FOLDING DERIVED CATEGORIES WITH FROBENIUS FUNCTORS

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Abstract. Following the work [4], we show that a Frobenius morphism F on an algebra A induces naturally a functor F on the (bounded) derived category $\mathscr{D}^b(A)$ of **mod**-A, and we further prove that the derived category $\mathscr{D}^b(A^F)$ of **mod**-A^F for the F-fixed point algebra A^F is naturally embedded as the triangulated subcategory $\mathscr{D}^b(A)^F$ of F-stable objects in $\mathscr{D}^b(A)$. When applying the theory to an algebra with a finite global dimension, we discover a folding relation between the Auslander-Reiten triangles in $\mathscr{D}^b(A^F)$ and those in $\mathscr{D}^b(A)$. Thus, the AR-quiver of $\mathscr{D}^b(A^F)$ can be obtained by folding the AR-quiver of $\mathscr{D}^b(A)$. Finally, we further extend this relation to the root categories $\mathscr{R}(A^F)$ of A^F and $\mathscr{R}(A)$ of A, and show that, when A is hereditary, this folding relation over the indecomposable objects in $\mathscr{R}(A^F)$ and $\mathscr{R}(A)$ results in the same relation on the associated root systems as induced from the graph folding relation.

CONTENTS

1. Introduction

In [4], we introduced the notion of Frobenius morphisms into the representation theory of algebras. The main advantage of using Frobenius morphisms is to link the structure and representations of an algebra B defined over a finite field $\mathbb{F}_q$ to that of an algebra A defined over the algebraic closure k of $\mathbb{F}_q$. Here A and B are related by a Frobenius morphism F on A such that $B = A^F$, the fixed point algebra. Thus, by regarding B-modules as F-stable A-modules, we proved that many nice properties such as heredity, finite representation type and so on are unchanged when passing from A to B, and, by applying the theory to quivers with automorphisms, and hence to Lie theory, we obtained a unified approach to both quiver and $\mathbb{F}_q$-species representations, and to Kac's theory (including both Kac's polynomials and Kac's theorem) for the symmetrizable case from that for the symmetric case. Furthermore, there is a close connection between the Auslander-Reiten theories of A and B which can be simply described by folding the Auslander-Reiten quiver of A to obtain the AR-quiver of B.

In this paper, we shall extend our investigation to the derived category level. Let $\mathscr{D}^b(B)$ (resp. $\mathscr{D}^b(A)$) be the bounded derived category of the category of finite dimensional B-modules (resp. A-modules). Then, base change (from $\mathbb{F}_q$ to k) induces a functor $\Psi = - \otimes \mathrm{id}_k : \mathscr{D}^b(B) \to \mathscr{D}^b(A)$.

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We shall prove that Ψ is faithful and its image can be characterized in terms of a Frobenius functor induced from the given Frobenius morphism F on A. More precisely, we lift the Frobenius (twist) functor $(\ )^{[1]}$ on the category of A-modules to its bounded derived category $\mathscr{D}^b(A)$, and then consider the subcategory $\mathscr{D}^b(A)^F$ consisting of objects satisfying $\mathsf{M} \cong \mathsf{M}^{[1]}$ in $\mathscr{D}^b(A)$ with morphisms compatible with these isomorphisms. We shall prove that Ψ gives rise to a triangulated category equivalence between $\mathscr{D}^b(A^F)$ and $\mathscr{D}^b(A)^F$. As an application of this result, we further extend the folding relation between the AR-quivers of A and B to a similar relation between the AR-quivers of $\mathscr{D}^b(A)$ and $\mathscr{D}^b(B)$ and between their associated root categories.

We organize the paper as follows. In §2, we introduce the Frobenius functor on the category of A-modules without using Frobenius maps on modules, and reformulate some definitions and results in [4] under this new setting. In the next three sections, we lift this functor step by step to the chain complex level, homotopy level and finally to the derived category level, and establish the relevant category equivalence at every level. In particular, the required derived category equivalence between the subcategory $\mathscr{D}^b(A)^F$ of F-stable objects in $\mathscr{D}^b(A)$ and the derived category $\mathscr{D}^b(A^F)$ is proved in §5. As applications of the equivalence to the case where A has a finite global dimension, we obtain a construction of Auslander-Reiten triangles in $\mathscr{D}^b(A^F)$ in terms of those in $\mathscr{D}^b(A)$ in §5, prove that the Auslander-Reiten quiver of $\mathscr{D}^b(A^F)$ can be obtained by folding the Auslander-Reiten quiver of $\mathscr{D}^b(A)$ in §6, and further embed the root category $\mathscr{R}(A^F)$ as a subcategory of F-stable objects of the root category $\mathscr{R}(A)$ of A in §8. Finally, in the Appendix section, we generalize the relation between finite dimensional hereditary algebras and quivers with automorphisms investigated in [4, §6] to arbitrary finite dimensional algebras, and give a list of all known relations between theories on A and theories on B.

Throughout the paper, $\mathbb{F}_q$ denotes a finite field of q elements, $k = \overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$ and $\mathfrak{f} : k \to k$ the field automorphism given by the formula $\mathfrak{f}(\lambda) = \lambda^q$. For an algebra A over a field, by **mod**-A we denote the category of all finite dimensional left A-modules. All modules considered here are finite dimensional left modules.

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2. Frobenius twists and F-stable modules

We start off on the definition of Frobenius morphisms.

Definition 2.1. Let A be an algebra over k with 1. A map $F : A \to A$ is called a *Frobenius morphism* on A if

(F1) F is a ring automorphism;
(F2) $F(\lambda a) = \lambda^q F(a)$ for all $a \in A$ and $\lambda \in k$;
(F3) for any $a \in A$, $F^n(a) = a$ for some $n > 0$.

