THE INTEGRAL QUANTUM LOOP ALGEBRA OF \( \mathfrak{gl}_n \)

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Abstract. We will construct the Lusztig form for the quantum loop algebra of \( \mathfrak{gl}_n \) by proving the conjecture [4, 3.8.6] and establish partially the Schur–Weyl duality at the integral level in this case. We will also investigate the integral form of the modified quantum affine \( \mathfrak{gl}_n \) by introducing an affine stabilisation property and will lift the canonical bases from affine quantum Schur algebras to a canonical basis for this integral form. As an application of our theory, we will also discuss the integral form of the modified extended quantum affine \( \mathfrak{sl}_n \) and construct its canonical basis to verify a conjecture of Lusztig in this case.

1. Introduction

Let \( Z = \mathbb{Z}[v, v^{-1}] \) be the integral Laurent polynomial ring. It is well known that the Lusztig form \( U_Z \) of a quantum enveloping \( Q(v) \)-algebra \( U \) associated with a Cartan matrix of finite or affine type is a \( Z \)-free subalgebra generated by divided powers of simple root vectors \( E_\alpha, F_\alpha \) together with group-like elements \( K_{\alpha, \alpha} \). In particular, there is a triangular decomposition \( U_Z = U_Z^+ \cdot U_Z^0 \cdot U_Z^- \) where, in the simply-laced case, the 0-part \( U_Z^0 \) of this form is generated by \( K_\alpha \) and \( [K_\alpha, 0] \).

We now consider the quantum loop algebra \( U(\widehat{\mathfrak{gl}}_n) \). It contains a proper subalgebra \( U = U_0(n) \) generated by \( E_i = E_{\alpha_i}, F_i = F_{\alpha_i} \) and \( K_{i, i}^\pm, 1 \leq i \leq n \), where \( K_i K_{i+1}^{-1} = K_{\alpha, \alpha} \) with \( K_{n+1} = K_1 \). This is called the “extended” quantum affine \( \mathfrak{sl}_n \) in [4] which is also investigated in [28] (cf. the definition in [28, 7.7]). Note that the subalgebra generated by \( E_i, F_i \) is usually called the quantum enveloping algebra of affine \( \mathfrak{sl}_n \) type or the quantum loop algebra of \( \mathfrak{sl}_n \) (see, e.g., [28, 9.3] or [4, §1.3]). If \( U_Z^+ \) (resp., \( U_Z^- \)) denotes the \( Z \)-subalgebra generated by divided powers \( E_i^{(m)} \) (resp., \( F_i^{(m)} \)) and \( U_Z^0 \) denotes the \( Z \)-subalgebra generated by \( K_i \) and \( [K_i, 0] \) (\( t \in \mathbb{N}, 1 \leq i \leq n \)), then the \( Z \)-submodule \( U_Z = U_Z^+ \cdot U_Z^0 \cdot U_Z^- \) is a \( Z \)-free subalgebra of \( U \) which is the Lusztig form of \( U \) mentioned above. Now, naturally, one would ask what is a natural Lusztig form for \( U(\widehat{\mathfrak{gl}}_n) \)?

By using Drinfeld’s presentation for \( U(\widehat{\mathfrak{gl}}_n) \), a so-called restricted integral form \( U^{\text{res}}_v(\widehat{\mathfrak{gl}}_n) \) was constructed over \( \mathbb{C}[v, v^{-1}] \) by Frenkel–Mukhin in [13, §7.2]. However, it is not clear from the construction whether \( U^{\text{res}}_v(\widehat{\mathfrak{gl}}_n) \) is a Hopf algebra. Another integral form is constructed in [4,
by using a double Ringel–Hall algebra presentation for \( \widehat{\mathfrak{gl}}_n \). This integral form is the tensor product of the Lusztig form \( 'U_Z \) of \( 'U \) with an integral central subalgebra. This is a Hopf subalgebra but not large enough to have integral affine quantum Schur algebras as its quotients; see example [4, 5.3.8].

However, there is a natural candidate constructed in [4, §3.8]. By the double Ringel–Hall algebra presentation, we have a triangular decomposition:

\[
U(\widehat{\mathfrak{gl}}_n) \cong \mathcal{D}_\triangle(n) = \mathcal{H}_\triangle(n) \cdot U^0 \cdot \mathcal{H}_\triangle(n)^{\text{op}},
\]

where \( \mathcal{H}_\triangle(n) \) is a Ringel–Hall algebra over \( \mathbb{Q}(v) \) associated with a cyclic quiver and \( U^0 = \mathbb{Q}(v)[K_n^{\pm1}, \ldots, K_n^{\pm1}] \) is the 0-part of \( U(\widehat{\mathfrak{gl}}_n) \). The candidate we proposed is to use the (integral) Ringel–Hall algebra \( \mathcal{H}_\triangle(n) \) over \( \mathbb{Z} \) and the 0-part \( U^0 \) defined above to form the \( \mathbb{Z} \)-free submodule

\[
\mathcal{D}_\triangle(n)_\mathbb{Z} := \mathcal{H}_\triangle(n) \cdot U^0 \cdot \mathcal{H}_\triangle(n)^{\text{op}}.
\]

We conjectured in [4, 3.8.6] that \( \mathcal{D}_\triangle(n)_\mathbb{Z} \) is a \( \mathbb{Z} \)-subalgebra of \( \mathcal{D}_\triangle(n) \). If the conjecture is true, then \( \mathcal{D}_\triangle(n)_\mathbb{Z} \) is a Hopf subalgebra having integral affine quantum Schur algebras as its quotients.

In this paper, we will prove this conjecture. The proof is a beautiful application of a recent resolution of another conjecture, a realisation conjecture for quantum affine \( \mathfrak{gl}_n \), by the authors [11], together with some successful attempts in the classical case [14, 15] (see also [16]). The realisation conjecture is a natural affine generalisation of a new construction for quantum \( \mathfrak{gl}_n \) via quantum Schur algebras by A.A. Beilinson, G. Lusztig and R. MacPherson (BLM) in [1]. This remarkable work has important applications to the investigation of integral quantum Schur–Weyl reciprocity [12]. This reciprocity at non-roots of unity was formulated in [20] and its integral version was given in [8, 12], built on the work [1] and the Kazhdan–Lusztig cell theory.

Attempts to generalise the BLM work have been made by Ginzburg–Vasserot [18], Lusztig [28], etc. These constructions are geometric in nature, following BLM’s geometric construction, but cannot resolve a realisation for the entire quantum affine \( \mathfrak{gl}_n \). The main obstacle is that \( U(\widehat{\mathfrak{gl}}_n) \) cannot be generated by simple root vectors or simple generators. In [11], we discovered certain key multiplication formulas by semisimple generators via the affine Hecke algebra and affine quantum Schur algebras. This allows, by modifying BLM’s approach, to introduce a new algebra \( \mathcal{V}_\triangle(n) \) by a basis together with explicit multiplication formulas of basis elements by semisimple generators. This algebra is isomorphic to \( \mathcal{D}_\triangle(n) \) and hence to \( U(\widehat{\mathfrak{gl}}_n) \).

We now construct an integral \( \mathbb{Z} \)-subalgebra \( \mathcal{V}_\triangle(n)_\mathbb{Z} \) of \( \mathcal{V}_\triangle(n) \) and then prove that the image of \( \mathcal{V}_\triangle(n)_\mathbb{Z} \) in \( \mathcal{D}_\triangle(n) \) coincides with \( \mathcal{D}_\triangle(n)_\mathbb{Z} \). In this way we prove that \( \mathcal{D}_\triangle(n)_\mathbb{Z} \) is a subalgebra. As an immediate application, the \( \mathbb{Q}(v) \)-algebra epimorphism \( \zeta_r \) given in [4, Th. 3.8.1] restricts to a \( \mathbb{Z} \)-algebra epimorphism \( \zeta_r \) from \( \mathcal{D}_\triangle(n)_\mathbb{Z} \) to the affine quantum Schur algebra \( S_{\triangle}(n, r)_\mathbb{Z} \). This establishes partially the Schur–Weyl duality at the integral level and, hence, at roots of unity.

\footnote{It is denoted by \( \tilde{\mathcal{D}}_\triangle(n) \) in [4, (3.8.1.1)], while \( \mathcal{D}_\triangle(n) \) denote the tensor product of \( 'U_Z \) with the integral central subalgebra in [4, 2.4.4].}
There is another application of the key multiplication formulas mentioned above. In [1], the \( \mathbb{Q}(v) \)-algebra \( K(n) \) was constructed as a result of a stabilisation property. The algebra \( K(n) \) is in fact isomorphic to the modified quantum group \( \hat{U}(gl_n) \). We will prove that a stabilisation property continue to hold in the affine case. Thus, we may also introduce a new \( \mathbb{Q}(v) \)-algebra \( K_\Delta(n) \), which is isomorphic to the modified quantum group \( \hat{U}(gl_n) \), and realise \( U(\hat{gl}_n) \) as a subalgebra of the completion algebra \( \hat{K} \). In this way, we obtain an (unmodified!) affine generalisation of BLM’s construction. We will further discuss the integral form \( K_\Delta(n)_{Z} \) of \( K_\Delta(n) \) which is a realisation of \( \hat{K}_\Delta(n)_{Z} \) (see Theorem 6.6) and construct its canonical basis as a lifting of the canonical bases for affine quantum Schur algebras. Applying our theory to the extended quantum affine \( s_{lt} \), we will introduce the canonical basis for the modified quantum group \( \hat{U}_\Delta(n) \) and verify in this case a conjecture of Lusztig [28, 9.3] which has been already proved in [32] (cf. [29, 7.9]).

The sections of the paper are organised as follows:

1. Introduction
2. The double Ringel–Hall algebra presentation
3. A BLM type presentation
4. Some integral multiplication formulas
5. Lusztig form of \( U(\hat{gl}_n) \) and integral affine quantum Schur–Weyl reciprocity
6. The affine BLM algebra \( K_\Delta(n)_{Z} \)
7. Canonical bases for the integral modified quantum affine \( gl_n \)
8. Application to a conjecture of Lusztig.

**Notation 1.1.** For a positive integer \( n \), let \( \Theta_{\Delta}(n) \) (resp., \( \tilde{\Theta}_{\Delta}(n) \)) be the set of all matrices \( A = (a_{i,j})_{i,j \in \mathbb{Z}} \) with \( a_{i,j} \in \mathbb{N} \) (resp. \( a_{i,j} \in \mathbb{Z}, a_{i,j} \geq 0 \) for all \( i \neq j \)) such that

\[
\begin{align*}
(a) & \quad a_{i,j} = a_{i+n,j+n} \quad \text{for} \quad i, j \in \mathbb{Z}; \\
(b) & \quad \text{for every} \quad i \in \mathbb{Z}, \text{both sets} \quad \{j \in \mathbb{Z} \mid a_{i,j} \neq 0\} \quad \text{and} \quad \{j \in \mathbb{Z} \mid a_{j,i} \neq 0\} \quad \text{are finite.}
\end{align*}
\]

Let \( \mathbb{Z}_{\Delta}^{n} = \{ (\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i-n} \quad \text{for} \quad i \in \mathbb{Z} \} \) and \( \mathbb{N}_{\Delta}^{n} = \{ (\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}_{\Delta}^{n} \mid \lambda_i \geq 0 \quad \text{for} \quad i \in \mathbb{Z} \} \). We will sometimes identify \( \mathbb{Z}_{\Delta}^{n} \) with \( \mathbb{Z}^{n} \) via the natural bijection \( b : \mathbb{Z}_{\Delta}^{n} \rightarrow \mathbb{Z}^{n} \) defined by sending \( j \) to \( b(j) = (j_1, \cdots, j_n) \). Define an order relation \( \leq \) and “dot” product on \( \mathbb{Z}_{\Delta}^{n} \) by

\[
(1.1.1) \quad \lambda \leq \mu \quad \iff \quad \lambda_i \leq \mu_i \quad (1 \leq i \leq n) \quad \text{and} \quad \lambda \cdot \mu = \lambda_1 \mu_1 + \cdots + \lambda_n \mu_n = b(\lambda) \cdot b(\mu).
\]

We say that \( \lambda < \mu \) if \( \lambda \leq \mu \) and \( \lambda \neq \mu \).

Let \( \mathbb{Q}(v) \) be the fraction field of \( \mathbb{Z} = \mathbb{Z}[v, v^{-1}] \). For integers \( N, t \) with \( t \geq 0 \), define Gaussian polynomials and their symmetric version in \( \mathbb{Z} \): \( \left[ \begin{array}{c} N \\ t \end{array} \right] = \prod_{1 \leq i \leq t} v^{(N+i-1)-1} \) and \( \left[ \begin{array}{c} N \\ t \end{array} \right] = v^{-t(N-t)} \left[ \begin{array}{c} N \\ t \end{array} \right] \). For \( \mu \in \mathbb{Z}_{\Delta}^{n} \) and \( \lambda \in \mathbb{N}_{\Delta}^{n} \), let \( \left[ \begin{array}{c} \mu \\ \lambda \end{array} \right] = \prod_{1 \leq i \leq n} \left[ \begin{array}{c} \mu_i \\ \lambda_i \end{array} \right] \) and let \( \left[ \begin{array}{c} \mu \\ \lambda \end{array} \right] = \prod_{1 \leq i \leq n} \left[ \begin{array}{c} \mu_i \\ \lambda_i \end{array} \right] \). The
following identity holds (see [16, 3.3]): for any \( \lambda, \mu \in \mathbb{N}_0^n \) and \( \alpha, \beta \in \mathbb{Z}_0^n \),

\[
\begin{array}{c}
\left[ \frac{\alpha + \beta}{\lambda} \right] = \sum_{\mu \in \mathbb{N}_0^n, \mu \leq \lambda} v^{\alpha(\lambda - \mu) - \mu, \beta} \left[ \frac{\alpha}{\mu} \right] \left[ \frac{\beta}{\lambda - \mu} \right]; \\
\left[ \frac{\alpha}{\lambda} \right] \left[ \frac{\alpha}{\mu} \right] = \sum_{\gamma \in \mathbb{N}_0^n, \gamma \leq \lambda, \gamma \leq \mu} v^{\beta, \alpha - \alpha, \gamma} \left[ \frac{\alpha}{\lambda + \mu - \gamma} \right] \left[ \frac{\alpha}{\gamma, \lambda - \gamma, \mu - \gamma} \right];
\end{array}
\]

(1.1.2)

see [5, Exercises 0.14 and 0.15] for a proof in the case of Gaussian polynomials.

2. The Double Ringel–Hall algebra presentation

Let \( \triangle(n) \) (\( n \geq 2 \)) be the cyclic quiver with vertex set \( I = \mathbb{Z}/n\mathbb{Z} = \{1, 2, \ldots, n\} \) and arrow set \( \{i \rightarrow i + 1 \mid i \in I\} \). Note that we will regard \( I \) as an abelian group as well as a subset of \( \mathbb{Z} \) depending on context.

Let \( \mathbb{F} \) be a field. For \( i \in I \) and \( j \in \mathbb{Z} \) with \( i < j \), let \( S_i \) denote the one-dimensional representation of \( \triangle(n) \) with \( (S_i)_i = \mathbb{F} \) and \( (S_i)_k = 0 \) for \( i \neq k \) and \( M^{i,j} \) the unique indecomposable nilpotent representation of dimension \( j - i \) with top \( S_i \). Let

\[
\Theta_\triangle^+(n) = \{ A \in \Theta_\triangle(n) \mid a_{i,j} = 0 \text{ for } i \geq j \}.
\]

**Lemma 2.1.** For any \( A = (a_{i,j}) \in \Theta_\triangle^+(n) \), let

\[
M(A) = M_\mathbb{F}(A) = \bigoplus_{1 \leq i < n, i < j} a_{i,j} M^{i,j}.
\]

Then \( \mathcal{M} = \{ [M(A)] \}_{A \in \Theta_\triangle^+(n)} \) forms a complete set of isomorphism classes of finite dimensional nilpotent representations of \( \triangle(n) \).

