j} + 1 \end{bmatrix}.
$$

Note that, when $\alpha$ defines a simple module or $A$ defines another semisimple module, the formula coincides with the formulas given in [3, Th. 5.4.1] (built on [8, Th. 4.2]) and [4, Cor. 1.5].
For \( w \in \mathfrak{S}_{\lambda,r} \) and \( t \in \mathbb{Z} \), let
\[
\text{Inv}(w,t) = \{(i,j) \in \mathbb{Z}^2 \mid 1 + t \leq i \leq r + t, \ i < j, \ w(i) > w(j)\}
\]
\[
\text{Inv}(w) = \text{Inv}(w,0).
\]
(5.0.1)

Then the number of inversions \( \ell'(w) := |\text{Inv}(w,t)| \) is clearly independent of \( t \).

**Proposition 5.1.** If \( w = yw \) with \( y,w \in \mathfrak{S}_{\lambda,r} \) and \( s \in S \) satisfies \( \ell(w) = \ell(y) + 1 \), then \( \ell'(w) = \ell'(y) + 1 \).

**Proof.** Suppose \( s = s_{i_0} \) for some \( 1 \leq i_0 \leq r \). By the hypothesis and [19, 4.2.3], we have \( y(i_0) < y(i_0+1) \). Fix \( t \in \{0,1\} \) with \( i_0, i_0+1 \in [1+t,r+t] \). Let \( W = \text{Inv}(w,t) \) and \( \mathcal{Y} = \text{Inv}(y,t) \). We want to prove that \( |W| = |\mathcal{Y}| + 1 \). For \( j \in \mathbb{Z} \) and \( i \in [i_0 + 1, i_0 + r] \), let
\[
c(i_0,i,j) := |\{i_0, i_0 + 1\} \cap \{j\}|
\]
where \( j \) denote the unique integer in \([1 + t, r + t]\) such that \( j \equiv j \mod r \). For each \( x \in \{0,1,2\} \), let
\[
\mathcal{W}_x = \{(i,j) \in W \mid c(i_0,i,j) = x\} \quad \text{and} \quad \mathcal{Y}_x = \{(i,j) \in \mathcal{Y} \mid c(i_0,i,j) = x\}
\]

Then we have disjoint unions \( W = \mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2 \) and \( \mathcal{Y} = \mathcal{Y}_0 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \).

For \( j \in \mathbb{Z} \) and \( i \in [i_0 + 1, i_0 + r] \), if \( c(i_0,i,j) = 0 \), then \( w(i) = y(i) \) and \( w(j) = y(j) \). This implies \( W_0 = \mathcal{Y}_0 \). Hence, \( |W_0| = |\mathcal{Y}_0| \). Since \( y(i_0) < y(i_0 + 1) \), it follows that
\[
W_2 = \{(i_0, i_0 + 1 + kr) \mid k \in \mathbb{Z}_{\geq 0}, \ y(i_0 + 1) > y(i_0 + kr)\},
\]
while
\[
\mathcal{Y}_2 = \{(i_0 + 1, i_0 + kr) \mid k \in \mathbb{Z}_{\geq 0}, \ y(i_0 + 1) > y(i_0 + kr)\}.
\]
Hence, \( |W_2| = |\mathcal{Y}_2| + 1 \). It remains to prove that \( |W_1| = |\mathcal{Y}_1| \). In this case, we have