We also call an abelian group automorphism $F = F_V$ on a vector space V over k a *Frobenius map* if both (F2) and (F3) are satisfied. Clearly, every $\mathbb{F}_q$-space V_0 defines naturally a Frobenius map $F = \mathrm{id}_{V_0} \otimes \mathfrak{f}$ on $V := V_0 \otimes k$.

An $\mathbb{F}_q$-linear isomorphism $F : V \to W$ between k-spaces V and W is called a *q-twist map* if $F(\lambda v) = \lambda^q F(v)$ for all $v \in V$ and $\lambda \in k$. The following lemma tells us that a q-twist map from V to itself is indeed a Frobenius map if V is finite dimensional (see [14, p.192,Theorem]).

Lemma 2.2. *Let V be a finite dimensional k-space. A q-twist map $F_V : V \to V$ is a Frobenius map.*

Proof. Without loss of generality, we assume that $\dim V = m$ and $V = k^m$. Then there is an invertible matrix $B \in GL_m(k)$ such that $F_V(\mathbf{x}) = B\mathbf{x}^{[1]}$ for all $\mathbf{x} = (x_i) \in V$, where $\mathbf{x}^{[1]} = (x_i^q)$. Let F denote the Frobenius map $GL_m(k) \to GL_m(k), X = (x_{ij}) \mapsto (x_{ij}^q) := X^{[1]}$. By Lang-Steinberg's theorem (see, e.g., [17]), there exists an invertible matrix C such that $B = CF(C^{-1})$. For each fixed $\mathbf{x}$, choose n such that $F^n(C) = C$ and $\mathbf{x}^{[n]} = \mathbf{x}$. Then

$$\begin{aligned} F_V^n\mathbf{x} &= BF(B)\cdots F^{n-1}(B)\mathbf{x}^{[n]} \\ &= CF(C^{-1})F(C)F^2(C^{-1})\cdots F^{n-1}(C)F^n(C^{-1})\mathbf{x} \\ &= CF^n(C)^{-1}\mathbf{x} = \mathbf{x}. \end{aligned}$$

So F_V is a Frobenius map. □

Given a Frobenius morphism F on A, let

$$A^F := \{a \in A \mid F(a) = a\}$$

be the set of F-fixed points. Then A^F is an $\mathbb{F}_q$-subalgebra of A, and $A = A^F \otimes_{\mathbb{F}_q} k$. Further, the Frobenius morphism F is defined by A^F, i.e., $F = \mathrm{id}_{A^F} \otimes \mathfrak{f}$.

The Frobenius twist of a module M over an algebra A with a Frobenius morphism F has been defined in [4] with respect to a given Frobenius map F_M on M. We now give an intrinsic definition of Frobenius twist of M and reformulate some definitions and results in [4, §4].

Let $\mathfrak{f} : k \to k$ be the field automorphism given by the formula $\mathfrak{f}(\lambda) = \lambda^q$. For each k-space V and $r \geqslant 1$, let $V^{(r)}$ be the new vector space obtained from V by base change via $\mathfrak{f}^r$:

$$V^{(r)} = V \otimes_{\mathfrak{f}^r} k.$$

Thus, for $v \in V$ and $\lambda \in k$, we have $\lambda v \otimes 1 = v \otimes \lambda^{q^r}$. In other words, putting $v^{(r)} = v \otimes 1$, we have

$$(u+v)^{(r)} = u^{(r)} + v^{(r)}, \qquad (\lambda v)^{(r)} = \lambda^{q^r} \centerdot v^{(r)}.$$

Note that $V^{(r)}$ may be identified as V with a twisted scalar multiplication

$$\lambda \centerdot v = \sqrt[q^r]{\lambda} v.$$

Further, for a k-linear map $\phi : U \to V$, the map $\phi^{(r)} := \phi \otimes 1 : U^{(r)} \to V^{(r)}$ is again a k-linear map. In this way, we obtain an exact additive functor $(\ )^{(r)}$ from the category of k-vector spaces onto itself (see [7][1]).

Let $\tau_V^{(r)} : V \to V^{(r)}$ be the $\mathbb{F}_{q^r}$-linear isomorphism sending v to $v^{(r)}$. In case $r = 1$, we write

$$\tau_V = \tau_V^{(1)}.$$

If A is a k-algebra, then $A^{(r)}$ is also a k-algebra, and $\tau_A^{(r)} : A \to A^{(r)}$ becomes an $\mathbb{F}_{q^r}$-algebra isomorphism. The following lemma is obvious.

Lemma 2.3. *Let A be finite dimensional. A map $F : A \to A$ is a Frobenius morphism if and only if $\tau_A \circ F^{-1} : A \to A^{(1)}$ is a k-algebra isomorphism.*

Definition 2.4. Let A be a k-algebra with Frobenius morphism F and let M be an A-module defined by the k-algebra homomorphism $\pi : A \to \mathrm{End}_k(M)$. This gives a k-algebra homomorphism $\pi^{(1)} : A^{(1)} \to \mathrm{End}_k(M)^{(1)}$. Thus, the composition of the following maps

$$A \xrightarrow{F^{-1}} A \xrightarrow{\tau_A} A^{(1)} \xrightarrow{\pi^{(1)}} \mathrm{End}_k(M)^{(1)} \cong \mathrm{End}_k(M^{(1)})$$

[1] We thank Wilberd Van der Kallen for the reference.

defines an A-module structure on $M^{(1)}$ with the following new action

(2.4.1) $$a \centerdot (m^{(1)}) = (F^{-1}(a)m)^{(1)}, \quad \forall\, a \in A, m \in M.$$

We denote this module by $M^{[1]}$ and call it the *Frobenius twist* of M.