Let \( d(A) \in NI = \mathbb{N}^n \) be the dimension vector of \( M(A) \). For \( a = (a_i) \in \mathbb{Z}_0^n \) and \( b = (b_i) \in \mathbb{Z}_0^n \), the Euler form associated with the cyclic quiver \( \triangle(n) \) is the bilinear form \( \langle -,- \rangle : \mathbb{Z}_0^n \times \mathbb{Z}_0^n \rightarrow \mathbb{Z} \) defined by

\[
\langle a, b \rangle = \sum_{i \in I} a_i b_i - \sum_{i \in I} a_i b_{i+1}.
\]

By [31], for \( A, B, C \in \Theta_\triangle^+(n) \), the Hall polynomial \( \varphi^C_{A,B} \in \mathbb{Z}[v^2] \) is defined such that, for any finite field \( \mathbb{F}_q \), \( \varphi^C_{A,B}|_{v^2=q} \) is equal to the number of submodules \( N \) of \( M_{\mathbb{F}_q}(C) \) satisfying \( N \cong M_{\mathbb{F}_q}(B) \) and \( M_{\mathbb{F}_q}(C)/N \cong M_{\mathbb{F}_q}(A) \).

The (generic) twisted Ringel–Hall algebra \( \mathcal{S}_\triangle(n)_{\mathbb{Z}} \) of \( \triangle(n) \) is, by definition, the \( \mathbb{Z} \)-algebra spanned by basis \( \{ u_A = u_{[M(A)]} \mid A \in \Theta_\triangle^+(n) \} \) whose multiplication is defined by, for all \( A, B \in \Theta_\triangle^+(n) \),

\[
u_A u_B = v^{(d(A), d(B))} \sum_{C \in \Theta_\triangle^+(n)} \varphi^C_{A,B} u_C.
\]

Base change gives the \( \mathbb{Q}(v) \)-algebra \( \mathcal{S}_\triangle(n) = \mathcal{S}_\triangle(n)_{\mathbb{Z}} \otimes \mathbb{Q}(v) \).
We now describe the semisimple generators $u_\lambda = u_{[S_\lambda]}$ $(\lambda \in \mathbb{N}_0^n)$ of $\mathfrak{S}_\lambda(n)\mathbb{Z}$, where $S_\lambda := \oplus_{i=1}^n \lambda_i S_i$ is a semisimple representation of $\triangle(n)$.\footnote{For emphasising on semisimple generators, we will use the same notation to denote the matrix in (2.1.1) defining $S_\lambda$; see, e.g., Theorem 3.3.}

On the set $\mathcal{M}$ of isoclasses of finite dimensional nilpotent representations of $\triangle(n)$, define a multiplication $\ast$ by $[M] \ast [N] = [M \ast N]$ for any $[M], [N] \in \mathcal{M}$, where $M \ast N$ is the generic extension of $M$ by $N$. By [3, 30] $\mathcal{M}$ is a monoid with identity $1 = [0]$.

An element $\lambda$ in $\mathbb{N}_0^n$ is called sincere if $\lambda_i > 0$ for all $i \in \mathbb{Z}$. For $1 \leq i \leq n$ let $e_i^\lambda \in \mathbb{N}_0^n$ be the element satisfying $(e_i^\lambda)_j = \delta_{i,j}$ for $j \in \mathbb{Z}$. Here $i$ is the congruence class of $i$ modulo $n$. Let

$$\tilde{I} = \{e_1^\lambda, e_2^\lambda, \ldots, e_n^\lambda\} \cup \{\text{all sincere vectors in } \mathbb{N}_0^n\}.$$ 

Let $\tilde{\Sigma}$ be the set of words on the alphabet $\tilde{I}$.

There is a natural surjective map $\varphi^+ : \tilde{\Sigma} \to \Theta_\lambda^+(n)$ ([6, 3.3]) by taking $w = a_1 a_2 \cdots a_m$ to $\varphi^+(w)$, where $\varphi^+(w) \in \Theta_\lambda^+(n)$ is defined by

$$[S_{a_1}] \ast \cdots \ast [S_{a_m}] = [M(\varphi^+(w))].$$

For $A \in \Theta_\lambda^+(n)$, let

$$\tilde{u}_A = u^{\dim \text{End}(M(A)) - \dim M(A)}u_A.$$ 

For $\lambda \in \mathbb{N}_0^n$ let $\tilde{u}_\lambda = \tilde{u}_{[S_\lambda]}$. Any word $w = a_1 a_2 \cdots a_m$ in $\tilde{\Sigma}$ can be uniquely expressed in the tight form $w = b_1^{x_1} b_2^{x_2} \cdots b_t^{x_t}$ where $x_i = 1$ if $b_i$ is sincere, and $x_i$ is the number of consecutive occurrences of $b_i$ if $b_i \in \{e_1^\lambda, e_2^\lambda, \ldots, e_n^\lambda\}$. For $w = a_1 a_2 \cdots a_m \in \tilde{\Sigma}$ with the tight form $b_1^{x_1} b_2^{x_2} \cdots b_t^{x_t}$, define the associated monomials:

$$\tilde{u}_{(w)} = \tilde{u}_{x_1 b_1} \tilde{u}_{x_2 b_2} \cdots \tilde{u}_{x_t b_t} \in \mathfrak{S}_\lambda(n)\mathbb{Z}.$$

Following [1, 3.5] and [10] we may define the order relation $\succeq$ on $M_{\lambda,n}(\mathbb{Z})$ as follows. For $A \in M_{\lambda,n}(\mathbb{Z})$ and $i \neq j \in \mathbb{Z}$, let

$$\sigma_{i,j}(A) = \begin{cases} \sum_{s \leq t \leq j} a_{s,t}, & \text{if } i < j; \\ \sum_{s \geq t \leq j} a_{s,t}, & \text{if } i > j. \end{cases}$$

For $A, B \in M_{\lambda,n}(\mathbb{Z})$, define

$$(2.1.2) \quad B \succeq A \text{ if and only if } \sigma_{i,j}(B) \leq \sigma_{i,j}(A) \text{ for all } i \neq j.$$ 

Put $B \prec A$ if $B \succeq A$ and, for some pair $(i, j)$ with $i \neq j$, $\sigma_{i,j}(B) < \sigma_{i,j}(A)$.

Associated each $A \in \Theta_\lambda^+(n)$ to a distinguished word $w_A$ (see [6, (9.1)]), the following triangular relation relative to $\succeq$ between the monomial basis $\{\tilde{u}_{(w_A)}\}_{A \in \Theta_\lambda^+(n)}$ and the defining basis $\{\tilde{u}_A\}_{A \in \Theta_\lambda^+(n)}$ holds (see [6, (9.2)], [10, 6.2]):
Theorem 2.3. \( A \in \Theta^+_\Delta(n) \), there exist \( w_A \in \widetilde{\sum} \) such that \( \varphi^+(w_A) = A \) and

\[
\tilde{u}_{(w_A)} = u_A + \sum_{B \in \Theta^+_\Delta(n), B < A, d(A) = d(B)} f_{B,A} \tilde{u}_B.
\]

where \( f_{B,A} \) is \( \in \mathbb{Z} \). In particular, \( \mathcal{H}_\Delta(n) \) is generated by \( u_\lambda \) for \( \lambda \in \mathbb{N}_0^n \) with a monomial basis \( \{ \tilde{u}_{(w_A)} \mid A \in \Theta^+_\Delta(n) \} \).

The Hall algebra and its opposite algebra can be used to describe the \( \pm \)-part of quantum affine \( \mathfrak{gl}_n \). Let \( \mathcal{D}_\Delta(n) \) be the (reduced) double Ringel–Hall algebra of the cyclic quiver \( \Delta(n) \) over \( \mathbb{Q}(v) \) (cf. [34] and [4, 2.1.3.2]). Then it has a triangular decomposition:

\[
\mathcal{D}_\Delta(n) \cong \mathcal{D}_\Delta^+(n) \otimes \mathcal{D}_\Delta^0(n) \otimes \mathcal{D}_\Delta^-(n)
\]

with \( \mathcal{D}_\Delta^+(n) = \mathcal{H}_\Delta(n) \), \( \mathcal{D}_\Delta^-(n) = \mathcal{H}_\Delta(n)^{\text{op}} \), and \( \mathcal{D}_\Delta^0(n) = \mathbb{Q}(v)[K_1^{\pm 1}, \ldots, K_n^{\pm 1}] \). We will add superscript \( + \) or \( - \) to \( u_A, \tilde{u}_A, u_\lambda, u_{(w)} \), etc. for the corresponding objects in \( \mathcal{D}_\Delta^+(n) \) or \( \mathcal{D}_\Delta^-(n) \). Thus, \( \mathcal{D}_\Delta(n) \) has basis \( \{ \tilde{u}_A \}_{A \in \Theta^+_\Delta(n)} \), generators \( u_\lambda, \lambda \in \mathbb{N}_0^n \) and monomials \( \tilde{u}_A \).

Note that it is also natural to use the notation \( \{ \tilde{u}_A : A \in \Theta^+_\Delta(n) \} \) for a basis for \( \mathcal{D}_\Delta^+(n) \) and the notation \( \{ \tilde{u}_B : B \in \Theta^-_\Delta(n) \} \) for the corresponding basis for \( \mathcal{D}_\Delta^-(n) \), where

\[
\Theta^+_\Delta(n) = \{ A \in \Theta_\Delta(n) \mid a_{i,j} = 0 \text{ for } i \leq j \}.
\]

With such notations, the matrix transpose induces the anti-isomorphism

\[
\tau : \mathcal{D}_\Delta^+(n) \to \mathcal{D}_\Delta^-(n), \quad \tilde{u}_A \mapsto \tilde{u}_A^T.
\]

For \( A \in \Theta_\Delta(n) \), we write

\[
A = A^+ + A^0 + A^- \quad \Rightarrow \quad A^\pm = A^+ + A^-
\]

where \( A^+ \in \Theta^+_\Delta(n) \), \( A^- \in \Theta^-_\Delta(n) \) and \( A^0 \) is a diagonal matrix.

We have the following (not so elegant) presentation for quantum affine \( \mathfrak{gl}_n \) via the double Ringel–Hall algebra; see [4, 2.5.3, 2.6.1, 2.6.7 and 2.3.6(2)].

Theorem 2.3. (1) The (Hopf) algebra \( \mathcal{D}_\Delta(n) \) is isomorphic to Drinfeld’s algebra \( \mathcal{U}^\wedge(\mathfrak{gl}_n) \). It is the algebra over \( \mathbb{Q}(v) \) which is spanned by basis

\[
\{ u^+_A K^j u^-_B \mid A, B \in \Theta^+_\Delta(n), j \in \mathbb{Z}_0^n \}, \quad \text{where } K^j = K_1^{j_1} \cdots K_n^{j_n},
\]

and is generated by \( u_\lambda^+, K_\mu^{\pm 1}, u_\lambda^- (\lambda, \mu \in \mathbb{N}_0^n, 1 \leq i \leq n) \), and whose multiplication is given by the following relations:

(a) \( K_i K_j = K_j K_i \), \( K_i K_i^{-1} = 1 \);
(b) \( K^j u^+_A = v^{(d(A),j)} u^+_A K^j \), \( u^+_A K^j = v^{(d(A),j)} K^j u^+_A \);
(c) \( u^+_A u^+_A = \sum_{C \in \Theta^+_\Delta(n)} v^{(\lambda,d(A),j)} C^C A^C u^+_C \);
(d) \( u^-_A u^-_A = \sum_{C \in \Theta^+_\Delta(n)} v^{(\mu,d(A),j)} C^C A^C u^-_C \);
(e) \( u_\mu^+ u_\lambda^- - u_\lambda^+ u_\mu^- = \sum_{\alpha, \gamma \in \mathbb{N}_0 \cup \{0\}} \sum_{0 \leq \gamma \leq \alpha} x_{\alpha, \gamma} K^{2\gamma - \alpha} u_{\lambda - \alpha}^- u_{\mu - \alpha}^- \), where the coefficients \( x_{\alpha, \gamma} \in \mathbb{Z} \) are rather complicated as given in [4, Cor. 2.6.7].

(2) There exists a central subalgebra \( \mathcal{D}_\lambda(n) = \mathbb{Q}(v)[z^+_m, z^-_m]_{m \geq 1} \) such that \( \mathcal{D}_\lambda(n) \cong U_\lambda(n) \otimes \mathcal{Z}_\lambda(n) \), where \( U_\lambda(n) \) is the subalgebra generated by \( E_i = u_{e_i^+}, F_i = u_{e_i^-}, K_i \) for all \( i \in I \).

We now define a candidate of the Lusztig form of \( \mathcal{D}_\lambda(n) \).

Let \( \mathcal{D}_\lambda(n) = \mathcal{D}_\lambda^+(n) \mathcal{Z} \mathcal{D}_\lambda^0(n) \mathcal{Z} \mathcal{D}_\lambda^-(n) \mathcal{Z} \) be a subalgebra of \( \mathcal{D}_\lambda(n) \) spanned by the elements \( u_\lambda^+ \) (resp., \( u_\lambda^- \)) for \( A \in \mathcal{D}_\lambda(n) \), and let \( \mathcal{D}_\lambda(n) \) be the subalgebra of \( \mathcal{D}_\lambda(n) \) generated by \( K_i^{\pm 1} \) and \( \left[ K_i^{t:0} \right] \), for \( i \in I \) and \( t \in \mathbb{N} \), where

\[
\left[ K_i^{t:0} \right] = \prod_{s=1}^{t} \frac{K_i v^{s+1} - K_i^{s-1}}{v^s - v^{-s}}.
\]

Let \( \mathcal{D}_\lambda(n) = \mathcal{D}_\lambda(n) \mathcal{Z} \mathcal{D}_\lambda^0(n) \mathcal{Z} \mathcal{D}_\lambda^-(n) \mathcal{Z} \). We will prove in Theorem 5.6 that \( \mathcal{D}_\lambda(n) \) is a \( \mathbb{Z} \)-subalgebra of \( \mathcal{D}_\lambda(n) \) and give a realisation for \( \mathcal{D}_\lambda(n) \).

3. A BLM type presentation

We now describe a better presentation for \( \mathcal{D}_\lambda(n) \). Let \( \mathcal{S}_{\lambda, r} \) be the group consisting of all permutations \( w : \mathbb{Z} \to \mathbb{Z} \) such that \( w(i + r) = w(i) + r \) for \( i \in \mathbb{Z} \). The extended affine Hecke algebra \( \mathcal{H}_\lambda(r) \) over \( \mathbb{Z} \) associated to \( \mathcal{S}_{\lambda, r} \) is the (unital) \( \mathbb{Z} \)-algebra with basis \( \{ T_w \}_{w \in \mathcal{S}_{\lambda, r}} \), and multiplication defined by

\[
\begin{align*}
T_{s_i} &= (v^2 - 1)T_{s_i} + v^2, & \text{for } 1 \leq i \leq r \\
T_w T_{w'} &= T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'),
\end{align*}
\]

where \( s_i \in \mathcal{S}_{\lambda, r} \) is defined by setting \( s_i(j) = j \) for \( j \neq i, i + 1 \text{ mod } r \), and \( s_i(j) = j - 1 \) for \( j \equiv i + 1 \text{ mod } r \). Let \( \mathcal{H}_\lambda(r) = \mathcal{H}_\lambda(r) \mathbb{Z} \otimes \mathbb{Q}(v) \).

For \( \lambda = (\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}^n_\lambda \), let \( \sigma(\lambda) = \sum_{1 \leq i \leq n} \lambda_i \). For \( r \geq 0 \) we set

\[
\Lambda_\lambda(n, r) = \{ \lambda \in \mathbb{Z}^n_\lambda \mid \sigma(\lambda) = r \}.
\]

For \( \lambda \in \Lambda_\lambda(n, r) \), let \( \mathcal{S}_\lambda := \mathcal{S}(\lambda_1, \ldots, \lambda_n) \) be the corresponding standard Young subgroup of \( \mathcal{S}_r \). For each \( \lambda \in \Lambda_\lambda(n, r) \), let \( x_\lambda = \sum_{w \in \mathcal{S}_\lambda} T_w \in \mathcal{H}_\lambda(r) \mathbb{Z} \). The endomorphism algebras

\[
\mathcal{S}_\lambda(n, r) := \text{End}_{\mathcal{H}_\lambda(r) \mathbb{Z}} \left( \bigoplus_{\lambda \in \Lambda_\lambda(n, r)} x_\lambda \mathcal{H}_\lambda(r) \right) \quad \text{and} \quad \mathcal{S}_\lambda(n, r) := \text{End}_{\mathcal{H}_\lambda(r) \mathbb{Z}} \left( \bigoplus_{\lambda \in \Lambda_\lambda(n, r)} x_\lambda \mathcal{H}_\lambda(r) \right).
\]

are called affine quantum Schur algebras (cf. [18, 19, 28]).