\[
W_1 = W_{1,(i_0,\bullet)} \cup W_{1,(i_0+1,\bullet)} \cup W_{1,(\bullet,i_0)} \cup W_{1,(\bullet,i_0+1)}
\]
where \( W_{1,(i_0,\bullet)} = \{(i,j) \in W_1 \mid i = i_0\} \), etc. Define \( \mathcal{Y}_{1,(\bullet,\bullet)} \) similarly to get a similar partition for \( \mathcal{Y}_1 \). Then the condition \( y(i_0) < y(i_0 + 1) \) implies the following
\[
\mathcal{Y}_{1,(i_0,\bullet)} \subseteq W_{1,(i_0,\bullet)}, \quad \mathcal{Y}_{1,(i_0+1,\bullet)} \supseteq W_{1,(i_0+1,\bullet)}
\]
\[
\mathcal{Y}_{1,(\bullet,i_0)} \supseteq W_{1,(\bullet,i_0)}, \quad \mathcal{Y}_{1,(\bullet,i_0+1)} \subseteq W_{1,(\bullet,i_0+1)}.
\]
But, since
\[
W_{1,(i_0,\bullet)} \setminus \mathcal{Y}_{1,(i_0,\bullet)} = \{(i_0,j) \mid i_0 < j, j \in \mathbb{Z}, j \notin \{i_0, i_0 + 1\}, y(i_0) < y(j) < y(i_0 + 1)\}
\]
and
\[
\mathcal{Y}_{1,(i_0+1,\bullet)} \setminus W_{1,(i_0+1,\bullet)} = \{(i_0 + 1,j) \mid i_0 + 1 < j, j \in \mathbb{Z}, j \notin \{i_0, i_0 + 1\}, y(i_0) < y(j) < y(i_0 + 1)\},
\]
it follows that
\[
|W_{1,(i_0,\bullet)}| + |W_{1,(i_0+1,\bullet)}| = |\mathcal{Y}_{1,(i_0,\bullet)}| + |\mathcal{Y}_{1,(i_0+1,\bullet)}|.
\]
Similarly, one proves that
\[
|W_{1,(\bullet,i_0)}| + |W_{1,(\bullet,i_0+1)}| = |\mathcal{Y}_{1,(\bullet,i_0)}| + |\mathcal{Y}_{1,(\bullet,i_0+1)}|.
\]
This completes the proof.

The following result given in [3, (3.2.1.1)] without proof follows immediately.

**Corollary 5.2.** For \( w \in \mathfrak{S}_{\lambda,r} \) we have \( \ell(w) = \ell'(w) \).
We now generalise the construction for the shortest representatives of double cosets of the symmetric group \([7, \S 3]\) to the affine case. For \(A \in \Theta_\lambda(n, r)\) with \(\lambda = \rho(A)\) and \(\mu = \co(A)\), define a pseudo matrix \(A^\perp\) as follows: the entry \(a_{i,j}\) is replaced by the sequence

\[
\xi_{i,j} = \xi_{i,j}(A) = \left(\lambda_{k_0,i_0}-1 + \sum_{t \leq j-1} a_{i,t} + 1, \ldots, \lambda_{k_0,i_0}-1 + \sum_{t \leq j-1} (a_{i,j}-1), \lambda_{k_0,i_0}-1 + \sum_{t \leq j} a_{i,t}\right)
\]

where \(i = i_0 + k_0 n, j \in \mathbb{Z}\) with \(1 \leq i_0 \leq n\) and \(k_0 \in \mathbb{Z}\). We define \(\widetilde{y}_A \in \mathcal{S}_\lambda^\perp\) by

\[
\widetilde{y}_A(i + kr) = a_i + kr, \text{ for all } 1 \leq i \leq r, k \in \mathbb{Z},
\]

where \((a_1, a_2, \ldots, a_r)\) is the sequence obtained by reading the numbers in column 1 inside the subsequences from left to right and from top to bottom, and then in column 2, etc., and then in column \(n\). In other words, it is the sequence obtained by ignoring 0’s from \(((\xi_{k,1})_{k \in \mathbb{Z}}, (\xi_{k,2})_{k \in \mathbb{Z}}, \ldots, (\xi_{k,n})_{k \in \mathbb{Z}})\) with \((\xi_{k,i})_{k \in \mathbb{Z}} = (\cdots, \xi_{1,i}, \xi_{2,i}, \cdots, \xi_{n,i}, \cdots)\).

We are ready to prove Lemma 3.2(2).