If $f : M \to N$ is an A-module homomorphism, then the k-linear map $f^{(1)} : M^{(1)} \to N^{(1)}$ becomes an A-module homomorphism $M^{[1]} \to N^{[1]}$ which is denoted by $f^{[1]}$ in the sequel. Thus, we obtain a functor

(2.4.2) $$(\ )^{[1]} = (\ )^{[1]}_{\mathbf{mod}\text{-}A} : \mathbf{mod}\text{-}A \to \mathbf{mod}\text{-}A.$$

This functor will be called the *Frobenius (twist) functor* on $\mathbf{mod}$-A. Clearly, it is a category equivalence.

Inductively, we can define the s-fold Frobenius twist $M^{[s]} := (M^{[s-1]})^{[1]}$ of M and $f^{[s]} = (f^{[s-1]})^{[1]}$ for $s \geqslant 1$, where $M^{[0]} = M$ and $f^{[0]} = f$ by convention. Further, we can define $M^{[-1]}$ to be the A-module N such that $M = N^{[1]}$ and similarly $f^{[-1]}$, and inductively, $M^{[s]}$ and $f^{[s]}$ for $s < 0$.

The Frobenius twist $M^{[1]}$ defined here coincides, up to isomorphism, with the Frobenius twist defined in [4]. Recall from [4, §4] that, for a given Frobenius map F_M on M, $M^{[F_M]}$ is the A-module by setting $M^{[F_M]} = M$ as a vector space with F-twisted action

(2.4.3) $$a * m := F_M\big(F^{-1}(a)F_M^{-1}(m)\big) \text{ for all } a \in A, m \in M.$$

Alternatively, if $\pi : A \to \mathrm{End}_k(M)$ and $\pi^{[F_M]} : A \to \mathrm{End}_k(M^{[F_M]})$ denote the corresponding representations, then

$$\pi^{[F_M]}(a) = F_M \circ \pi(F^{-1}(a)) \circ F_M^{-1} \text{ for all } a \in A.$$

In other words, $\pi^{[F_M]}$ is uniquely defined by the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \pi\ } & \mathrm{End}_k(M) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_{(M,M)}} \\ A & \xrightarrow{\ \pi^{[F_M]}\ } & \mathrm{End}_k(M^{[F_M]}) \end{array}$$

where $F_{(M,M)}$ is the induced Frobenius map on $\mathrm{End}_k(M)$ sending f to $F_M \circ f \circ F_M^{-1}$.

The A-module $M^{[F_M]}$ is called the *F_M-twist* of M. Inductively, we define the s-fold F_M-twist $M^{[F_M]^s}$ of M by $M^{[F_M]^s} = (M^{[F_M]^{s-1}})^{[F_M]}$ for every $s \geqslant 1$. It is clear from the definition that the s-fold F_M-twist $M^{[F_M]^s}$ of M agrees with the F_M^s-twist $M^{[F_M^s]}$ with respect to F^s on A and F_M^s on M.

Lemma 2.5. *For an A-module M, the Frobenius twist $M^{[1]}$ and the F_M-twist $M^{[F_M]}$ are isomorphic.*

Proof. Since $M = M^{[F_M]}$ as k-spaces, we have that $\varphi_M = \tau_M \circ F_M^{-1} : M^{[F_M]} \to M^{[1]}$ is a k-linear isomorphism. It is straightforward to check that φ_M is an A-module homomorphism. □

We also need a notation for twisting A-module homomorphisms relative to given Frobenius maps. If F_M and F_N are Frobenius maps on A-modules M and N, respectively, then they induce a Frobenius map (see [4, 2.4])

(2.5.1)
$$F_{(M,N)} : \mathrm{Hom}_k(M, N) \to \mathrm{Hom}_k(M^{[F_M]}, N^{[F_N]});\ \ f \mapsto f^{[F]} := F_N \circ f \circ F_M^{-1}.$$

Moreover, if $f : M \to N$ is an A-module homomorphism, then so is $f^{[F]}$. Likewise, the homomorphism $f^{[F^s]} : M^{[F^s_M]} \to N^{[F^s_N]}$ can be regarded as the s-fold twist of $f^{[F]}$. Note that $f^{[F]}$ agrees, up to the isomorphism in 2.5, with the morphism twist $f^{[1]}$ as seen by the following commutative diagram

$$\begin{array}{ccc} M^{[F_M]} & \xrightarrow{f^{[F]}} & N^{[F_N]} \\ \varphi_M \downarrow & & \downarrow \varphi_N \\ M^{[1]} & \xrightarrow{f^{[1]}} & N^{[1]} \end{array}$$

Definition 2.6. An A-module M is *Frobenius periodic* (or simply *F-periodic*) if $M \cong M^{[r]}$ for some $r \geqslant 1$. Call the minimal r with this property the *F-period* of M, denoted by $p_F(M)$. If $p_F(M) = 1$, M is said to be *Frobenius stable* (or simply *F-stable*).

By Lemma 2.5, these definitions of F-stable and F-periodic modules coincide with those defined in [4]. It is proved in [4, 4.6] that, for a finite dimensional A, every finite dimensional A-module is F-periodic. In fact, we may say a bit more.