For \( A \in \mathcal{S}_\lambda(n) \) and \( r \geq 0 \), let

\[
\sigma(A) = \sum_{1 \leq i \leq n, j \in \mathbb{Z}} a_{i,j} \text{ and } \Theta_\lambda(n, r) = \{ A \in \mathcal{S}_\lambda(n) \mid \sigma(A) = r \}.
\]
For \( \lambda \in \Lambda_{\triangle}(n, r) \), let \( \mathcal{D}_{\triangle}^\lambda = \{ d \mid d \in \mathcal{S}_{\triangle}, \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathcal{S}_{\triangle} \} \) and \( \mathcal{D}_{\triangle}^{\lambda, \mu} = \mathcal{D}_{\triangle}^\lambda \cap \mathcal{D}_{\triangle}^{\mu} \). By [33, 7.4] (see also [10, 9.2]), there is a bijective map
\[
\mathcal{J}_\circ : \{ (\lambda, d, \mu) \mid d \in \mathcal{D}_{\triangle}^{\lambda, \mu}, \lambda, \mu \in \Lambda_{\triangle}(n, r) \} \rightarrow \Theta_{\triangle}(n, r)
\]
sending \( (\lambda, d, \mu) \) to the matrix \( A = (|\mathcal{R}^i_k| \cap dR^i_k)|_{k,l \in \mathbb{Z}} \), where
\[
R^\prime_{i+kn} = \{ \nu_{k,i-1} + 1, \nu_{k,i-1} + 2, \ldots, \nu_{k,i-1} + \nu_i = \nu_{k,i} \} \text{ with } \nu_{k,i-1} = kr + \sum_{1 \leq t \leq i-1} \nu_t,
\]
for all \( 1 \leq i \leq n, k \in \mathbb{Z} \) and \( \nu \in \Lambda_{\triangle}(n, r) \).

For \( \lambda, \mu \in \Lambda_{\triangle}(n, r) \) and \( d \in \mathcal{D}_{\triangle}^{\lambda, \mu} \) satisfying \( A = \mathcal{J}_\circ(\lambda, d, \mu) \in \Theta_{\triangle}(n, r) \), define \( e_A \in \mathcal{S}_{\triangle}(n, r) \) by
\[
e_A(x_v h) = \delta_{\mu v} \sum_{w \in \mathcal{S}_{\triangle} d \mathcal{S}_{\mu}} T_i h,
\]
where \( \nu \in \Lambda_{\triangle}(n, r) \) and \( h \in \mathcal{H}_{\triangle}(r) \), and let
\[
[A] = v^{-d_A} e_A, \quad \text{where } d_A = \sum_{1 \leq i \leq n, k \geq k_j, j < l} a_{i,j} a_{k,l}.
\]
Note that the sets \( \{ e_A \mid A \in \Theta_{\triangle}(n, r) \} \) and \( \{ [A] \mid A \in \Theta_{\triangle}(n, r) \} \) form \( \mathbb{Z} \)-bases for \( \mathcal{S}_{\triangle}(n, r) \).

Let
\[
\Theta_{\triangle}^+(n) = \{ A \in \Theta_{\triangle}(n) \mid a_{i,i} = 0 \text{ for all } i \}.
\]
For \( A \in \Theta_{\triangle}^+(n) \), \( j \in \mathbb{Z}_n^+ \) and \( \lambda \in \mathbb{N}_{\triangle}^+ \) let
\[
A(j, r) = \sum_{\mu \in \Lambda_{\triangle}(n, r - \sigma(A))} v^{\mu \lambda} [A + \text{diag}(\mu)] \in \mathcal{S}_{\triangle}(n, r) \mathbb{Z}.
\]
\[
A(j, \lambda, r) = \sum_{\mu \in \Lambda_{\triangle}(n, r - \sigma(A))} v^{\mu \lambda} [\mu^T] [A + \text{diag}(\mu)] \in \mathcal{S}_{\triangle}(n, r) \mathbb{Z}
\]
The relationship between \( \mathcal{D}_{\triangle}(n) \) and \( \mathcal{S}_{\triangle}(n, r) \) can be seen from the following (cf. [18, 28] and [33, Prop. 7.6]).

**Theorem 3.1** ([4, 3.6.3, 3.8.1]). For \( r \geq 0 \), the map \( \zeta_r : \mathcal{D}_{\triangle}(n) \rightarrow \mathcal{S}_{\triangle}(n, r) \) satisfying
\[
\zeta_r(K^j) = 0(j, r), \quad \zeta_r(\tilde{u}^A) = A(0, r), \quad \text{and } \zeta_r(\tilde{u}^A) = (tA)(0, r),
\]
for all \( j \in \mathbb{Z}_n^+ \), \( A \in \Theta_{\triangle}^+(n) \) and the transpose \( tA \) of \( A \), is a surjective algebra homomorphism.

The map \( \zeta_r \) defined in Theorem 3.1 induce an algebra homomorphism
\[
\zeta = \prod_{r \geq 0} \zeta_r : \mathcal{D}_{\triangle}(n) \rightarrow \mathcal{S}_{\triangle}(n).
\]
We now describe the image of \( \zeta \).

Let
\[
\mathcal{S}_{\triangle}(n) = \prod_{r \geq 0} \mathcal{S}_{\triangle}(n, r).
\]
For $A \in \Theta_\alpha^\pm(n)$, $j \in \mathbb{Z}_0^n$, and $\lambda \in \mathbb{N}_0^n$, define elements in $S_\lambda(n)$

$$A(j) = (A(j, r))_{r \geq 0}, \quad A(j, \lambda) = (A(j, \lambda, r))_{r \geq 0}.$$ 

We set, for $A \in M_{d_{\lambda}(\mathbb{Z})}$ with $a_{i,j} < 0$ for some $i \neq j$, $A(j, \lambda) = A(j) = 0$.

Let $V_\lambda(n)$ be the $\mathbb{Q}(v)$-subspace of $S_\lambda(n)$ spanned by $A(j, \lambda)$ for $A \in \Theta_\alpha^+(n)$, $j \in \mathbb{Z}_0^n$ and $\lambda \in \mathbb{N}_0^n$. By [11, Lem. 4.1], $\{A(j) \mid A \in \Theta_\alpha^+(n), j \in \mathbb{N}_0^n\}$ forms a basis for $V_\lambda(n)$.

**Theorem 3.2** ([11, 4.4]). The $\mathbb{Q}(v)$-space $V_\lambda(n)$ is a subalgebra of $S_\lambda(n)$. Furthermore, the restriction of $\zeta$ to $V_\lambda(n)$ induces a $\mathbb{Q}(v)$-algebra isomorphism $\zeta : \mathcal{D}_\lambda(n) \to V_\lambda(n)$. In particular, we have

$$\zeta(K^j) = 0(j), \quad \zeta(\tilde{u}^+_j) = A(0), \quad \text{and} \quad \zeta(\tilde{u}^-_j) = (^tA)(0),$$

for all $A \in \Theta_\alpha^+(n)$ and $j \in \mathbb{Z}_0^n$.

We shall identify $\mathcal{D}_\lambda(n)$ with $V_\lambda(n)$ via the map $\zeta$ and identify $\mathcal{D}_\lambda(n)$ with $U(\widehat{\mathfrak{gl}}_n)$ under the isomorphism given in Theorem 2.3. The following better presentation for $U(\widehat{\mathfrak{gl}}_n)$, called a modified BLM type realisation of quantum affine $\mathfrak{gl}_n$, is given in [11, Th. 1.1].

For $T = (t_{i,j}) \in \tilde{\Theta}_\alpha(n)$ let $\delta_T = (t_{i,i})_{i \in \mathbb{Z}} \in \mathbb{Z}_{\alpha}^n$, the “diagonal” of $T$ and let $\bar{T} = (\bar{t}_{i,j})$, where $\bar{t}_{i,j} = t_{i-1,j}$ for all $i,j \in \mathbb{Z}$.

For $A \in \tilde{\Theta}_\alpha(n)$, let $\text{ro}(A) = (\sum_{j \in \mathbb{Z}} a_{i,j})_{i \in \mathbb{Z}}$ and $\text{co}(A) = (\sum_{i \in \mathbb{Z}} a_{i,j})_{j \in \mathbb{Z}}$.

**Theorem 3.3.** The quantum loop algebra $U(\widehat{\mathfrak{gl}}_n)$ is the $\mathbb{Q}(v)$-algebra which is spanned by the basis $\{A(j) \mid A \in \Theta_\alpha^+(n), j \in \mathbb{Z}_0^n\}$ and generated by $0(j)$, $S_\alpha(0)$ and $S_\alpha(0)$ for all $j \in \mathbb{Z}_0^n$ and $\alpha \in \mathbb{N}_0^n$, where $S_\alpha = \sum_{1 \leq i \leq n} a_{i,i} E_{i,i+1}$ and $S_\alpha$ is the transpose of $S_\alpha$, and whose multiplication rules are given by:

1. $0(j')A(j) = v^{r_0(\text{co}(A))} A(j' + j)$ and $A(j)0(j') = v^{r_0(\text{co}(A))} A(j' + j)$;
2. $S_\alpha(0)A(j) = \sum_{T \in \Theta_\alpha(n)} v^{r(T)} \prod_{1 \leq i < n} a_{i,j} \begin{bmatrix} a_{i,j} + t_{i,j} - t_{i-1,j} \\ t_{i,j} \end{bmatrix} (A + T_+ - \bar{T}_+)(j_T, \delta_T)$, where $j_T = j + \sum_{1 \leq i \leq n} (\sum_{j < l} (t_{i,j} - t_{i-1,j})) e_i$ and

$$f_T = \sum_{1 \leq i < n} a_{i,j} t_{i,j} - \sum_{1 \leq i < n} a_{i+1,j} t_{i+1,j} - \sum_{1 \leq i < n} t_{i-1,j} t_{i,j} + \sum_{1 \leq i < n} t_{i,j} t_{i,1} + \sum_{1 \leq i < n} t_{i,j} t_{i+1,j}$$

and $j_T = j + \sum_{1 \leq i \leq n} (\sum_{j < l} (t_{i-1,j} - t_{i,j})) e_i$;

3. $S_\alpha(0)A(j) = \sum_{T \in \Theta_\alpha(n)} v^{f_T} \prod_{1 \leq i < n} a_{i,j} \begin{bmatrix} a_{i,j} - t_{i,j} + t_{i-1,j} \\ t_{i-1,j} \end{bmatrix} (A - T_+ + \bar{T}_+)(j_T, \delta_T)$,
where \( j'_T = j + \sum_{1 \leq i \leq n} (\sum_{j > i} (t_{i-1,j} - t_{i,j})) e_i^e \) and

\[
 f'_T = \sum_{1 \leq i \leq n, j > i} a_{i,j} t_{i-1,j} - \sum_{1 \leq i \leq n, j \neq i} t_{i-1,j} t_{i,j} + \sum_{1 \leq i \leq n, j > i+1} t_{i,j} t_{i,l}
 + \sum_{1 \leq i \leq n, i < j} t_{i,j} t_{i-1,i} + \sum_{1 \leq i \leq n} j_i (t_{i,i} - t_{i-1,i}).
\]

4. SOME INTEGRAL MULTIPLICATION FORMULAS

Let \( \tilde{\cdot} : \mathbb{Z} \to \mathbb{Z} \) be the ring homomorphism defined by \( \tilde{v} = v^{-1} \). The following result is proved in [11, 3.6].

**Proposition 4.1.** Let \( A \in \Theta_\Delta(n, r) \) and \( \alpha, \gamma \in \mathbb{N}_\Delta^n \).

1. If \( B \in \Theta_\Delta(n, r) \) satisfies that \( B - \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\Delta \) is a diagonal matrix and \( \co(B) = \ro(A) \), then in \( S_\Delta(n, r) \) :

\[
 [B][A] = \sum_{T \in \Theta_\Delta(n), \ro(T) = 0} \prod_{1 \leq i \leq n, j \in \mathbb{Z}^\Delta} v^{\beta(T,A)} \left[ a_{i,j} + t_{i,j} - t_{i-1,j} \right] [A + T - \tilde{T}],
\]

where \( \beta(T,A) = \sum_{1 \leq i \leq n} \sum_{j \geq 2} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i,j}) t_{i,l} \).

2. If \( C \in \Theta_\Delta(n, r) \) satisfies that \( C - \sum_{1 \leq i \leq n} \gamma_i E_{i+1,i}^\Delta \) is a diagonal matrix and \( \co(C) = \ro(A) \), then in \( S_\Delta(n, r) \) :

\[
 [C][A] = \sum_{T \in \Theta_\Delta(n), \ro(T) = 0} \prod_{1 \leq i \leq n, j \in \mathbb{Z}^\Delta} v^{\beta'(T,A)} \left[ a_{i,j} - t_{i,j} + t_{i-1,j} \right] [A - T + \tilde{T}],
\]

where \( \beta'(T,A) = \sum_{1 \leq i \leq n} \sum_{j \geq 2} (a_{i,j} - t_{i,j}) t_{i-1,l} - \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i,j}) t_{i,l} \).

We now derive some integral version of the multiplication formulas.

**Proposition 4.2.** Let \( A \in \Theta_\Delta^\pm(n) \), \( S_\alpha = \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\Delta \) and \( 'S_\alpha = \sum_{1 \leq i \leq n} \alpha_i E_{i+1,i}^\Delta \) with \( \alpha \in \mathbb{N}_\Delta^n \). Let \( \lambda, \mu \in \mathbb{N}_\Delta^n \), \( j, j' \in \mathbb{Z}_\Delta^+ \). The following identities holds in \( S_\Delta(n) \):

1. \( 0(j', \mu) A(j, \lambda) = \sum_{\nu \in \mathbb{N}_\Delta^n, \nu \leq \mu} a_\nu A(j' + j - \nu, \lambda + \mu - \nu) \);

where

\[ a_\nu = \sum_{j'' \in \mathbb{N}_\Delta^n} v^{\ro(A), (j'+\mu-j'')+(\mu-j'')} \left[ \ro(A) \right] \left[ j'' \right] \left[ \nu - j'' \right] \left[ \lambda + \mu - \nu \right] \left[ \nu - j'', \lambda - \nu + j'', \mu - \nu \right] \];

2. \( S_\alpha(0) A(j, \lambda) = \sum_{T \in \Theta_\Delta(n), \ro(T) = 0} g_{\beta,\eta,T} \cdot (A + T^\pm - \tilde{T}^\pm)(j_T + \lambda - \eta - 2\beta, \delta_T + \eta) \);
where

\[ g_{\beta, \eta, T} = u^{f_T + (\eta + \beta)T - \delta_T} \left[ \delta_T - \delta_T \right] \left[ \delta_T + \eta \right] \prod_{1 \leq i < n, j \neq i, j \in \mathbb{Z}} \left[ a_{i, j} + t_{i, j} - t_{i-1, j} \right] \in \mathbb{Z}, \]

and \( j_T, f_T \) are defined as in Theorem 3.3(2);

(3) \( S_\alpha(0) A(j, \lambda) = \sum_{T \in \Theta(n), \alpha(T) = \alpha} \sum_{1 \leq i < n, j \neq i, j \in \mathbb{Z}} g_{\beta, \eta, T} \cdot (A - T^\pm + \tilde{T}^\pm) (j_T' + \lambda - \eta - 2\beta, \delta_T - \eta), \)

where

\[ g_{\beta, \eta, T} = u^{f_T' + (\eta + \beta)T - \delta_T'} \left[ \delta_T - \delta_T' \right] \left[ \delta_T' + \eta \right] \prod_{1 \leq i < n, j \neq i, j \in \mathbb{Z}} \left[ a_{i, j} - t_{i, j} + t_{i-1, j} \right] \in \mathbb{Z}, \]

and \( j_T', f_T' \) are defined as in Theorem 3.3(3). The same formulas hold in \( S_\alpha(n, r) \mathbb{Z} \) with \( A(j, \lambda) \) etc. replaced by \( A(j, \lambda, r) \), etc.