**Proposition 5.3.** Let \(A \in \Theta_\lambda(n, r)\). Then \(\widetilde{y}_A\) is the shortest representative of the double coset defined by \(A\), i.e., \(y_A = \widetilde{y}_A\), and

\[
\ell(y_A) = \sum_{1 \leq i \leq n, 1 \leq j \leq \ell} a_{ij}a_{kl}.
\]

**Proof.** Let \(\lambda = \rho(A)\) and \(\mu = \co(A)\). For \(w \in \mathcal{S}_\lambda y_A \mathcal{S}_\mu\), let

\[
\mathcal{N} = \bigcup_{1 \leq i \leq n, 1 \leq j \leq \ell} (R^\lambda_i \cap w(R^\mu_j)) \times (R^\lambda_k \cap w(R^\mu_l)) \text{ and } N = |\mathcal{N}| = \sum_{1 \leq i \leq n, 1 \leq j \leq \ell} a_{ij}a_{kl}.
\]

Then there is a injective map \(\varphi_w\) defined as follows:

\[
\varphi_w : \mathcal{N} \longrightarrow \text{Inv}(w), \; (c, d) \longmapsto (w^{-1}(d), w^{-1}(c)).
\]

Hence by 5.2 we have \(N \leq \ell(w)\). In particular, we have \(\ell(y_A) \geq N\).

For \(i, j \in \mathbb{Z}\) let \(C_{ij}(A)\) denote the set of members of the sequence \(\xi_{ij}(A)\). By definition we have, for \(i \in \mathbb{Z}\) and \(1 \leq j \leq n\),

\[
R^\lambda_i = \bigcup_{t \in \mathbb{Z}} C_{it}(A), \quad \text{and} \quad \widetilde{y}_A(R^\mu_j) = \bigcup_{k \in \mathbb{Z}} C_{kj}(A).
\]

It is easy to see that \(C_{ij}(A) + tr = C_{i+tn,j+tn}(A)\) for \(i, j, t \in \mathbb{Z}\). Hence, for \(j = j_0 + tn\) with \(1 \leq j_0 \leq n\) and \(t \in \mathbb{Z}\),

\[
\widetilde{y}_A(R^\mu_j) = tr + w_A(R^\mu_{j_0}) = tr + \bigcup_{k \in \mathbb{Z}} C_{kj_0}(A) = \bigcup_{k \in \mathbb{Z}} C_{k+tjn, j+tn}(A) = \bigcup_{k \in \mathbb{Z}} C_{kj}(A).
\]

Thus, we have \(C_{ij}(A) = R^\lambda_i \cap \widetilde{y}_A(R^\mu_j)\) for \(i, j \in \mathbb{Z}\) and so \(a_{ij} = |R^\lambda_i \cap \widetilde{y}_A(R^\mu_j)|\) for \(i, j \in \mathbb{Z}\). This implies that \(\widetilde{y}_A \in \mathcal{S}_\lambda y_A \mathcal{S}_\mu\) and, hence, \(\ell(\widetilde{y}_A) \geq \ell(y_A) \geq N\). Observe that \(\widetilde{y}_A(C_{ji}(A)) = C_{ij}(A)\) for \(i, j \in \mathbb{Z}\), where \(A^\perp\) is the transpose matrix of \(A\).

We now prove that \(\ell(\widetilde{y}_A) = N\) by showing that \(\varphi_{\widetilde{y}_A}\) is surjective. Let \((a, b) \in \text{Inv}(\widetilde{y}_A)\). Since \(\mathbb{Z} = \bigcup_{s, t \in \mathbb{Z}} C_{st}(A)\), there exist \(i, j, k, l \in \mathbb{Z}\) such that \(\widetilde{y}_A(a) \in C_{kt}(A)\) and \(\widetilde{y}_A(b) \in C_{ij}(A)\). Since \(1 \leq a \leq r\) we have \(1 \leq l \leq n\). Since \(\widetilde{y}_A(a) > \widetilde{y}_A(b)\) we have either \(i < k\) or \(i = k\) and \(j < l\). On
the other hand, the conditions $a \in \tilde{y}_A^{-1}(C_{kl}(A)) = C_{lk}('A)$, $b \in \tilde{y}_A^{-1}(C_{ij}(A)) = C_{ji}('A)$ and $a < b$ imply either $l < j$ or $l = j$ and $k < i$. Hence, we must have $i < k$ and $l < j$. Therefore, the map $\varphi_{\tilde{y}_A}$ is bijective, proving $\ell(\tilde{y}_A) = N$. □

References


School of Mathematics and Statistics, University of New South Wales, Sydney 2052, Australia.
E-mail address: j.du@unsw.edu.au

Department of Mathematics, Tongji University, Shanghai, 200092, China.
E-mail address: q.fu@hotmail.com, q.fu@tongji.edu.cn