Proposition 2.7. *Let A be a finitely generated k-algebra with a Frobenius morphism F. Then every finite dimensional A-module is F-periodic.*

Proof. Let $\{a_1, \dots, a_s\}$ be a set of generators of A. Since F is a Frobenius map on the k-space A, there is an $l \geqslant 1$ such that $F^l(a_i) = a_i$ for all i. The rest of the proof is entirely similar to the proof of [4, 4.6] with F^l in the position of F there. □

The next result generalizes part (b) of [4, 4.3] with a proof independent of the use of Lang-Steinberg's Theorem.

Proposition 2.8. *Let M be an A-module. Then M is F-periodic if and only if there exists a Frobenius map F_M on M and an integer $r \geqslant 1$ such that $M = M^{[F^r_M]}$ as A-modules.*

Proof. By (2.4.3), $M = M^{[F^r_M]}$ means that the A-module structure on M satisfies

$$F^r_M(am) = F^r_A(a)F^r_M(m) \text{ for all } a \in A, m \in M.$$

The sufficiency follows directly from Lemma 2.5. Conversely, suppose $\phi : M^{[r]} \xrightarrow{\sim} M$ is an isomorphism. Then the composition $F' := \phi \circ \tau$ is a Frobenius map on M, where $\tau = \tau^{(r)}_M : M \to M^{(r)} = M^{[r]}$. By taking a basis for the $\mathbb{F}_{q^r}$-structure $M^{F'}$ of M, we may define a Frobenius map $F_M : M \to M$ such that $F^r_M = F' = \phi \circ \tau$. Then, applying (2.4.1) yields, for all $a \in A$ and $m \in M$,

$$\begin{aligned} F^r_M(am) &= (\phi \circ \tau)(am) = \phi((am)^{(r)}) = \phi(F^r_A(a) \cdot m^{(r)}) \\ &= F^r_A(a)(\phi \circ \tau)(m) = F^r(a)F^r_M(m). \end{aligned}$$

□

The Frobenius functor given in (2.4.2) determines a new category $\mathbf{mod}^F$-A whose objects are A-modules M with $\phi_M : M^{[1]} \xrightarrow{\sim} M$ and whose morphisms are compatible with the isomorphisms ϕ_M, i.e.,

$$\mathrm{Hom}_{\mathbf{mod}^F\text{-}A}(M, N) = \{f \in \mathrm{Hom}_A(M, N) \mid \phi_N \circ f^{[1]} = f \circ \phi_M\}.$$

Taking $r = 1$ in the proof above, we see immediately the following.

Corollary 2.9. *The category $\mathbf{mod}^F$-A is the same category as defined right above* [4, 3.2] *if we take $F_M = \phi_M \circ \tau_N$ and $F_N = \phi_N \circ \tau_M$ as the Frobenius maps on M and N, respectively.*

Given an F-stable A-module M, Proposition 2.8 asserts that there is a Frobenius map F_M such that $M = M^{[F_M]}$ as A-modules. Thus we obtain an A^F-module

$$M^{F_M} = \{m \in M \mid F_M(m) = m\}.$$

We sometimes use the pair (M, F_M) to denote the F-stable module M. Note from [4, 4.1] that, up to A^F-module isomorphism, M^{F_M} is independent of the choice of F_M. Thus, for notational simplicity, we shall write M^F for M^{F_M}.

In general, if M is F-periodic with F-period r, then there is an F-stable module $\tilde{M}$ such that $M \mid \tilde{M}$ (meaning that M is isomorphic to a direct summand of $\tilde{M}$). To see this, we apply Proposition 2.8 to choose a Frobenius map F_M on M such that $M = M^{[F_M^r]}$ as A-modules. Then let

$$\tilde{M} = M \oplus M^{[F_M]} \oplus \cdots \oplus M^{[F_M^{r-1}]}$$

and define a Frobenius map $F_{\tilde{M}} : \tilde{M} \to \tilde{M}$ by

$$F_{\tilde{M}}(x_0, x_1, \ldots, x_{r-1}) = (F_M(x_{r-1}), F_M(x_0), \ldots, F_M(x_{r-2})).$$

Then it is clear that $\tilde{M} = \tilde{M}^{[F_{\tilde{M}}]}$. In particular, this gives rise to an A^F-module $\tilde{M}^F$. Moreover, by [4, Th. 5.1], $\tilde{M}^F$ is indecomposable whenever M is so, and every indecomposable A^F-module can be obtained in this way.

The following category equivalence at the module level is our starting point to establish further category equivalence at the chain complex level and triangulated category equivalence at homotopy and derived category level. We refer to [4, 3.2] for its proof.

Theorem 2.10. *The category* $\mathbf{mod}^F$-A *is equivalent to the category* $\mathbf{mod}$-A^F *of* A^F*-modules.*

3. Frobenius twists of chain complexes

From now on, we assume that A is a finite dimensional k-algebra with a Frobenius morphism F (though part of the theory continues to hold for infinite dimensional algebras). We refer to [19, 12] for the basics concerning the derived category of $\mathbf{mod}$-A.