**Proof.** The fact \( [A][B] \neq 0 \implies \text{ro}(B) = \text{co}(A) \) gives

\[ 0(j', \mu, r) A(j, \lambda, r) = \sum_{\alpha \in \Lambda_{\alpha}(n, r - \sigma(A))} v^{(\text{ro}(A) + \alpha)j' + \alpha j} \left[ \begin{array}{c} \text{ro}(A) + \alpha \\ \mu \end{array} \right] \left[ \begin{array}{c} \lambda \\ A + \text{diag}(\alpha) \end{array} \right]. \]

Applying (1.1.2) yields the required formula. For more details, see [16, 3.4].

Similarly, by Proposition 4.1, the left hand side of (2) at level \( r \) becomes

\[ S_\alpha(0, r) A(j, \lambda, r) = \sum_{\gamma \in \Lambda_{\alpha}(n, r - \sigma(A))} v^{\gamma j} \left[ \begin{array}{c} \gamma \\ \lambda \end{array} \right] \left[ S_\alpha + \text{diag}(\gamma - \text{ro}(A) + \sum_{1 \leq i \leq n} \alpha_i e_{n+1}^i) \right] \left[ A + \text{diag}(\gamma) \right] \]

\[ = \sum_{T \in \Theta(n), \alpha(T) = \alpha} \prod_{1 \leq i < n, j \neq i, j \in \mathbb{Z}} \left[ a_{i, j} + t_{i, j} - t_{i-1, j} \right] x_T, \]

where

\[ x_T = \sum_{\gamma \in \Lambda_{\alpha}(n, r - \sigma(A))} v^{\gamma j + \beta(T, A + \text{diag}(\gamma))} \left[ \begin{array}{c} \gamma + \delta_T - \delta_T' \\ \delta_T \end{array} \right] \left[ A + T^\pm - \tilde{T}^\pm + \text{diag}(\gamma + \delta_T - \delta_T') \right]. \]

Let \( \nu \gamma + \delta_T - \delta_T'. \) Then \( \beta(T, A + \text{diag}(\gamma)) = \beta_{A, T} + \beta_{\nu, T}, \) where \( \beta_{\nu, T} = \sum_{1 \leq i \leq n, i \neq l} \nu_i t_{i, l} - \sum_{1 \leq i \leq n, i + 1 \neq l} \nu_{i+1} t_{i+1, l} \) and

\[ \beta_{A, T} = \sum_{1 \leq i \leq n, j \neq l, j \neq i} (a_{i, j} - t_{i-1, j}) t_{i, l} - \sum_{1 \leq i \leq n, j \neq l, j \neq i + 1} a_{i+1, j} t_{i, l} + \sum_{1 \leq i \leq n, j \neq l, j \neq i + 1} t_{i, j} t_{i, l} \]

\[ - \sum_{1 \leq i \leq n} t_{i, l}^2 + \sum_{1 \leq i \leq n, i + 1} t_{i+1, l} t_{i, l}. \]
Applying the identities in (1.1.2) yields
\[ x_T = \sum_{\nu \in \Lambda_0(n,r - \sigma(A + T^z - T^z))} v^{f_T + \nu \delta T} \left[ \begin{array}{c} \nu \\ \delta T \end{array} \right] \left[ \begin{array}{c} \nu - \delta T + \delta T^- \lambda \\ \lambda - x \end{array} \right] \]
Applying the identities in (1.1.2) yields
\[ x_T = \sum_{\nu \in \Lambda_0(n,r - \sigma(A + T^z - T^z))} v^{f_T + \nu \delta T} \left[ \begin{array}{c} \nu \\ \delta T \end{array} \right] \left[ \begin{array}{c} \nu - \delta T + \delta T^- \lambda \\ \lambda - x \end{array} \right] \]
Thus,
\[ x_T = \sum_{\nu \in \Lambda_0(n,r - \sigma(A + T^z - T^z))} v^{f_T + \nu \delta T} \left[ \begin{array}{c} \nu \\ \delta T \end{array} \right] \left[ \begin{array}{c} \nu - \delta T + \delta T^- \lambda \\ \lambda - x \end{array} \right] \]

5. LUSZTIG FORM OF $\hat{U}(\mathfrak{gl}_n)$ AND INTEGRAL AFFINE QUANTUM SCHUR–WEYL RECIPROCITY

We are now ready to determine the Lusztig form of $\hat{U}(\mathfrak{gl}_n)$ by proving the conjecture [4, 3.8.6].

Let $\mathcal{V}_\lambda(n)_{\mathbb{Z}}$ be the $\mathbb{Z}$-submodule of $\mathcal{S}_\lambda(n)$ spanned by $\{ A(j, \lambda) \mid A \in \Theta^+_\lambda(n), j \in \mathbb{Z}^n_0, \lambda \in \mathbb{N}^n_0 \}$. As seen above, $\mathcal{V}_\lambda(n)_{\mathbb{Z}}$ is a $\mathbb{Z}$-submodule of $\mathcal{V}_\lambda(n)$. Our aim is to show that $\mathcal{V}_\lambda(n)_{\mathbb{Z}}$ is a realisation of $\mathcal{D}_\lambda(n)_{\mathbb{Z}}$ (see Theorem 5.6 below). The following result is [16, 4.8].

Lemma 5.1. The set $\{ A(j, \lambda) \mid A \in \Theta^+_\lambda(n), j, \lambda \in \mathbb{N}^n_0, j_i \in \{0,1\}, \forall i \}$ forms a $\mathbb{Z}$-basis for $\mathcal{V}_\lambda(n)_{\mathbb{Z}}$.

Proof. Since the 0-part of $\hat{U}(\mathfrak{gl}_n)$ is the same as that of $U(\mathfrak{gl}_n)$, the proof in the finite case [16, 4.2] carries over. \qed

Let $\mathcal{V}_\lambda^+(n)_{\mathbb{Z}} = \text{span}\{ A(0) \mid A \in \Theta^+_\lambda(n) \}$, $\mathcal{V}_\lambda^-(n)_{\mathbb{Z}} = \text{span}\{ A(0) \mid A \in \Theta^-_\lambda(n) \}$ and $\mathcal{V}_\lambda^0(n)_{\mathbb{Z}} = \text{span}\{ 0(j, \lambda) \mid j \in \mathbb{Z}^n_0, \lambda \in \mathbb{N}^n_0 \}$. By Proposition 4.2(1), $\mathcal{V}_\lambda^0(n)_{\mathbb{Z}}$ is a $\mathbb{Z}$-subalgebra of $\mathcal{S}_\lambda(n)$. 
Lemma 5.2. The $\mathbb{Z}$-module $V_\lambda^+(n)_{\mathbb{Z}}$ (resp., $V_\lambda^{-}(n)_{\mathbb{Z}}$) is a subalgebra of $S_\lambda(n)$ which is generated by $(\sum_{1 \leq i \leq n} \alpha_i E_{i+1,i}^{\Delta})(0)$ (resp., $(\sum_{1 \leq i \leq n} \alpha_i E_{i+1,i}^{\Delta})(0)$) for $\alpha \in \mathbb{N}_0^n$ as a $\mathbb{Z}$-algebra.

Proof. Since $D_\lambda^{\pm}(n)_{\mathbb{Z}} = H_\lambda(n)_{\mathbb{Z}}$ is a $\mathbb{Z}$-subalgebra of $D_\lambda(n)$ and $V_\lambda^{+}(n) = \zeta(D_\lambda^{+}(n)_{\mathbb{Z}})$ by Theorem 3.2, we conclude the first assertion which together with Proposition 2.2 gives the second assertion. \hfill \Box

We now recall the triangular relation for affine quantum Schur algebras. For $A, B \in \Theta_\lambda(n)$ define

$$B \sqsubseteq A \text{ if and only if } B \not< A, \text{ co}(B) = \text{co}(A) \text{ and } \text{ro}(B) = \text{ro}(A).$$

Put $B \sqsubseteq A$ if $B \sqsubseteq A$ and $B \neq A$. According to [10, 6.1] the order relation $\sqsubseteq$ is a partial order relation on $\Theta_\lambda(n)$ with finite intervals $(-\infty, A]$ for all $A$; see Lemma 7.5 below.

For $A \in \Theta_\lambda(n)$ with $\sigma(A) = r$, we denote $[A] = 0 \in S_\lambda(n, r)_{\mathbb{Z}}$ if $a_{i,i} < 0$ for some $i \in \mathbb{Z}$. For $A \in \Theta_\lambda(n)$ let $\sigma(A) = (\sigma_i(A))_{i \in \mathbb{Z}} \in \mathbb{N}_0^n$ where $\sigma_i(A) = a_{i,i} + \sum_{j < i}(a_{i,j} + a_{j,i})$. The following triangular relation for affine quantum Schur algebras is given in [4, 3.7.7]. The first assertion can be seen easily from the proof of loc. cit.

Proposition 5.3. For $A \in \Theta_\lambda^{\pm}(n)$ and $\lambda \in \Lambda_\lambda(n, r)$, we have

$$A^+(0, r)[\text{diag}(\lambda)]A^-(0, r) = [A + \text{diag}(\lambda - \sigma(A))] + A \text{-linear comb. of } [A'] \text{ with } A' \sqsubseteq A.$$

In particular, the set

$$\{A^+(0, r)[\text{diag}(\lambda)]A^-(0, r) \mid A \in \Theta_\lambda^{\pm}(n), \lambda \in \Lambda_\lambda(n, r), \lambda \geq \sigma(A)\}$$

forms a $\mathbb{Z}$-basis for $S_\lambda(n, r)_{\mathbb{Z}}$, where the order relation $\leq$ is defined in (1.1.1).

For $w \in \tilde{\Sigma}$, let

$$m_+^{(\lambda)}(w) = \zeta(\tilde{u}_+^{(\lambda)}(w)) \in S_\lambda(n) \quad \text{and} \quad m_-^{(\lambda)}(w) = \zeta(\tilde{u}_-^{(\lambda)}(w)) \in S_\lambda(n).$$

The triangular relation for affine quantum Schur algebras can be lifted to the $S_\lambda(n)$ level as follows.

Lemma 5.4. Let $A \in \Theta_\lambda^{\pm}(n)$, $j \in \mathbb{Z}_\lambda^n$ and $\lambda \in \mathbb{N}_0^n$.

1. We have

$$A^+(0, r)[j, \lambda]A^-(0, r) = \sum_{\delta \in \mathbb{N}_0^n \delta \leq \lambda} \nabla^{(j - \delta) \sigma(A)}_{\lambda - \delta} \cdot [\sigma(A)]_{\lambda - \delta} A(j + \lambda - \delta, \delta) + f$$

where $f$ is a $\mathbb{Z}$-linear combination of $B(j', \delta)$ such that $B \in \Theta_\lambda^{\pm}(n)$, $B \not< A$, $\delta \in \mathbb{N}_0^n$ and $j' \in \mathbb{Z}_\lambda^n$.

In particular, We have $V_\lambda(n)_{\mathbb{Z}} = V_\lambda^{+}(n)_{\mathbb{Z}}V_\lambda^{0}(n)_{\mathbb{Z}}V_\lambda^{-}(n)_{\mathbb{Z}}$. 

There exist \( w_{A^+}, w_{A^-} \in \tilde{Z} \) such that \( \varphi^+(w_{A^+}) = A^+ \), \( \varphi^-(w_{A^-}) = A^- \) and
\[
\mathfrak{m}^+_0(\mathfrak{j}) \mathfrak{m}^-_0 = \sum_{\delta \in \mathbb{N}^+} v(\mathfrak{j}, \lambda + \delta, \delta) + g
\]
where \( g \) is a \( Z \)-linear combination of \( B(j', \delta) \) such that \( B \in \Theta_0^+(n) \), \( B \prec A \), \( \delta \in \mathbb{N}^0 \) and \( j' \in \mathbb{Z}_0^n \).

**Proof.** According to Proposition 5.3, for any \( \mu \in \Lambda_\delta(n, r) \), we have
\[
A^+(0, r) \cdot [\text{diag}(\mu)] A^- (0, r) = [A + \text{diag}(\mu - \sigma(A))] + f_{\mu, r}
\]
where \( f_{\mu, r} \) is a \( Z \)-linear combination of \( [B] \) such that \( B \in \Theta_\delta(n, r) \) and \( B \sqsupseteq A + \text{diag}(\mu - \sigma(A)) \).

Thus,
\[
A^+(0, r) 0(\mathfrak{j}, \lambda, r) A^- (0, r) = \sum_{\mu \in \Lambda_\delta(n, r)} v^\mu \left[ \begin{array}{c} \mu \\ \lambda \end{array} \right] \left[ [A + \text{diag}(\mu - \sigma(A))] + f_{\mu, r} \right]
\]
\[=
\sum_{\nu \in \mathbb{Z}_0^n} v^\nu \left[ \begin{array}{c} \nu + \sigma(A) \\ \lambda \end{array} \right] \left[ A + \text{diag}(\nu) \right] f_r,
\]
where \( f_r = \sum_{\mu \in \Lambda_\delta(n, r)} v^\mu \left[ \begin{array}{c} \mu \\ \lambda \end{array} \right] f_{\mu, r} \). By (1.1.2), we have
\[
A^+(0, r) 0(\mathfrak{j}, \lambda, r) A^- (0, r) = \sum_{\nu \in \mathbb{Z}_0^n} v^\nu \left[ \begin{array}{c} \nu + \sigma(A) \\ \lambda \end{array} \right] \left[ A + \text{diag}(\nu) \right] + f_r
\]
\[=
\sum_{\delta \in \mathbb{N}^+} v(\mathfrak{j}, \lambda - \delta, \delta) A(\mathfrak{j}, \lambda - \delta, \delta) + f_r.
\]

On the other hand, by Lemma 5.2 and Proposition 4.2, we see that \( (f_r)_{r \geq 0} \in \mathcal{V}_\delta(n)_Z \). Hence, \( (f_r)_{r \geq 0} \) must be a \( Z \)-linear combination of \( B(j', \delta) \) such that \( B \in \Theta_\delta^+(n) \), \( B \prec A \), \( \delta \in \mathbb{N}^0 \) and \( j' \in \mathbb{Z}_0^n \). This proves (1). The assertion (2) follows from (1), Proposition 2.2 and Theorem 3.1. \( \square \)

For \( A \in \widetilde{\Theta}_\delta(n) \), let
\[
\|A\| = \sum_{1 \leq j \leq n} \left( \frac{j - i + 1}{2} \right) (a_{i,j} + a_{j,i}).
\]
Then, \( A \prec B \) implies \( \|A\| < \|B\| \). The following result is the affine version of [16, Prop. 4.3] which is conjectured in [16, 4.9].

**Proposition 5.5.** The \( Z \)-module \( \mathcal{V}_\delta(n)_Z \) is a subalgebra of \( \mathcal{S}_\delta(n) \) which is generated by the elements \( (\sum_{1 \leq i \leq n} \alpha_i E^\delta_{i,i+1})(0), \sum_{1 \leq i \leq n} \alpha_i E^\delta_{i+1,i}(0), \sum_{0 \leq i \leq n} \alpha_i E^\delta_{i,i}(0), 0(e^\delta_i), 0(0, te^\delta_i) \) for all \( \alpha \in \mathbb{N}_0^n \), \( t \in \mathbb{N} \), \( 1 \leq i \leq n \).

**Proof.** Let \( \mathcal{V}_\alpha(n)_Z \) be the \( Z \)-subalgebra of \( \mathcal{S}_\delta(n) \) generated by the indicated elements. According to Proposition 4.2, we have \( \mathcal{V}_\alpha(n)_Z \subseteq \mathcal{V}_\alpha(n)_Z \subseteq \mathcal{V}_\alpha(n)_Z \subseteq \mathcal{V}_\alpha(n)_Z \). We shall show by induction on \( \|A\| \) that \( A(j, \lambda) \in \mathcal{V}_\alpha(n)_Z \) for all \( A \in \Theta_\alpha^+(n) \), \( j \in \mathbb{Z}_0^n \) and \( \lambda \in \mathbb{N}_0^n \). If \( \|A\| = 0 \), then \( A = 0 \) and
0(j, λ) = \prod_{1 \leq i \leq n} 0(e_i^{A})^{0(j, \lambda, e_i^{A})} \in \nu^{\prime}(n)_{Z}. Now we assume that \|A\| > 0 and A'(j, λ) \in \nu^{\prime}(n)_{Z} for all A', j, \lambda with \|A'\| < \|A\|. By Lemma 5.4(2) and [4, 3.7.6], there exist w_{A+}, w_{A-} \in \Delta such that

\begin{equation}
m_{(w_{A+})}m_{(w_{A-})} = A(0) + g
\end{equation}

where g is a \mathbb{Z}-linear combination of B(j', \delta) with B \in \Theta^{\times}_{\Delta}(n), \|B\| < \|A\|, \delta \in \mathbb{N}^{n} and j' \in \mathbb{Z}^{n}. By the induction hypothesis we have g \in \nu^{\prime}(n)_{Z}. It follows that A(0) \in \nu^{\prime}(n)_{Z} and so A(j) \in \nu^{\prime}(n)_{Z} by Theorem 5.6. Furthermore, by Proposition 4.2(1) (setting j' = \mu - \nu there),

\begin{equation}
0(j, \lambda)A(0) = v^{\text{ro}(A), \mu+\lambda}A(j, \lambda) + \sum_{j' < \lambda} v^{\text{ro}(A), \mu+\lambda} \left[ \text{ro}(A) \right] \lambda - j' A(j + j' - \lambda, j')
\end{equation}

(5.5.1)

Thus, by induction on \sigma(\lambda), we conclude that A(j, \lambda) \in \nu^{\prime}(n)_{Z} for all j \in \mathbb{Z}^{n} and \lambda \in \mathbb{N}^{n}. □

As indicated in [16, Rem. 4.10(3)], we now use Proposition 5.5 to prove the conjecture formulated in [4, 3.8.6]. Recall from Theorem 3.2 that the homomorphism \zeta in (3.1.1) induces an isomorphism \zeta : \mathcal{D}_{\Delta}(n) \rightarrow \nu^{\prime}(n).

Theorem 5.6. We have \zeta^{-1}(\nu^{\prime}(n)_{Z}) = \mathcal{D}_{\Delta}(n)_{Z}. In particular, \mathcal{D}_{\Delta}(n)_{Z} is a subalgebra of \mathcal{D}_{\Delta}(n) isomorphic to \nu^{\prime}(n)_{Z}. Moreover, \mathcal{D}_{\Delta}(n)_{Z} is a Hopf subalgebra of \mathcal{D}_{\Delta}(n).

Proof. Since \zeta(\mathcal{D}_{\Delta}(n)_{Z}) = \zeta(\mathcal{D}_{\Delta}^{\prime}(n)_{Z})\zeta(\mathcal{D}_{\Delta}^{0}(n)_{Z})\zeta(\mathcal{D}_{\Delta}^{-}(n)_{Z}) = \nu^{\prime}(n)_{Z}\nu^{0}(n)_{Z}\nu^{-}(n)_{Z}, it follows from Lemma 5.4(1) that \zeta(\mathcal{D}_{\Delta}(n)_{Z}) = \nu^{\prime}(n)_{Z}. Hence, by Proposition 5.5 and Theorem 3.2, \mathcal{D}_{\Delta}(n)_{Z} is a subalgebra. By using the semisimple generators for \mathcal{D}_{\Delta}(n)_{Z}, the last assertion follows from [4, 3.5.7]. □

Remark 5.7. (1) A different integral form \text{U}_{\text{res}}^{\text{res}}(\hat{\mathfrak{g}}_{n}) of \text{U}(\hat{\mathfrak{g}}_{n}) was constructed in [13, 7.2]. As pointed out in [13], it is not known if \text{U}_{\text{res}}^{\text{res}}(\hat{\mathfrak{g}}_{n}) is a Hopf subalgebra. It would be interesting to find a relation between \mathcal{D}_{\Delta}(n)_{Z} and \text{U}_{\text{res}}^{\text{res}}(\hat{\mathfrak{g}}_{n}).

(2) There is another form using the Lusztig form of \text{U}(\hat{\mathfrak{g}}_{n}) tensoring with an integral central algebra; see [4, 2.4.4]. However, this form does not map onto the integral affine quantum Schur algebras; see Example 5.3.8 in [4].

We end this section with an application to the affine quantum Schur–Weyl reciprocity at the integer level. The proof of the following result is the same as that of [4, Th. 3.8.1(1)].

Theorem 5.8. The restriction of \zeta_{t} to \mathcal{D}_{\Delta}(n)_{Z} gives a surjective \mathbb{Z}-algebra homomorphism

\zeta_{t} : \mathcal{D}_{\Delta}(n)_{Z} \rightarrow \mathcal{S}_{\theta}(n, r)_{Z}.
Lemma 6.1. Let $\mathfrak{k}$ be a commutative ring containing an invertible element $\varepsilon$. We will regard $\mathfrak{k}$ as a $\mathbf{Z}$-module by specializing $v$ to $\varepsilon$. Let $\mathcal{D}_\mathfrak{k}(n) = \mathcal{D}_\mathfrak{k}(n) \mathbf{Z} \otimes \mathfrak{k}$, $\mathcal{S}_\mathfrak{k}(n,r) = \mathcal{S}_\mathfrak{k}(n,r) \mathbf{Z} \otimes \mathfrak{k}$. Then we have $\mathcal{S}_\mathfrak{k}(n,r) \cong \text{End}_{\mathcal{H}_\mathfrak{k}(r)}(\mathcal{T}_\mathfrak{k}(n,r))$, where $\mathcal{T}_\mathfrak{k}(n,r) = \oplus_{\lambda \in \Lambda(n,r)}(x_\lambda \mathcal{H}_\mathfrak{k}(r))$ with $\mathcal{H}_\mathfrak{k}(r) = \mathcal{H}(r) \mathbf{Z} \otimes \mathfrak{k}$.

**Corollary 5.9.** For any commutative ring $\mathfrak{k}$, there is an algebra epimorphism

$$
\zeta_r \otimes 1 : \mathcal{D}_\mathfrak{k}(n) \to \mathcal{S}_\mathfrak{k}(n,r).
$$

6. The affine BLM algebra $\mathcal{K}_\mathfrak{k}(n)_{\mathbf{Z}}$

We first derive in Proposition 6.3 the affine stabilisation property for affine quantum Schur algebras, which is the affine analogue of [1, 4.2]. We then construct the affine BLM algebra $\mathcal{K}_\mathfrak{k}(n)$ and prove that it is isomorphic to the modified quantum group $\mathcal{D}_\mathfrak{k}(n)$.

Observe the structure constants in Proposition 4.1 and separate the Gaussian polynomial $\left[ a_{i,j} + t_{i,j} - t_{i-1,j} \right]$ from the product. We now introduce, for a second indeterminate $v'$, $T \in \Theta_\mathfrak{k}(n)$ and $A \in \Theta_\mathfrak{k}(n)$, the polynomials

$$
P_{T,A}(v,v') = v^{\beta(T,A)} \prod_{1 \leq i \leq n, j \neq 1} \left[ a_{i,j} + t_{i,j} - t_{i-1,j} \right] \prod_{1 \leq i \leq n, 1 \leq j \leq t} \frac{v^{-2(a_{i,j} - t_{i,j} + t_{i-1,j})}}{v^{2s-1}} \frac{v^{2(a_{i,j} - t_{i,j} + t_{i-1,j})}}{v^{2s-1}}
$$

and

$$
Q_{T,A}(v,v') = v^{\beta'(T,A)} \prod_{1 \leq i \leq n, j \neq 1} \left[ a_{i,j} - t_{i,j} + t_{i-1,j} \right] \prod_{1 \leq i \leq n, 1 \leq j \leq t} \frac{v^{-2(a_{i,j} - t_{i,j} + t_{i-1,j})}}{v^{2s-1}} \frac{v^{2(a_{i,j} - t_{i,j} + t_{i-1,j})}}{v^{2s-1}}
$$

in the subring $\mathcal{Z}_1$ of $\mathbb{Q}(v)[v', v'^{-1}]$, where

$$(6.0.1) \quad \mathcal{Z}_1 \text{ is generated (over } \mathbb{Z}[t]) \text{ by } \prod_{1 \leq i \leq t} \frac{v^{2(a-i)}}{v^{2i-1}}, \prod_{1 \leq i \leq t} \frac{v^{2(a-i)}}{v^{2i-1}}, \text{ and } v^j$$

for all $a \in \mathbb{Z}$, $t \geq 1$ and $j \in \mathbb{Z}$. Note that $\mathcal{Z}_1|_{v'=1} = \mathcal{Z}$.

For $A \in \tilde{\Theta}_\mathfrak{k}(n)$ and $p \in \mathbb{Z}$, let

$$
pA = A + pI
$$

where $I \in \Theta_\mathfrak{k}(n)$ is the identity matrix. Then it is clear that $\beta(T,A) = \beta(T,pA)$ and $\beta'(T,A) = \beta'(T,pA)$. Thus, Proposition 4.1 can be generalised as follows.

**Lemma 6.1.** Let $A, B \in \tilde{\Theta}_\mathfrak{k}(n)$ and assume $\text{co}(B) = \text{ro}(A)$ and $b = \sigma(A) = \sigma(B)$.

1. If $B - \sum_{1 \leq i \leq n} \alpha_i E_{i+1,i+1}$ is diagonal for some $\alpha \in \mathbb{N}_\mathfrak{k}$ then, for large $p$ and $r = pn + b$, we have in $\mathcal{S}_\mathfrak{k}(n,r)_{\mathbf{Z}}$:

$$
[pB][pA] = \sum_{T \in \tilde{\Theta}_\mathfrak{k}(n), \text{ro}(T) = \alpha} P_{T,A}(v,v^{-p})[p(A + T - \overline{T})].
$$
 Finally, by writing the words \( x, y \preceq B \), we have in \( S_\alpha(n, r)_Z \):

\[
[pB][pA] = \sum_{\substack{T \in T_\alpha(n, r), \rho(T) = \alpha \\
\alpha_{i,j} - l_{i,j} + 1, \quad j \geq 0, \quad \forall \not=i}} Q_{T,A}(v, v^{-p})[p(A - T + \tilde{T})].
\]

Let \( \tilde{\Theta}_\alpha(n)^{ss} \) be the set of \( X \in \tilde{\Theta}_\alpha(n) \) such that either \( X - \sum_{1 \leq i \leq n} \alpha_i E_{i+1,i}^\Delta \) or \( X - \sum_{1 \leq i \leq n} \alpha_i E_{i+1,i}^\Delta \) is diagonal for some \( \alpha \in \mathbb{N}_\alpha \). We have the following affine version of \([1, 3.9]\) (see \([15, 4.5]\) for a slightly different version). For completeness, we include a proof.

**Proposition 6.2.** Let \( A \in \Theta_\alpha(n, r) \). Then there exist upper triangular matrices \( A_1, A_2, \ldots, A_s \) and lower triangular matrices \( A_{s+1}, A_{s+2}, \ldots, A_t \) in \( \tilde{\Theta}_\alpha(n)^{ss} \cap \Theta_\alpha(n, r) \) such that \( \rho(A_i) = \rho(A_{i+1}) \) (1 \( \leq i \leq t - 1 \)) and the following identity holds in \( S_\alpha(n, pn + r)_Z \): for \( p \geq 0 \),

\[
[p(A_1)] \cdots [p(A_s)] \cdot [p(A_{s+1})] \cdots [p(A_t)] = [pA] + \text{lower terms relative to } \preceq.
\]

**Proof.** By Proposition 2.2, there is a distinguished words \( w_B \) for every \( B \in \Theta_\alpha^+(n) \) satisfying the triangular relation (2.2.1). Let \( x = w_{A^+} \) and \( y = w_{A^-} \). By Theorem 3.1 and Proposition 2.2, we have in \( S_\alpha(n, r)_Z \)

\[
m^+_{(x),r} := \zeta_r(\tilde{u}^+(x)) = A^+(0, r) + f \quad \text{and} \quad m^-_{(y),r} := \zeta_r(\tilde{u}^-(y)) = A^-(0, r) + g,
\]

where \( f \) (resp., \( g \)) is a linear combination of \( B(0, r) \) with \( B \in \Theta_\alpha^+(n) \) (resp., \( B \in \Theta_\alpha^-(n) \)) and \( B \preceq A^+ \) (resp., \( B \preceq A^- \)). By Proposition 5.3, we have for \( p \geq 0 \)

\[
m^+_{(x),r}([\text{diag}(\sigma(pA))]m^-_{(y),r} = [pA] + \text{lower terms}.
\]

Finally, by writing the words \( x, y \) in full, it is clear to see that there exist upper triangular matrices \( A_1, A_2, \ldots, A_s \) and lower triangular matrices \( A_{s+1}, A_{s+2}, \ldots, A_t \) in \( \tilde{\Theta}_\alpha(n)^{ss} \) such that

\[
m^+_{(x),r}([\text{diag}(\sigma(pA))] = [p(A_1)] \cdots [p(A_s)] \quad \text{and} \quad [\text{diag}(\sigma(pA))]m^-_{(y),r} = [p(A_{s+1})] \cdots [p(A_t)],
\]

as desired. \( \square \)

We can now prove the following stabilization property for affine quantum Schur algebras.

**Proposition 6.3.** Let \( A, B \in \tilde{\Theta}_\alpha(n) \) and assume \( \rho(B) = \rho(A) \). Then there exist unique \( X_1, \ldots, X_m \in \tilde{\Theta}_\alpha(n) \), unique \( P_1(v, v'), \ldots, P_m(v, v') \in Z_1 \) and an integer \( p_0 \geq 0 \) such that, in \( S_\alpha(n, pn + \sigma(A))_Z \),

\[
[pB][pA] = \sum_{1 \leq i \leq m} P_i(v, v^{-p})[pX_i] \quad \text{for all } p \geq p_0.
\]

**Proof.** The proof can be conducted by induction on \( \|B\| \). With Lemma 6.1 and Proposition 6.2, the proof is entirely similar to that of \([1, 3.9]\) or \([5, \text{Prop. 14.1}]\). \( \square \)
Let $\tilde{K}_\alpha(n) \mathcal{Z}$ be the free $\mathcal{Z}$-module with basis $\{A \mid A \in \tilde{\Theta}_\alpha(n)\}$. Then, by Proposition 6.3, we may make $\tilde{K}_\alpha(n) \mathcal{Z}$ into an associative $\mathcal{Z}$-algebra (without unit) by the multiplication:

\begin{equation}
B \cdot A = \begin{cases} \sum_{1 \leq i \leq m} P_i(v, v') X_i, & \text{if } \text{co}(B) = \text{ro}(A); \\ 0, & \text{otherwise.} \end{cases}
\end{equation}

Let

\[ K_\alpha(n) = \tilde{K}_\alpha(n) \mathcal{Z} \otimes_{\mathcal{Z}} \mathcal{Z}, \]

where $\mathcal{Z}$ is regarded as a $\mathcal{Z}$-module by specializing $v'$ to 1. Then $K_\alpha(n) \mathcal{Z}$ becomes an associative $\mathcal{Z}$-algebra with basis $\{[A] := A \otimes 1 \mid A \in \tilde{\Theta}_\alpha(n)\}$. Let $K_\alpha(n) = K_\alpha(n) \mathcal{Z} \otimes \mathcal{Q}(v)$.

Following [1, 5.1], let $K_\alpha(n)$ be the vector space of all formal (possibly infinite) $\mathcal{Q}(v)$-linear combinations $\sum_{A \in \tilde{\Theta}_\alpha(n)} \beta_A[A]$ such that, for any $x \in \mathcal{Z}^n$, the sets $\{A \in \tilde{\Theta}_\alpha(n) \mid \beta_A \neq 0, \text{ro}(A) = x\}$ and $\{A \in \tilde{\Theta}_\alpha(n) \mid \beta_A = 0, \text{co}(A) = x\}$ are finite. We can define the product of two elements $\sum_{A \in \mathcal{Z}^n} \beta_A[A], \sum_{B \in \mathcal{Z}^n} \gamma_B[B]$ in $K_\alpha(n)$ to be $\sum_{A, B} \beta_A \gamma_B[A][B]$. This defines an associative algebra structure on $K_\alpha(n)$. The algebra $\mathcal{V}_\alpha(n)$ can also be realized as a $\mathcal{Q}(v)$-subalgebra of $K_\alpha(n)$, which we now describe.

The following result can be proved in a way similar to the proof of [9, 6.7] (cf. [15, 6.3]).

**Lemma 6.4.** The linear map $\hat{\zeta}_r : K_\alpha(n) \mathcal{Z} \to S_\alpha(n, r) \mathcal{Z}$ defined by

\begin{equation}
\hat{\zeta}_r([A]) = \begin{cases} [A] & \text{if } A \in \Theta_\alpha(n, r); \\ 0 & \text{otherwise} \end{cases}
\end{equation}

is an algebra epimorphism.