Let $\mathscr{C}(A) := \mathscr{C}(\mathbf{mod}\text{-}A)$ denote the category of (chain) complexes of A-modules

$$\mathsf{M} = (M^i, d^i) = \cdots \longrightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots$$

where $d^2 = 0$. Applying the Frobenius functor to each M^i, we obtain a new chain complex

$$\mathsf{M}^{[1]} = \cdots \longrightarrow (M^{i-1})^{[1]} \xrightarrow{(d^{i-1})^{[1]}} (M^i)^{[1]} \xrightarrow{(d^i)^{[1]}} (M^{i+1})^{[1]} \xrightarrow{(d^{i+1})^{[1]}} \cdots .$$

This will be called the *Frobenius twist* of M. Further, each complex morphism $f = (f^i) : \mathsf{M} \to \mathsf{N}$ induces a morphism $f^{[1]} := ((f^i)^{[1]}) : \mathsf{M}^{[1]} \to \mathsf{N}^{[1]}$. Thus, the Frobenius functor on $\mathbf{mod}$-A defined in (2.4.2) induces a functor

$$(\ )^{[1]} = (\ )^{[1]}_{\mathscr{C}(A)} : \mathscr{C}(A) \to \mathscr{C}(A),$$

which we still call the *Frobenius (twist) functor* (on complexes). As in the module case, we can inductively define the s-fold Frobenius twist $\mathsf{M}^{[s]}$ of M for all $s \in \mathbb{Z}$. Note that $\mathsf{M}^{[0]} = \mathsf{M}$ and $\mathsf{M}^{[-1]}$ is defined to be the complex N such that $\mathsf{M} = \mathsf{N}^{[1]}$.

The *shift functor* $\mathcal{T}$ on $\mathscr{C}(A)$ is defined by $(\mathcal{T}\mathsf{M})^i = M^{i+1}$, $d^i_{\mathcal{T}\mathsf{M}} = -d^{i+1}_{\mathsf{M}}$ and $\mathcal{T}(f)^i = f^{i+1}$ if f is a morphism in $\mathscr{C}(A)$. The following Lemma is obvious.

Lemma 3.1. *The shift functor* $\mathcal{T}$ *commutes with the Frobenius functor, i.e.,* $(\mathcal{T}\mathsf{M})^{[1]} = \mathcal{T}(\mathsf{M}^{[1]})$ *and* $\mathcal{T}(f)^{[1]} = \mathcal{T}(f^{[1]})$ *for each object* M *and morphism* f *in* $\mathscr{C}(A)$.

Let $\mathscr{C}^b(A)$ (resp. $\mathscr{C}^+(A)$, $\mathscr{C}^-(A)$) be the full subcategory of $\mathscr{C}(A)$ consisting of bounded complexes (resp. complexes bounded below, complexes bounded above). It is obvious to see that the Frobenius functor on $\mathscr{C}(A)$ restricts to functors on $\mathscr{C}^b(A)$, $\mathscr{C}^+(A)$, and $\mathscr{C}^-(A)$.

We may construct $\mathsf{M}^{[1]}$ via Frobenius maps on M^i. Let $\mathsf{M} = (M^i, d^i)$ be a complex in $\mathscr{C}(A)$ and let $\mathcal{F} := \{F_i : M^i \to M^i \mid i \in \mathbb{Z}\}$ be a family of Frobenius maps. We shall call $\mathcal{F}$ a *Frobenius map* on M. For each $i \in \mathbb{Z}$, let $(M^i)^{[F_i]}$ denote the F_i-twist of M^i. Then each $d^i : M^i \to M^{i+1}$ gives an A-module homomorphism (see (2.5.1) for the notation)

$$d^{i[F]} = F_{i+1} \circ d^i_M \circ F_i^{-1} : (M^i)^{[F_i]} \to (M^{i+1})^{[F_{i+1}]}.$$

Thus we obtain a complex $((M^i)^{[F_i]}, d^{i[F]})$, which is called the $\mathcal{F}$-*twist* of M and is denoted by $\mathsf{M}^{[\mathcal{F}]}$. Inductively, we can define for each $s \in \mathbb{Z}$ the s-fold twist $\mathsf{M}^{[\mathcal{F}]^s}$ of M which coincides with the $\mathcal{F}^s$-twist $\mathsf{M}^{[\mathcal{F}^s]}$ of M. Here $\mathcal{F}^s = \{F_i^s : M^i \to M^i \mid i \in \mathbb{Z}\}$.

Lemma 3.2. *Let M be a complex and $\mathcal{F} = \{F_i : M^i \to M^i \mid i \in \mathbb{Z}\}$ be a Frobenius map on M. Then the complexes $\mathsf{M}^{[\mathcal{F}]}$ and $\mathsf{M}^{[1]}$ are isomorphic.*

Proof. For each $i \in \mathbb{Z}$, the k-linear map $f^i = \tau_{M^i} \circ F_i^{-1} : (M^i)^{[F_i]} \to (M^i)^{[1]}$ is an A-module isomorphism (see Lemma 2.5). It is easy to see that $f := (f^i)$ is an isomorphism from $\mathsf{M}^{[\mathcal{F}]}$ to $\mathsf{M}^{[1]}$. □

The lemma implies that, up to isomorphism, $\mathsf{M}^{[\mathcal{F}]}$ is independent of the choice of the Frobenius map $\mathcal{F}$.

If $\mathsf{M} = (M^i, d^i_{\mathsf{M}})$ and $\mathsf{N} = (N^i, d^i_{\mathsf{N}})$ are objects in $\mathscr{C}(A)$ together with families of Frobenius maps $\mathcal{F}_1 = \{F_{1,i} : M^i \to M^i \mid i \in \mathbb{Z}\}$ and $\mathcal{F}_2 = \{F_{2,i} : N^i \to N^i \mid i \in \mathbb{Z}\}$, respectively. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ induce a q-twist map (not necessarily a Frobenius map)

(3.2.1)

$$F_{(\mathsf{M},\mathsf{N})} : \mathrm{Hom}_{\mathscr{C}(A)}(\mathsf{M},\mathsf{N}) \longrightarrow \mathrm{Hom}_{\mathscr{C}(A)}(\mathsf{M}^{[\mathcal{F}_1]}, \mathsf{N}^{[\mathcal{F}_2]});\ f = (f^i) \longmapsto f^{[\mathcal{F}]} := (f^{i[F]}),$$

where $f^{i[F]} = F_{2,i} \circ f^i \circ F_{1,i}^{-1}$ for all i are defined in (2.5.1). In general, for each $s \in \mathbb{Z}$, a morphism $f = (f^i) : \mathsf{M} \to \mathsf{N}$ gives rise to a morphism $f^{[\mathcal{F}^s]} : \mathsf{M}^{[\mathcal{F}_1^s]} \to \mathsf{N}^{[\mathcal{F}_2^s]}$.

Replacing modules by complexes in Definition 2.6, we may introduce F-periodic complexes and F-stable complexes.

Definition 3.3. A complex M in $\mathscr{C}(A)$ is *F-periodic* if $\mathsf{M} \cong \mathsf{M}^{[r]}$ for some integer $r \geqslant 1$. Such a minimum r, denoted by $p_{\mathscr{C}}(\mathsf{M})$, is called the *$F$-period* of M. If $p_{\mathscr{C}}(\mathsf{M}) = 1$, M is said to be F-stable.

Examples 3.4. (a) There is a full embedding of **mod**-A into $\mathscr{C}(A)$ which sends each A-module M to the stalk complex $M^\bullet = (M^i, d^i)$ with $M^0 = M$, $M^i = 0$ for all $i \neq 0$ concentrated in the 0-position. If M is an F-stable A-module, then the corresponding stalk complex is clearly F-stable.

(b) Let $\mathsf{X} = (X^i, d^i)$ be a complex in $\mathscr{C}(A^F) = \mathscr{C}(\mathbf{mod}\text{-}A^F)$. It is easy to see that the complex

$$\mathsf{X}_k = \mathsf{X} \otimes k := \cdots \longrightarrow X^{i-1} \otimes k \xrightarrow{d^{i-1}\otimes 1} X^i \otimes k \xrightarrow{d^i \otimes 1} X^{i+1} \otimes k \xrightarrow{d^{i+1}\otimes 1} \cdots$$

is an F-stable complex in $\mathscr{C}(A)$.

Lemma 3.5. *Each bounded complex M in $\mathscr{C}^b(A)$ is F-periodic.*

Proof. Let $\mathsf{M} = (M^i, d^i)$ be a bounded complex. Since by Corollary 2.7, each M^i is an F-periodic A-module, there exists a Frobenius map $F_i : M^i \to M^i$ such that $(M^i)^{[F_i^{r_i}]} = M^i$ for some $r_i \geqslant 1$. Further, for each $i \in \mathbb{Z}$, the induced map

$$F_{(M^i,M^{i+1})} : \mathrm{Hom}_A(M^i, M^{i+1}) \longrightarrow \mathrm{Hom}_A(M^i, M^{i+1}), f \longmapsto f^{[F]} = F_{i+1} \circ f \circ F_i^{-1}$$

is a Frobenius map. Thus, there is an integer $s_i \geqslant 1$ such that $F^{s_i}_{M^i,M^{i+1}}(d^i) = d^i$, that is, $d^i = F^{s_i}_{i+1} \circ d^i \circ F_i^{-s_i} = (d_i)^{[F^{s_i}]}$. Let r be the least common multiple of r_i and s_j with $M^i \neq 0$ and $d^j \neq 0$. Then it is clear that $\mathsf{M}^{[\mathcal{F}^r]} = \mathsf{M}$, where $\mathcal{F} = \{F_i \mid i \in \mathbb{Z}\}$. Hence M is F-periodic. □

The following is a complex version of 2.8.

Lemma 3.6. *Let $\mathsf{M} = (M^i, d^i)$ be a complex in $\mathscr{C}(A)$. Then M is F-periodic if and only if there exists a Frobenius map $\mathcal{F} = \{F_i : M^i \to M^i \mid i \in \mathbb{Z}\}$ on M and an integer $r \geqslant 1$ such that $\mathsf{M}^{[\mathcal{F}^r]} = \mathsf{M}$ as complexes of A-modules.*

Proof. The sufficiency follows directly from Lemma 3.2. We now prove the necessity. Let $f = (f^i) : \mathsf{M}^{[r]} \to \mathsf{M}$ be an isomorphism of complexes. For each i, let $\tau_i = \tau^{(r)}_{M^i} : M^i \to (M^i)^{[r]}$. Then the composition $f^i \circ \tau_i$ is a Frobenius map on M^i which defines an $\mathbb{F}_{q^r}$-structure on M^i. Since M^i is finite dimensional, there is a Frobenius map $F_i : M^i \to M^i$ which defines an $\mathbb{F}_q$-structure on M^i and satisfies $F_i^r = f^i \circ \tau_i$. Then, for each i, $(M^i)^{[F_i^r]} = M^i$ as A-modules (see the proof of Proposition 2.8). Moreover, for each i, we have

$$\begin{aligned}(d^i)^{[F^r]} &= F^r_{i+1} \circ d^i \circ (F_i)^{-r} = f^{i+1} \circ \tau_{i+1} \circ d^i \circ \tau_i^{-1} \circ (f^i)^{-1} \\ &= f^{i+1} \circ (d^i)^{[r]} \circ (f^i)^{-1} = d^i,\end{aligned}$$

that is, $\mathsf{M}^{[\mathcal{F}^r]} = \mathsf{M}$ as complexes, where $\mathcal{F} = \{F_i \mid i \in \mathbb{Z}\}$. □

In particular, if $\mathsf{M} = (M^i, d^i)$ is an F-stable complex in $\mathscr{C}(A)$, then there is a Frobenius map $\mathcal{F} = \{F_i : M^i \to M^i \mid i \in \mathbb{Z}\}$ on M such that $\mathsf{M}^{[\mathcal{F}]} = \mathsf{M}$ as complexes. Thus we obtain A^F-modules $(M^i)^{F_i}$ with $M^i = (M^i)^{F_i} \otimes k$ for all $i \in \mathbb{Z}$. The restriction of each d^i gives an A^F-module homomorphism $d^i : (M^i)^{F_i} \to (M^{i+1})^{F_{i+1}}$. We finally get a complex $\mathsf{X} = ((M^i)^{F_i}, d^i)$ in $\mathscr{C}(A^F)$ such that $\mathsf{M} = \mathsf{X} \otimes k$. We will denote X by $\mathsf{M}^{\mathcal{F}}$.