The map $\hat{\zeta}_r : K_\alpha(n) \mathcal{Z} \to S_\alpha(n, r) \mathcal{Z}$ induces a surjective algebra homomorphism

\begin{equation}
\hat{\zeta}_r : K_\alpha(n) \to S_\alpha(n, r)
\end{equation}

sending $\sum_{A \in \tilde{\Theta}_\alpha(n)} \beta_A[A]$ to $\sum_{A \in \tilde{\Theta}_\alpha(n)} \beta_A \hat{\zeta}_r([A])$. Consequently, we get a surjective algebra homomorphism

\begin{equation}
\hat{\zeta} : K_\alpha(n) \to S_\alpha(n).
\end{equation}

defined by sending $x$ to $\hat{\zeta}(x) := (\hat{\zeta}_r(x))_{r \geq 0}$. It is clear that we have $\hat{\zeta}(K_\alpha(n)) = S_\alpha \mathcal{Q}(n)$ where $S_\alpha \mathcal{Q}(n) = \bigoplus_{r \geq 0} S_\alpha(n, r)$. Thus, by restriction $\hat{\zeta}$ to $K_\alpha(n)$, we get a surjective algebra homomorphism from $K_\alpha(n)$ to $S_\alpha \mathcal{Q}(n)$.

For $A \in \Theta_\alpha^\pm(n), j \in \mathbb{Z}_+^n$ and $\lambda \in \mathbb{N}_+^n$, let

\[ A(j) := \sum_{\mu \in \mathbb{Z}_+^n} v^{\mu j}[A + \text{diag}(\mu)] \text{ and } A(j, \lambda) := \sum_{\mu \in \mathbb{Z}_+^n} v^{\mu j} \left[ \mu \right] [A + \text{diag}(\mu)]. \]
By Proposition 5.3, the stabilisation property Proposition 6.3 implies that for any \( A \in \tilde{\Theta}(n) \),
\[
A^+(0)_\lambda \mathbf{diag}(\sigma(A))A^- (0) = [A] + a \text{-linear comb. of } [A'] \text{ with } A' \subset A.
\]

Let \( \mathcal{V}(n) \) be the \( \mathbb{Q}(v) \)-subspace of \( \mathcal{K}(n) \) spanned by all \( A(j) (A \in \Theta^\pm(n) \text{ and } j \in \mathbb{Z}^n) \). Let \( \mathcal{V}(n)_Z \) be the \( \mathbb{Z} \)-submodule of \( \mathcal{K}(n) \) spanned by \( A(j, \lambda) \) for all \( A, j, \lambda \) as above.

**Theorem 6.5.** (1) \( \mathcal{V}(n) \) is a subalgebra of \( \mathcal{K}(n) \) and the restriction of \( \hat{\zeta} \) to \( \mathcal{V}(n) \) induces an algebra isomorphism \( \hat{\zeta} : \mathcal{V}(n) \to \mathcal{V}(n) \), \( A(j) \mapsto A(j) \).

(2) The \( \mathbb{Z} \)-module \( \mathcal{V}(n)_Z \) is a subalgebra of \( \mathcal{K}(n) \) and the restriction of \( \hat{\zeta} \) to \( \mathcal{V}(n)_Z \) induces an algebra isomorphism \( \hat{\zeta} : \mathcal{V}(n)_Z \to \mathcal{V}(n)_Z \), \( A(j, \lambda) \mapsto A(j, \lambda) \).

*Proof.* By looking at the kernel of \( \hat{\zeta} \) (cf. [10, §8]), it is clear that the restriction of \( \hat{\zeta} \) to \( \mathcal{V}(n) \) is injective. Note that \( \hat{\zeta}(\mathcal{V}(n)) = \mathcal{V}(n) \) and \( \hat{\zeta}(\mathcal{V}(n)_Z) = \mathcal{V}(n)_Z \). Now the assertion follows from Theorem 3.2 and Proposition 5.5. \( \square \)

This result together with Theorem 3.2 gives another realisation of \( \hat{U}(\mathfrak{g}t(n)) \). This is an unmodified affine generalisation of the BLM construction in [1]. In particular, we will identify \( \mathfrak{D}(n) \) with \( \mathcal{V}(n) \) and \( \mathfrak{D}(n)_Z \) with \( \mathcal{V}(n)_Z \) in the sequel.

We end this section with a discussion on a realisation of the modified quantum group \( \hat{\mathfrak{D}}(n) \). We will prove that \( \hat{\mathfrak{D}}(n) \) and its integral form \( \hat{\mathfrak{D}}(n)_Z \) is isomorphic the affine BLM algebras \( \mathfrak{K}(n) \) and \( \mathfrak{K}(n)_Z \), respectively.

Let \( \Pi(n) = \{ e^\Delta_j - e^\Delta_{j+1} \ | \ 1 \leq j \leq n \} \). According to [14, 3.5.2], the algebra \( \mathfrak{D}(n) \) is a \( \mathbb{Z}^n \)-graded algebra with \( \text{deg}(u^\lambda_{A}) = \text{ro}(A) - \text{co}(A) \), \( \text{deg}(u^\Delta_{A}) = \text{co}(A) - \text{ro}(A) \) and \( \text{deg}(K^\pm_1) = 0 \) for \( A \in \Theta^\pm(n) \) and \( 1 \leq i \leq n \). For \( \nu \in \mathbb{Z}^n \), let \( \mathfrak{D}(n)_{\nu} \) be the set of homogeneous elements in \( \mathfrak{D}(n) \) of degree \( \nu \). Then we have \( \mathfrak{D}(n) = \bigoplus_{\nu \in \Pi(n)} \mathfrak{D}(n)_{\nu} \).

For \( \lambda, \mu \in \mathbb{Z}^n \), we define \( \lambda I_{\mu} = \mathfrak{D}(n)_{\mu} / \lambda I_{\mu} \), where
\[
\lambda I_{\mu} = \left( \sum_{j \in \mathbb{Z}^n} (K^j - v^{\lambda^j}) \mathfrak{D}(n) + \sum_{j \in \mathbb{Z}^n} \mathfrak{D}(n)(K^j - v^{\mu^j}) \right).
\]
Let \( \pi_{\lambda, \mu} : \mathfrak{D}(n) \to \lambda \mathfrak{D}(n)_{\mu} \) be the canonical projection. Since \( \pi_{\lambda, \mu} (\mathfrak{D}(n)_{\lambda - \mu}) = \lambda \mathfrak{D}(n)_{\mu} \) (cf. [9, Lemma 6.2]), it follows that \( \lambda \mathfrak{D}(n)_{\mu} \) is spanned by the elements \( \pi_{\lambda, \mu} (u^A_B u^B_B) \) for all \( A, B, \lambda, \mu \) with \( \lambda - \mu = \text{deg}(u^A_B u^B_B) \). Let
\[
\hat{\mathfrak{D}}(n) := \bigoplus_{\lambda, \mu \in \mathbb{Z}^n} \lambda \mathfrak{D}(n)_{\mu}.
\]
We define the product in \( \hat{\mathfrak{D}}(n) \) as follows. For \( \lambda', \mu', \mu'' \in \mathbb{Z}^n \) with \( \lambda' - \mu', \lambda'' - \mu'' \in \Pi(n) \) and any \( t \in \mathfrak{D}(n)_{\lambda' - \mu'} \), \( s \in \mathfrak{D}(n)_{\lambda'' - \mu''} \), the product \( \pi_{\lambda', \mu'}(t) \pi_{\lambda'', \mu'}(s) \) is equal to \( \pi_{\lambda', \mu''}(ts) \) if \( \mu' = \lambda'' \), and it is zero, otherwise. Then \( \hat{\mathfrak{D}}(n) \) becomes an associative \( \mathbb{Q}(v) \)-algebra with this product. The algebra \( \hat{\mathfrak{D}}(n) \) is naturally a \( \mathfrak{D}(n) \)-bimodule defined by \( t' \pi_{\lambda', \mu'}(s) t'' = \pi_{\lambda' + \mu', \lambda'' - \mu''}(t's't'') \), for \( t' \in \mathfrak{D}(n)_{\nu'} \), \( s \in \mathfrak{D}(n) \), \( t'' \in \mathfrak{D}(n)_{\nu''} \) and \( \lambda', \lambda'' \in \mathbb{Z}^n \) (cf. [27, 14]). In
particular, putting \( 1_\lambda = \pi_{\lambda,\lambda}(1) \), we have \( u_1^A \lambda u_B^- = \pi_{\lambda + \operatorname{deg}(u_A^+),\lambda - \operatorname{deg}(u_B^-)}(u_1^A u_B^-) \) and \( \mathcal{D}_\lambda(n) \) is spanned by the elements \( u_1^A \lambda u_B^- \) for all \( A, B, \lambda \).

Let \( \mathcal{D}_\lambda(n)_Z \) be the \( Z \)-submodule of \( \mathcal{D}_\lambda(n) \) spanned by the elements \( u_1^A \lambda u_B^- \) for \( A, B \in \Theta^+_\lambda(n) \) and \( \lambda \in \mathbb{Z}_n^\lambda \). It is proved in [14, Th. 4.2] that \( \mathcal{D}_\lambda(n)_Z \) is a \( Z \)-subalgebra of \( \mathcal{D}_\lambda(n) \). We now can realise \( \mathcal{D}_\lambda(n)_Z \) and \( \mathcal{D}_\lambda(n) \) as \( K_\lambda(n) \) and \( K_\lambda(n)_Z \), respectively; cf. [9, Th. 6.3].

**Theorem 6.6.** The linear map \( \Phi : \mathcal{D}_\lambda(n) \to K_\lambda(n) \) sending \( \pi_{\lambda,\mu}(u) \) to \( \left[ \text{diag}(\lambda) \right] u \left[ \text{diag}(\mu) \right] \) for all \( u \in \mathcal{D}_\lambda(n) \) and \( \lambda, \mu \in \mathbb{Z}_n^\lambda \), is an algebra isomorphism. Furthermore we have \( \Phi(\mathcal{D}_\lambda(n)_Z) = K_\lambda(n)_Z \).

**Proof.** By a proof similar to that of [9, 6.3], it is easy to see that \( \Phi \) is an algebra homomorphism. In particular, \( \Phi(1_\lambda) = \left[ \operatorname{diag}(\lambda) \right] \). By (6.4.4), the image of the spanning set \( \{ u_1^A \lambda u_B^- \mid A, B \in \Theta^+_\lambda(n), \lambda \in \mathbb{Z}_n^\lambda \} \) is in fact a basis for \( K_\lambda(n) \), proving the first assertion which implies the last assertion by definition.

We will identify \( \mathcal{D}_\lambda(n) \) with \( K_\lambda(n) \) and \( \mathcal{D}_\lambda(n)_Z \) with \( K_\lambda(n)_Z \) via the map \( \Phi \) defined in Theorem 6.6 and identify \( \mathcal{D}_\lambda(n) \) with \( V_\lambda(n) \) and \( \mathcal{D}_\lambda(n)_Z \) with \( V_\lambda(n)_Z \) as in Theorem 6.5. Then the \( \mathcal{D}_\lambda(n) \)-bimodule structure on \( \mathcal{D}_\lambda(n) \) satisfies the following simple formula: for all \( A \in \Theta^+_\lambda(n), j, \lambda \in \mathbb{Z}_n^\lambda \),

\[
(6.6.1) \quad A(j) [\text{diag}(\lambda)] = [A + \text{diag}(\lambda - \text{co}(A))], \quad [\text{diag}(\lambda)] A(j) = [A + \text{diag}(\lambda - \text{ro}(A))].
\]

For \( A \in \mathcal{D}_\lambda(n) \), choose words \( w_{A^+}, w_{A^-} \in \mathcal{S} \) such that (2.2.1) and its opposite version (obtained by applying (2.2.2) to (2.2.1)) hold. Then, by (6.4.4),

\[
(6.6.2) \quad M(A) := \tilde{u}^+_w(\sigma(A)) \tilde{u}^-_{w_A} = \sum_{B \in A \Theta^+_\lambda(n)} h_{A,B}[B],
\]

where \( h_{A,B} \in Z \). Thus, we have immediately:

**Corollary 6.7.** The set \( \{ M(A) \mid A \in \mathcal{D}_\lambda(n) \} \) forms a \( Z \)-basis for \( \mathcal{D}_\lambda(n)_Z \).

7. **Canonical bases for the integral modified quantum affine \( \mathfrak{gl}_n \)**

It is well known that the positive part of a quantum enveloping algebra \( \mathbf{U} \) has a canonical basis with remarkable properties (see [21], [23], [24]). In contrast, there is no canonical basis for \( \mathbf{U} \). However, the modified form \( \tilde{\mathbf{U}} \) of \( \mathbf{U} \) can have a canonical basis (see [22], [26], [27]). We now define the canonical basis relative the basis \( \{ A \} \) for \( \mathcal{D}_\lambda(n)_Z = K_\lambda(n)_Z \). Our strategy is to use a stabilisation property for the bar involution on \( S_\lambda(n,r)_Z \) to define a bar involution on \( \tilde{K}_\lambda(n)_Z \) (see (6.3.2)) which then induces a bar involution on \( K_\lambda(n)_Z \).
We first define the bar involution on $S_\delta(n, r)$ via the one on the Hecke algebra, following [7] (cf. [33]). Let $W_r$ be the subgroup of $S_\delta(n, r)$ generated by $s_i$ for $1 \leq i \leq r$. Let $\rho$ be the permutation of $\mathbb{Z}$ sending $j$ to $j + 1$ for all $j \in \mathbb{Z}$. Let $H(W_r)$ be the $\mathbb{Z}$-subalgebra of $H_\delta(r)\mathbb{Z}$ generated by $T_{s_i}$ for $1 \leq i \leq r$. Let $\{C'_w \mid w \in W_r\}$ be the canonical basis of $H(W_r)$ defined in [23, 1.1(e)]. For $w = \rho^a x \in S_\delta(n, r)$ with $a \in \mathbb{Z}$ and $x \in W_r$, let $C'_w = T_\rho^a C'_x$. Then the set $\{C'_w \mid w \in S_\delta(n, r)\}$ forms a $\mathbb{Z}$-basis for $H_\delta(r)\mathbb{Z}$. Note that $C'_{w, \mu} = v^{-\ell(w, \mu)}x_\mu$. Let $^- : H_\delta(r)\mathbb{Z} \to H_\delta(r)\mathbb{Z}$ be the ring involution defined by $v = v^{-1}$ and $T_w = T_{w^{-1}}$. We define a map $^\lambda : S_\delta(n, r) \to S_\delta(n, r)$ such that $v = v^{-1}$ and $\bar{f}(C'_{w, \mu} h) = \bar{f}(C'_{w, \mu} h)$ for $f \in \text{Hom}_{H_\delta(r)\mathbb{Z}}(x_\mu H_\delta(r)\mathbb{Z}, x_\lambda H_\delta(r)\mathbb{Z})$ and $h \in H_\delta(r)\mathbb{Z}$. Then the map $^- : S_\delta(n, r) \to S_\delta(n, r)$ is a ring involution.\(^3\) We need to look some first properties of the bar involution in Lemma 7.2 before proving its stabilisation property in Proposition 7.3.

Given $A \in \Theta_\delta(n, r)$, write $y_A = w$ if $A = \delta_\lambda(\lambda, w, \mu)$, and also write $y_A^+$ for the unique longest element in $S_\lambda w S_{\mu}$. For $\lambda \in \Lambda_\delta(n, r)$, let $w_{0, \lambda}$ be the longest element in $S_{\lambda}$. We have $d_A = \ell(y_A^+) = \ell(x) + \ell(y_A) = \ell(w_{0, \mu}) + \ell(y_A)$ given in (3.0.2).

**Lemma 7.1.** For $A \in \Theta_\delta(n, r)$ we have $\ell(y_A^+) = d_A + \ell(w_{0, \mu})$ where $\mu = \text{co}(A)$ and $d_A$ is given in (3.0.2).

**Proof.** For $1 \leq i \leq n$, let $\nu^{(i)}$ be the composition of $\mu_i$ obtained by removing all zeros from column $i$ of $A$. Let $\lambda = \text{ro}(A)$. According to [4, 3.2.3], $y_A^{-1} S_{\lambda} y_A \cap S_\mu = S_\nu$, where $\nu = (\nu^{(1)}, \ldots, \nu^{(n)})$. Let $x$ be the longest element in $S_\nu \cap S_\mu$. Then $y_A = w_{0, \lambda} x$ and $\ell(y_A^+) = \ell(w_{0, \mu}) + \ell(y_A) + \ell(x)$. Since $w_{0, \mu} x$ is the longest element in $S_\mu$, it follows that $w_{0, \mu} = w_{0, \mu} x$ and

$$\ell(x) = \ell(w_{0, \mu}) - \ell(w_{0, \mu}) = \sum_{1 \leq i \leq n} \left( \nu^{(i)} \right)_{k=2} = \sum_{1 \leq i \leq n} \nu^{(i)}.$$ 

Hence,

$$\ell(y_A^+) = \ell(w_{0, \mu}) + \ell(y_A) + \ell(x) = \ell(w_{0, \mu}) + \ell(y_A) + \sum_{1 \leq i \leq n} a_{s, i} a_{t, i}$$

By [11, 5.3], $d_A - \ell(y_A) = \sum_{1 \leq i \leq n; j < l} a_{s, j} a_{t, i}$. Furthermore, we have

$$\ell(w_{0, \mu}) - \ell(w_{0, \mu}) = \sum_{1 \leq i \leq n} \left( \lambda_i (\lambda_i - 1) \right)_{k=2} - \mu_i (\mu_i - 1) = \sum_{1 \leq i \leq n} (a_{s, i} a_{t, i} - a_{k, i} a_{l, i}).$$

Thus, by (7.1.1), we conclude that $d_A = \ell(y_A) - (\ell(w_{0, \mu}) = \ell(y_A^+) - \ell(w_{0, \mu}) - \ell(y_A)$. Consequently, $\ell(y_A^+) = d_A + \ell(w_{0, \mu})$. \(\square\)

For $d \in \mathcal{D}_\lambda \mu$ let

$$\bar{T}_{\Theta_\lambda d S_{\mu}} = v^{-\ell(d^+)} T_{\Theta_\lambda d S_{\mu}}.$$
where $d^+$ is the unique longest element in $\mathfrak{S}_d d^+ \mathfrak{S}_d$. Recall from Theorem 3.3 and Proposition 4.2 that the matrix $S_α = \sum_{1 \leq i \leq n} α_i E_{i,i+1}^*$ defines a semisimple representation of the cyclic quiver $Δ(n)$.

**Lemma 7.2.** For $α, β \in \mathbb{N}^n$, let $A = S_α + \text{diag}(β) \in Θ(Δ(n, r))$. Then, in $S_α(0, r)Z$, $[A] = [A]$ and $[A] = [A]$. In particular, we have $S_α(0, r) = S_α(0, r), S_α(0, r) = S_α(0, r)$ for $α \in \mathbb{N}^n$.

**Proof.** Let $λ = \text{ro}(A)$ and $μ = \text{co}(A)$. Then, by Lemma 7.1, we have $[A](C^ρ_{w, λ}) = T \mathfrak{S}_{λ} S_ρ$ and $[A](C^ρ_{w, λ}) = T \mathfrak{S}_{λ} S_ρ$ (note that $y^{−1}_A = y_A$). By [11, (2.0.2)] (cf. the proof of [11, Prop. 3.5]), we have $y_A = \rho^{−α}$ and $y_A = \rho^{α}$. It follows from [2, (1.10)] that $C^ρ_{y_A} = T \mathfrak{S}_{λ} S_ρ$ and $C^ρ_{y_A} = T \mathfrak{S}_{λ} S_ρ$. Thus,

$$[A](C^ρ_{w, λ}) = [A](C^ρ_{w, λ}) = C^ρ_{y_A} = [A](C^ρ_{w, λ})$$

$$[A](C^ρ_{w, λ}) = [A](C^ρ_{w, λ}) = C^ρ_{y_A} = [A](C^ρ_{w, λ}).$$

Consequently $[A] = [A]$ and $[A] = [A]$. The last assertion is clear. □

The stabilisation property developed at the beginning of last section gives the following stabilisation property.

**Proposition 7.3.** For $A \in \Theta(Δ(n))$ there exist $C_1, \cdots, C_m \in \Theta(Δ(n))$, elements $H_i(v, v') \in \mathbb{Z}_1$ ($1 \leq i \leq m$) and an integer $p_0 \geq 0$ such that, in $S_α(n, pn + σ(A))Z$,

$$[pA] = \sum_{1 \leq i \leq m} H_i(v, v^{-p})[pC_i] \quad \text{for all } p \geq p_0.$$

**Proof.** We prove the assertion by induction on $‖A‖$. If $‖A‖ = 0$ then $[pA] = [pA]$ for all large enough $p$. Assume now that $‖A‖ \geq 1$ and the result is true for all $A'$ with $‖A'‖ < ‖A‖$. By Lemma 6.1 and Proposition 6.2, there exist $A_i \in \Theta(Δ(n))$, $Z_j \in \Theta(Δ(n))$ and $Q_j(v, v') \in \mathbb{Z}_1$ ($1 \leq i \leq N$, $1 \leq j \leq m$) such that the following identity holds in $S_α(n, pn + σ(A))Z$

$$[pA] = [pA_1] \cdots [pA_N] - \sum_{1 \leq j \leq m} Q_j(v, v^{-p})[pZ_j]$$

for all large enough $p$, where $‖Z_i‖ < ‖A‖$ for $1 \leq i \leq m$. It follows from Lemma 7.2 that

$$[pA] = [pA_1] \cdots [pA_N] - \sum_{1 \leq j \leq m} Q_j(v, v^{-p})[pZ_j].$$

Now the assertion follows from the induction hypothesis. □

Recall the ring $\mathbb{Z}_1$ defined in (6.0.1). It admits a ring involution (i.e., a ring automorphism of order two) $\bar{v} = v^{-1}$ and $\bar{v}' = v'^{-1}$. Extend the bar involution on $\mathbb{Z}_1$ to define a ring involution $\bar{K}(n) \mathbb{Z}_1 \rightarrow \bar{K}(n) \mathbb{Z}_1$ by setting $\bar{A} = \sum_{1 \leq i \leq m} H_i(v, v')C_i$ (notation of Proposition 7.3). This involution induces a ring involution
proving the first assertion. The last assertion is clear from (5.2.1).

Lemma 7.5. For algebras (see [28] for a geometric construction). We need the following interval finite condition.

Lemma 7.6. There exist

Proof. (1) follows from Proposition 7.3 and Lemma 7.2. (2) follows from (1), Theorems 5.6 and 6.6. Finally, (3) is clear as the bimodule structure on \( K_\theta(n) \)

Corollary 7.4. (1) For \( \alpha, \beta \in \mathbb{N}^n \), if \( A = S_\alpha + \text{diag}(\beta) \in \Theta_\theta(n, r) \), then \([A] = [A]\) and \([\overline{A}] = [A]\).

In particular, for any \( \alpha \in \mathbb{N}^n \), \( S_\alpha(0) = S_\alpha(0) \), \( S_\alpha(0) = S_\alpha(0) \).

(2) There is a unique \( \mathbb{Q}\)-algebra involution\(^4\) \( \bar{\cdot} : K_\theta(n) \rightarrow K_\theta(n) \) such that \( [A] = [A] \) and \( \overline{\sum_{A \in \Theta_\theta(n)} \beta[A]} = \sum_{A \in \Theta_\theta(n)} \beta[A] \).

(3) The bar involution on \( K_\theta(n) \) preserves the bimodule structure on \( K_\theta(n) \).

Proof. Clearly, by the definition of the bar involution on \( K_\theta(n) \), (1) follows from Proposition 7.3 and Lemma 7.2. (2) follows from (1), Theorems 5.6 and 6.6. Finally, (3) is clear as the bimodule structure on \( K_\theta(n) \) is induced by the algebra structure of \( \hat{K}_\theta(n) \) on which the bar involution is an ring automorphism.

We first look at an algebraic construction of the canonical basis for affine quantum Schur algebras (see [28] for a geometric construction). We need the following interval finite condition.

Lemma 7.5. For \( A \in \Theta_\theta^+(n) \), the set \( \{ B \in \Theta_\theta^+(n) \mid B < A \} \) is finite. Hence, the intervals \((-\infty, A] := \{ B \in \Theta_\theta(n) \mid B \subseteq A \} \) for all \( A \in \Theta_\theta(n) \) are finite.

Proof. There exist \( j_0 \geq n \) such that \( a_{s, j} = 0 \) for \( 1 \leq s \leq n \) and \( j \in \mathbb{Z} \) with \( |j| > j_0 \). Let \( X_A = \{ B \in \Theta_\theta^+(n) \mid b_{s, j} = 0 \text{ for } 1 \leq s \leq n \text{ and } |j| > j_0, \sigma(B) < |A| \} \). Then, \( X_A \) is a finite set.

If \( B < A, 1 \leq i \leq n \) and \( j_0 < j \), then

This implies that if \( B < A \), then \( b_{i, j} = 0 \) for \( 1 \leq i \leq n \) and \( j > j_0 \). Similarly, if \( B < A \), then \( b_{i, j} = 0 \) for \( 1 \leq i \leq n \) and \( j < -j_0 \). Furthermore, by [4, 3.7.6], we conclude that \( \sigma(B) < |B| \) for \( B \in \Theta_\theta^+(n) \) with \( B < A \). Consequently, \( \{ B \in \Theta_\theta^+(n) \mid B < A \} \subseteq X_A \), proving the first assertion. The last assertion is clear from (5.2.1).

Proposition 7.6. (1) There is a unique \( \mathbb{Z}\)-basis \( \{ \theta_{A, r} \mid A \in \Theta_\theta(n, r) \} \) for \( S_\theta(n, r) \) such that \( \overline{\theta_{A, r}} = \theta_{A, r} \) and

\[
\theta_{A, r} - [A] = \sum_{B \subseteq \Theta_\theta(n, r)} g_{B, A, r}[B] \in \sum_{B \subseteq \Theta_\theta(n, r)} [\overline{B}][B].
\]

\(^4\)This bar involution can also be induced from the bar involutions on \( S_\theta(n, r) \) via \( S_\theta(n) \) and \( V_\theta(n) \). Thus, we may avoid using the stabilisation property.
Theorem 7.7. Let \( M \) be the \( \Theta \)-affine case. In particular, \( g_{B,A,r} \) can be described in terms of Kazhdan–Lusztig polynomials.

Proof. By Proposition 2.2, for each \( A \in \Theta_\Theta(n,r) \), we may choose words \( w_{A^+} \in \varSigma \) such that (2.2.1) hold. Let \( w_{A^-} = \tilde{w}_{B^-} \). By (2.2.1) and its opposite version for \( \tilde{w}_{A^-} = \tau(\tilde{w}_{A^-}) \) (see (2.2.2)) and Proposition 5.3, we have

\[
\begin{align*}
(7.6.2) \quad m^A := \zeta_r(\tilde{u}_{w_{A^+}}^r)[\text{diag}(\sigma(A))]\zeta_r(\tilde{w}_{A^-}) = [A] + \sum_{B \subseteq A} h_{A,B}[B] \quad (h_{A,B} \in \mathcal{Z}).
\end{align*}
\]

Now the interval finite condition in Lemma 7.5 implies that there exist \( h_{A,B}' \in \mathcal{Z} \) such that

\[
[A] = m^A + \sum_{B \subseteq A} h_{A,B}'m(B).
\]

Furthermore, by Lemma 7.2, we have \( \overline{m^A} = m^A \) for \( A \in \Theta_\Theta(n,r) \). Thus, (7.6.2) implies

\[
\begin{align*}
[\overline{A}] &= m^A + \sum_{B \subseteq A} h_{A,B}'m(B) = [A] + \sum_{B \subseteq A} k_{A,B}[B],
\end{align*}
\]

where \( k_{A,B} \in \mathcal{Z} \). Now (1) follows from a standard argument; see, e.g., [25, 7.10]. Let \( \leq \) be the partial order on \( \Theta_\Theta(n,r) \) defined in [28, 4.1]. According to [29, §7], if \( A, B \in \Theta_\Theta(n,r) \) and \( B < A \) then \( B \subseteq A \). Thus, by [28, 4.1(e)] and [33, Remark 7.6], we conclude (2).

We now construct the canonical basis for \( \mathcal{K}_\Theta(n) \) as follows. See [17] for a construction in the non-affine case.

Theorem 7.7. (1) There exists a unique \( \mathcal{Z} \)-basis \( \{ \theta_A \mid A \in \tilde{\Theta}_\Theta(n) \} \) for \( \mathcal{K}_\Theta(n) = \mathcal{S}_\Theta(n) \) such that \( \bar{\theta}_A = \theta_A \) and \( \theta_A - [A] \in \sum_{B \subseteq A} v^{-1}Z[v^{-1}][B] \).

(2) The algebra homomorphism \( \zeta_r : \mathcal{K}_\Theta(n) \to \mathcal{S}_\Theta(n) \) given in (6.4.1) preserves the bar involution and the canonical bases:

\[
\begin{align*}
(a) \zeta_r(\bar{u}) = \frac{\zeta_r(u)}{\bar{u}} \text{ for all } u \in \mathcal{K}_\Theta(n) ; \quad (b) \zeta_r(\theta_A) = \begin{cases}
\theta_{A,r}, & \text{if } A \in \Theta_\Theta(n,r) ; \\
0, & \text{otherwise}.
\end{cases}
\end{align*}
\]

(3) There is an anti-automorphism \( \hat{\tau} \) on \( \mathcal{K}_\Theta(n) \) such that \( \hat{\tau}(A) = [A] \) and \( \hat{\tau}(\theta_A) = \theta_{A,r} \).

Proof. Consider the monomial basis \( \{ \mathcal{M}^A \mid A \in \tilde{\Theta}_\Theta(n) \} \) given in Corollary 6.7. Then Lemma 7.2 implies \( \bar{\mathcal{M}}^A = \mathcal{M}^A \) and (6.6.2) together with the interval finite property Lemma 7.5 implies \( [A] = \mathcal{M}^A + h \), where \( h \) is a \( \mathcal{Z} \)-linear combination of \( \mathcal{M}^C \) with \( C \in \tilde{\Theta}_\Theta(n) \) and \( C \subseteq A \). Thus, we conclude that \( [\overline{A}] - [A] \in \sum_{C < A} \mathcal{Z}[C] \). Hence, like the proof of Proposition 7.6, a standard argument proves (1).

According to (6.4.1) and Lemma 7.2 we see that \( \zeta_r(\overline{\mathcal{M}^A}) = \bar{\zeta}_r(\mathcal{M}^A) \) for \( A \in \tilde{\Theta}_\Theta(n) \). Furthermore, by Corollary 6.7, the set \( \{ \mathcal{M}^A \mid A \in \tilde{\Theta}_\Theta(n) \} \) forms a \( \mathcal{Z} \)-basis for \( \mathcal{K}_\Theta(n) \). Thus,
Also, by Corollary 7.4(3),

$$\zeta_r(\bar{u}) = \overline{\zeta_r(u)}$$

for $u \in K_\zeta(n)\mathbb{Z}$. The second assertion in (2) follows from the argument for the uniqueness of canonical basis.

By [28, 1.11], the $\mathbb{Z}$-linear map $\tau_r : S_\zeta(n, r) \to S_\zeta(n, r)$, $[A] \mapsto [\bar{A}]$ is an algebra anti-automorphism, where $\bar{A}$ is the transpose of $A$. By Proposition 6.3, the maps $\tau_r$ induce an algebra anti-automorphism $\hat{\tau} : K_\zeta(n)\mathbb{Z} \to K_\zeta(n)\mathbb{Z}$ such that $\hat{\tau}([A]) = [\bar{A}]$ for $A \in \Theta_\zeta(n)$. Finally, applying $\hat{\tau}$ to $\theta_A - [A]$ yields $\hat{\tau}(\theta_A) = \theta_{\bar{A}}$ by the uniqueness of canonical bases. \hfill \Box

**Remark 7.8.** The basis constructed in Theorem 7.7(1) is the canonical basis for the integral modified quantum affine $\mathfrak{gl}_n$. Theorem 7.7(2b) shows that this basis is the lifting of the canonical bases for affine quantum Schur algebras. A similar basis with a similar property for the modified quantum affine $\mathfrak{sl}_n$ was conjectured by Lusztig in [28, 9.3]. This conjecture (rather its slight modified version) was proved by Vasserot and Schiffmann in [32]. Thus, Theorems 6.5, 6.6 and 7.7 can be regarded as of a generalisation of the conjecture of Lusztig to the quantum loop algebra $U(\mathfrak{gl}_n)$. We will address an extension of our approach to the extended quantum affine $\mathfrak{sl}_n$ case in the last section.