Let $\mathscr{C}^b(A)^F$ denote the category whose objects are F-stable complexes M in $\mathscr{C}^b(A)$ together with a Frobenius map $\mathcal{F} = \mathcal{F}_{\mathsf{M}}$ such that $\mathsf{M}^{[\mathcal{F}]} = \mathsf{M}$ and whose morphisms are morphisms of complexes compatible with the Frobenius maps. Moreover, if $(\mathsf{M}, \mathcal{F}_1)$ and $(\mathsf{N}, \mathcal{F}_2)$ are bounded F-stable complexes, then, by Lemma 2.2, the q-twist map $F = F_{(\mathsf{M},\mathsf{N})}$ is a Frobenius map and induces an isomorphism

$$\mathrm{Hom}_{\mathscr{C}(A)^F}(\mathsf{M}, \mathsf{N}) = \mathrm{Hom}_{\mathscr{C}(A)}(\mathsf{M}, \mathsf{N})^F \cong \mathrm{Hom}_{\mathscr{C}(A^F)}(\mathsf{M}^{\mathcal{F}_1}, \mathsf{N}^{\mathcal{F}_2}).$$

Combining all observations above, we have

Proposition 3.7. *(1) The embedding $\mathscr{C}(A^F) \to \mathscr{C}(A)$ sending X to $\mathsf{X}_k = \mathsf{X} \otimes k$ induces a category equivalence*

$$\mathscr{C}^b(A^F) \cong \mathscr{C}^b(A)^F.$$

(2) Let $(\mathsf{M}, \mathcal{F})$ be an F-stable complex in $\mathscr{C}^b(A)$. Then we have $\mathbb{F}_q$-algebra isomorphisms

$$\big(\mathrm{End}_{\mathscr{C}^b(A)}(\mathsf{M})\big)^F \cong \mathrm{End}_{\mathscr{C}^b(A^F)}(\mathsf{M}^{\mathcal{F}})$$

and

$$\big(\mathrm{End}_{\mathscr{C}^b(A)}(\mathsf{M})/\mathrm{Rad}\,\mathrm{End}_{\mathscr{C}^b(A)}(\mathsf{M})\big)^F \cong \mathrm{End}_{\mathscr{C}^b(A^F)}(\mathsf{M}^{\mathcal{F}})/\mathrm{Rad}\,\mathrm{End}_{\mathscr{C}^b(A^F)}(\mathsf{M}^{\mathcal{F}}).$$

Remark 3.8. Alternatively, we may define $\mathscr{C}^b(A)^F$ without using the Frobenius morphisms $\mathcal{F}_{\mathsf{M}}$ on M: the objects consist of complexes M satisfying $\mathsf{M}^{[1]} \overset{\phi_{\mathsf{M}}}{\cong} \mathsf{M}$ and the morphisms are compatible with these isomorphisms ϕ_{M} (cf. Corollary 2.9). In particular, we have

$$\mathrm{End}_{\mathscr{C}^b(A)^F}(\mathsf{M}) \cong \big(\mathrm{End}_{\mathscr{C}^b(A)}(\mathsf{M})\big)^F.$$

The method of constructing F-stable modules from F-periodic modules given at the end of §2 can be generalized to complexes. Let M be F-periodic complex in $\mathscr{C}(A)$ with F-period r. By Lemma 3.6, there is a Frobenius map $\mathfrak{F} = \{F_i : M^i \to M^i \mid i \in \mathbb{Z}\}$ such that $\mathsf{M} = \mathsf{M}^{[\mathfrak{F}^r]}$ as complexes. For each i, let

$$\tilde{M}^i = M^i \oplus (M^i)^{[F_i]} \oplus \cdots \oplus (M^i)^{[F_i^{r-1}]}$$

and define a Frobenius map $\tilde{F}_i : \tilde{M}^i \to \tilde{M}^i$ by

$$\tilde{F}_i(x_0, x_1, \dots, x_{r-1}) = (F_i(x_{r-1}), F_i(x_0), \dots, F_i(x_{r-2})). \tag{3.8.1}$$

Further, let $\tilde{d^i} = \mathrm{diag}(d^i, (d^i)^{[F]}, \dots, (d^i)^{[F^{r-1}]}) : \tilde{M}^i \to \tilde{M}^{i+1}$. Then we obtain an F-stable complex $\tilde{\mathsf{M}} = (\tilde{M}^i, \tilde{d^i})$ satisfying $\tilde{\mathsf{M}}^{[\tilde{\mathfrak{F}}]} = \tilde{\mathsf{M}}$, where $\tilde{\mathfrak{F}} = \{\tilde{F}_i \mid i \in \mathbb{Z}\}$. This gives a complex $\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}}$ in $\mathscr{C}(A^F)$. By Lemma 3.5, this construction applies to every object in $\mathscr{C}^b(A)$.