We end this section with a comparison of this canonical basis and the canonical basis for the Ringel–Hall algebra of a cyclic quiver. According to [33, Prop 7.5] (see also [24]), there is a unique $\mathbb{Z}$-basis $\{\theta^+_A \mid A \in \Theta^+_\zeta(n)\}$ for the Ringel–Hall algebra $R_\zeta(n)\mathbb{Z} = \mathcal{O}^+_\zeta(n)\mathbb{Z}$ such that $\theta^+_A = \theta_A$ and

$$
\theta^+_A - \bar{u}^+_A \in \sum_{B \prec A, B \in \Theta^+_\zeta(n) \atop d(B) = d(A)} v^{-1}Z[v^{-1}]\bar{u}^+_B.
$$

**Proposition 7.9.** Assume $A \in \Theta^+_\zeta(n)$ and $\lambda \in \mathbb{Z}^n_\zeta$. Then we have $\theta^+_A[\text{diag}(\lambda)] = \theta_{A+\text{diag}(\lambda-\text{co}(A))}$. In particular, we have $\theta^+_A = \sum_{\mu \in \mathbb{Z}^n_\zeta} \theta_{A+\text{diag}(\mu)}$.

**Proof.** By (6.6.1) and (7.8.1),

$$
\theta^+_A[\text{diag}(\lambda)] - [A + \text{diag}(\lambda - \text{co}(A))] \in \sum_{B \prec A, B \in \Theta^+_\zeta(n) \atop d(B) = d(A)} v^{-1}Z[v^{-1}][B + \text{diag}(\lambda - \text{co}(B))].
$$

It is direct to check that, for $d(B) = d(A)$ and $B \in \Theta^+_\zeta(n)$, $\text{ro}(B) - \text{co}(B) = \text{ro}(A) - \text{co}(A)$. Hence,

$$
\theta^+_A[\text{diag}(\lambda)] - [A + \text{diag}(\lambda - \text{co}(A))] \in \sum_{C \in \Theta^+_\zeta(n) \atop C \prec A+\text{diag}(\lambda-\text{co}(A))} v^{-1}Z[v^{-1}]C.
$$

Also, by Corollary 7.4(3), $\theta^+_A[\text{diag}(\lambda)] = \overline{\theta^+_A[\text{diag}(\lambda)]} = \theta^+_A[\text{diag}(\lambda)]$. Hence, the first assertion follows from the uniqueness of the canonical basis. Now, the identity element $1 = \sum_{\lambda \in \mathbb{Z}^n_\zeta} [\text{diag}(\lambda)]$ gives the last assertion. \hfill \Box
8. Application to a Conjecture of Lusztig

Let \( U_\triangle(n) \) be the extended affine \( \mathfrak{sl}_n \) as defined in Theorem 2.3(2) and let \( \hat{U}_\triangle(n) = \oplus_{\lambda, \mu \in \mathbb{Z}_n^a} U_\triangle(n)/\lambda \mathcal{I}_\mu \), where \( \lambda \mathcal{I}_\mu := \sum_{j \in \mathbb{Z}_n^a} (K^j - v^\lambda j) U_\triangle(n) + \sum_{j \in \mathbb{Z}_n^a} U_\triangle(n)(K^j - v^\mu j) \). Since \( \lambda \mathcal{I}_\mu = \lambda \mathcal{I}_\mu \cap U_\triangle(n) \) (see Theorem 2.3(2)), it follows that \( \hat{U}_\triangle(n) \cong \oplus_{\lambda, \mu \in \mathbb{Z}_n^a} U_\triangle(n)/\lambda \mathcal{I}_\mu \), where \( \lambda U_\triangle(n) \mu = \pi_{\lambda, \mu}(U_\triangle(n)) \). Thus, we will regard \( \hat{U}_\triangle(n) \) as this subalgebra of \( \mathfrak{D}_\triangle(n) = K_\triangle(n) \).

We now look at an application to the conjecture given in [28, 9.3] which is proved in [32].

Let \( \hat{U}_\triangle(n) Z \) be the \( Z \)-subalgebra of \( \mathfrak{D}_\triangle(n) \) generated by

\[
\tilde{u}_{me_i^+}^{\triangle}([\text{diag}(\lambda)] = L_i^{(m)}[\text{diag}(\lambda)], \quad \tilde{u}_{me_i^-}^{\triangle}([\text{diag}(\lambda)] = F_i^{(m)}[\text{diag}(\lambda)]
\]

for all \( 1 \leq i \leq n, m \in \mathbb{N} \) and \( \lambda \in \mathbb{Z}_n^a \). Then \( \hat{U}_\triangle(n) Z \) is a subalgebra of \( \mathfrak{D}_\triangle(n) Z = K_\triangle(n) Z \).

Call a matrix \( A = (a_{i,j}) \in \hat{\Theta}_\triangle(n) \) to be aperiodic if for every integer \( l \neq 0 \) there exists \( 1 \leq i \leq n \) such that \( a_{i,i+l} = 0 \). Let \( \hat{\Theta}_\triangle^{ap}(n) \) be the set of all aperiodic matrices in \( \hat{\Theta}_\triangle(n) \).

Recall the monomial basis for \( \mathfrak{D}_\triangle(n) Z \) given in Corollary 6.7.

**Lemma 8.1.** The set \( \{ \mathcal{M}(A) \mid A \in \hat{\Theta}_\triangle^{ap}(n) \} \) forms a \( Z \)-basis for \( \hat{U}_\triangle(n) Z \).

**Proof.** By [6, Th. 7.5(1)], the elements \( \tilde{u}_{me_i}^{\triangle}(\lambda, \mu) \in \hat{\Theta}_\triangle(n) \cap \hat{\Theta}_\triangle^{ap}(n) \), form a basis for the +-part \( \hat{U}_\triangle^+ Z \) generated by all \( L_i^{(m)} \). Hence, the set \( \{ \mathcal{M}(A) \mid A \in \hat{\Theta}_\triangle^{ap}(n) \} \) spans \( \hat{U}_\triangle(n) Z \). By (6.6.2), the set is linearly independent. 

For each \( A \in \hat{\Theta}_\triangle^{ap}(n) \), use the coefficients \( h_{A,B} \) given in (6.6.2) and the order \( \sqsubseteq \) given in (5.2.1) to define (cf. [6, Def. 7.2]) recursively the elements \( \mathcal{E}_A \in \hat{U}_\triangle(n) Z \) by

\[
\mathcal{E}_A = \begin{cases} 
\mathcal{M}(A) , & \text{if } A \text{ is minimal relative to } \sqsubseteq; \\
\mathcal{M}(A) - \sum_{B \in \hat{\Theta}_\triangle^{ap}(n)} h_{A,B} \mathcal{E}_B, & \text{otherwise.}
\end{cases}
\]

**Lemma 8.2.** (1) The set \( \{ \mathcal{E}_A \mid A \in \hat{\Theta}_\triangle^{ap}(n) \} \) forms a \( Z \)-basis for \( \hat{U}_\triangle(n) Z \).

(2) For \( A \in \hat{\Theta}_\triangle^{ap}(n) \) we have \( \mathcal{E}_A - [A] \in \sum_{B \in \hat{\Theta}_\triangle^{ap}(n)} \hat{\Theta}_\triangle^{ap}(n) \mathcal{Z}[B] \).

**Proof.** Statement (1) follows from Lemma 8.1 and the definition \( \mathcal{E}_A \) (8.1.1). We prove (2) by induction on \( |A| \). The assertion is clear for by \( |A| = 0 \). Assume now \( |A| \geq 1 \) By (6.6.2) and (8.1.1), we have

\[
\mathcal{E}_A - [A] + \sum_{B \in \hat{\Theta}_\triangle^{ap}(n)} h_{A,B}(\mathcal{E}_B - [B]) = \sum_{B \in \hat{\Theta}_\triangle^{ap}(n)} h_{A,B}[B],
\]

Now the assertion follows from induction since \( B \sqsubseteq A \) implies \( |B| < |A| \). 

Note that the restriction of the bar involution (7.3.1) gives a bar involution on \( \hat{U}_\triangle(n) Z \).
Proposition 8.3. There exists a unique $\mathcal{Z}$-basis $\{\theta'_A \mid A \in \tilde{\Theta}_\delta^{ap}(n)\}$ for $\hat{U}_\delta(n)_\mathcal{Z}$ such that $\theta'_A = \theta'_A$ and

$$\theta'_A - \mathcal{E}_A \in \sum_{B \in \tilde{\Theta}_\delta^{ap}(n), B \subseteq A} v^{-1}\mathcal{Z}[v^{-1}]\mathcal{E}_B.$$  

Proof. Since, by (8.1.1),

$$\mathcal{E}_A = \mathcal{M}(A) + a \mathcal{Z}\text{-linear combination of } \mathcal{M}(C) \text{ with } C \in \tilde{\Theta}_\delta^{ap}(n) \text{ and } C \supseteq A,$$

it follows that $\mathcal{E}_A - \mathcal{E}_A \in \sum_{C \in \tilde{\Theta}_\delta^{ap}(n), C \supset \Theta} \mathcal{Z}\mathcal{E}_C$. Now the assertion follows from a standard argument. □

Remark 8.4. Motivated by [28, Th. 8.2], it would be natural to conjecture that $\theta_A \in \hat{U}_\delta(n)_\mathcal{Z}$ for all $A \in \tilde{\Theta}_\delta^{ap}(n)$. Equivalently, $\theta'_A = \theta_A$ if $A \in \tilde{\Theta}_\delta^{ap}(n)$ (cf. [6, Th. 8.5]). In the rest of the section, we show some strong evidence for the truth of this conjecture.

Let $\mathcal{L}_r = \sum_{A \in \Theta_\delta(n,r)} \mathcal{Z}[v^{-1}][A] \in \mathcal{S}_\delta(n,r)_\mathcal{Z}$ and let $\mathcal{P}$ be the $\mathcal{Z}$-submodule of $\hat{\mathcal{S}}_\delta(n)_\mathcal{Z}$ spanned by the periodic elements $[B]$ with $B \in \tilde{\Theta}_\delta(n) \setminus \tilde{\Theta}_\delta^{ap}(n)$. Recall the algebra homomorphisms $\zeta$ in Theorem 3.2 and $\hat{\zeta}_r$ in (6.4.1) and note that $\hat{\zeta}_r(\mathcal{P}) \cap \mathcal{U}_\delta(n,r) = 0$, where $\zeta_r(\mathcal{U}_\delta(n)) = \mathcal{U}_\delta(n,r)$.

Let $\Theta_\delta^{ap}(n,r) = \tilde{\Theta}_\delta^{ap}(n) \cap \Theta_\delta(n,r)$.

Lemma 8.5. Assume $A \in \tilde{\Theta}_\delta^{ap}(n)$.

1. If $A \not\in \Theta_\delta(n,r)$ then we have $\hat{\zeta}_r(\mathcal{E}_A) = 0$.

2. If $A \in \Theta_\delta(n,r)$ then we have $\hat{\zeta}_r(\mathcal{E}_A) - [A] \in v^{-1}\mathcal{L}_r$.

Proof. If $A \not\in \Theta_\delta(n,r)$, Lemma 8.2(2) implies $\hat{\zeta}_r(\mathcal{E}_A) = \hat{\zeta}_r(\mathcal{E}_A) - \hat{\zeta}_r([A]) \in \hat{\zeta}_r(\mathcal{P}) \cap \mathcal{U}_\delta(n,r) = 0$, proving (1).

Now we assume $A \in \Theta_\delta(n,r)$. If $[A] = 0$ then $\mathcal{E}_A = [A]$ and $\hat{\zeta}_r(\mathcal{E}_A) - [A] = 0$. Now we assume $\|A\| > 0$. We write $\theta_{A,r}$ as in (7.6.1). By Lemma 8.2 and [28, 8.2], we see that

$$\theta_{A,r} - \left(\hat{\zeta}_r(\mathcal{E}_A) + \sum_{B \in \Theta_\delta^{ap}(n,r)} g_{B,A,r} \hat{\zeta}_r(\mathcal{E}_B)\right) = ([A] - \hat{\zeta}_r(\mathcal{E}_A)) + \sum_{B \in \Theta_\delta^{ap}(n,r)} g_{B,A,r}([B] - \hat{\zeta}_r(\mathcal{E}_B))$$

$$+ \sum_{B \in \Theta_\delta(n,r) \setminus \Theta_\delta^{ap}(n,r)} g_{B,A,r}[B],$$

which belongs to $\hat{\zeta}_r(\mathcal{P}) \cap \mathcal{U}_\delta(n,r) = 0$. Thus, by the induction hypothesis,

$$\hat{\zeta}_r(\mathcal{E}_A) - [A] = \sum_{B \in \Theta_\delta^{ap}(n,r)} g_{B,A,r}([B] - \hat{\zeta}_r(\mathcal{E}_B)) + \sum_{B \in \Theta_\delta(n,r) \setminus \Theta_\delta^{ap}(n,r)} g_{B,A,r}[B] \in v^{-1}\mathcal{L}_r$$

as required. □

We now show that the basis $\theta'_A$ satisfies a property similar to Theorem 7.7(2b).
Theorem 8.6. Let \( A \in \tilde{\Theta}_\Delta^{\text{ap}}(n) \). Then we have
\[
\hat{\zeta}_r(\theta'_A) = \begin{cases} 
\theta_{A,r} & \text{if } A \in \Theta_\Delta(n, r); \\
0 & \text{if } A \not\in \Theta_\Delta(n, r).
\end{cases}
\]
Hence, we have \( \hat{\zeta}_r(\theta'_A) = \hat{\zeta}_r(\theta_A) \) for \( A \in \tilde{\Theta}_\Delta^{\text{ap}}(n) \).

Proof. If \( A \not\in \Theta_\Delta(n, r) \) then, by Proposition 8.3 and Lemma 8.5, we see that
\[
\hat{\zeta}_r(\theta'_A) = \hat{\zeta}_r(\theta'_A - E_A) \in \sum_{B \in \Theta_\Delta^{\text{ap}}(n, r) \subseteq B \subseteq A} v^{-1} \mathbb{Z}[v^{-1}] \hat{\zeta}_r(E_B) \subseteq v^{-1} \mathcal{L}_r.
\]
If \( A \in \Theta_\Delta(n, r) \) then, by loc. cit., we have
\[
\hat{\zeta}_r(\theta'_A) \in \hat{\zeta}_r(E_A) + \sum_{B \in \Theta_\Delta^{\text{ap}}(n, r) \subseteq B \subseteq A} v^{-1} \mathbb{Z}[v^{-1}] \hat{\zeta}_r(E_B) \subseteq [A] + v^{-1} \mathcal{L}_r.
\]
Furthermore, we have \( \hat{\zeta}_r(\theta'_A) = \hat{\zeta}_r(\theta'_A) \) for all \( A \in \tilde{\Theta}_\Delta^{\text{ap}}(n) \). The assertion follows the uniqueness of the canonical basis. \( \square \)

Theorem 8.6 gives an algebraic construction of the conjecture of Lusztig stated at the end of [28, §9.3] for the modified extended quantum affine \( sl_n, \hat{U}_\Delta(n) \mathbb{Z} \), idempotented on \( \mathbb{Z}^n \); see [32] for a proof for the (polynomial weighted) modified quantum affine \( sl_n \) which is idempotented on \( \mathbb{N}^n \) (compare the construction in [29, §7] for the modified quantum affine \( sl_n \) idempotented on \( \mathbb{Z}^{n-1} \)). Note that, by the presentation for \( \hat{U}_\Delta(n) \mathbb{Z} \) given in [27, 31.1.3], this modified algebra of Schiffmann–Vasserot is a homomorphic image of \( \hat{U}_\Delta(n) \mathbb{Z} \).

References


This conjecture was made for quantum affine \( sl_n \) with associated modified quantum group idempotented on \( \mathbb{Z}^{n-1} \).
THE INTEGRAL QUANTUM LOOP ALGEBRA OF $\mathfrak{gl}_n$


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