Since $\mathscr{C}^b(A)$ is a Krull-Schmidt category (a category in which the theorem of Krull-Schmidt holds; see [18, p.52]), by using an analogous argument as in the proof of Theorem 2.10(2), we obtain the following result.

Theorem 3.9. *Maintain the notation above. Let M be an F-periodic indecomposable complex in $\mathscr{C}^b(A)$ with F-period r. Then $\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}}$ is indecomposable in $\mathscr{C}^b(A^F)$ and*

$$\mathrm{End}_{\mathscr{C}^b(A^F)}(\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{C}^b(A^F)}(\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}})) \cong \mathrm{End}_{\mathscr{C}^b(A)^F}(\tilde{\mathsf{M}})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{C}^b(A)^F}(\tilde{\mathsf{M}})) \cong \mathbb{F}_{q^r}.$$

Moreover, every indecomposable complex in $\mathscr{C}^b(A^F)$ is isomorphic to a complex of the form $\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}}$ for some F-periodic indecomposable complex M in $\mathscr{C}^b(A)$.

Remark 3.10. For each $n \geqslant 1$, let $T_n(A)$ denote the lower triangular matrix algebra

$$T_n(A) = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ A & A & 0 & \cdots & 0 \\ A & A & A & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A & A & A & \cdots & A \end{pmatrix}$$

and let I be the ideal of $T_n(A)$ consisting of all matrices whose entries on the diagonal are zero. By A_n we denote the quotient algebra of $T_n(A)$ by the ideal I^2. Then the Frobenius morphism F on A induces naturally a Frobenius morphism F_{A_n} on A_n. Then **mod**-A_n can be identified with the full subcategory $\mathscr{C}_n(A)$ of $\mathscr{C}(A)$ of complexes $\mathsf{M} = (M^i, d^i)$ satisfying $M^i = 0$ for all $i < 0$ or $i \geqslant n$. Under this identification, the Frobenius twist of a complex is the same as the Frobenius twist of the corresponding A_n-module. Thus, Proposition 3.7 , Lemma 3.5, and Theorem 3.9 follow directly from the relevant results for A_n-modules.

4. Homotopy theory with a Frobenius functor

A morphism $f : \mathsf{M} \to \mathsf{N}$ in $\mathscr{C}(A)$ is said to be *homotopic to zero* if there exist morphisms $s^i : M^i \to N^{i-1}$ in **mod**-A such that

$$f^i = s^{i+1} \circ d^i_{\mathsf{M}} + d^{i-1}_{\mathsf{N}} \circ s^i \text{ for all } i \in \mathbb{Z}.$$

Two morphisms $f, g : \mathsf{M} \to \mathsf{N}$ are said to be *homotopic* if $f - g$ is homotopic to zero. By $\mathrm{Ht}(\mathsf{M}, \mathsf{N})$ we denote the subspace of $\mathrm{Hom}_{\mathscr{C}(A)}(\mathsf{M}, \mathsf{N})$ consisting of morphisms homotopic to zero. Then the homotopy category $\mathscr{K}(A) := \mathscr{K}(\mathbf{mod}\text{-}A)$ is defined by

$$\begin{cases} \mathrm{Ob}(\mathscr{K}(A)) = \mathrm{Ob}(\mathscr{C}(A)), \text{ and for } \mathsf{M}, \mathsf{N} \in \mathrm{Ob}(\mathscr{K}(A)), \\ \mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M}, \mathsf{N}) = \mathrm{Hom}_{\mathscr{C}(A)}(\mathsf{M}, \mathsf{N})/\mathrm{Ht}(\mathsf{M}, \mathsf{N}). \end{cases}$$

We shall write $\bar{f}$ for the image of f in $\mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M},\mathsf{N})$. Similarly, we can define the full subcategories $\mathscr{K}^b(A)$, $\mathscr{K}^+(A)$, and $\mathscr{K}^-(A)$ of $\mathscr{K}(A)$. Note that $\mathscr{K}(A)$, and $\mathscr{K}^b(A)$, $\mathscr{K}^+(A)$, $\mathscr{K}^-(A)$ as well, are triangulated categories whose distinguished triangles are induced from mapping cones (see, e.g., [16, 1.4]).

Clearly, a morphism $f:\mathsf{M}\to\mathsf{N}$ is homotopic to zero if and only if so is $f^{[1]}$. Thus the Frobenius functor $(\ )^{[1]}$ on $\mathscr{C}(A)$ induces a functor

$$(\ )^{[1]}=(\ )^{[1]}_{\mathscr{K}(A)}:\mathscr{K}(A)\to\mathscr{K}(A)$$

which clearly preserves distinguished triangles. Therefore, the Frobenius functor on $\mathscr{K}(A)$ is an equivalence of triangulated categories. Moreover, the restrictions of this functor to the subcategories $\mathscr{K}^b(A)$, $\mathscr{K}^+(A)$, and $\mathscr{K}^-(A)$ are also triangulated category equivalences.

Similarly, since the shift functor $\mathcal{T}$ stabilizes every $\mathrm{Ht}(\mathsf{M},\mathsf{N})$, it induces a shift functor $\mathcal{T}$ on $\mathscr{K}(A)$ which clearly commutes with the Frobenius functor.

Let $\mathsf{M}=(M^i,d^i_{\mathsf{M}})$ be an object in $\mathscr{K}(A)$ and let $\mathcal{F}=\{F_i:M^i\to M^i\mid i\in\mathbb{Z}\}$ be a Frobenius map on M. From Lemma 3.2, we have that $\mathsf{M}^{[1]}$ and $\mathsf{M}^{[\mathcal{F}]}$ are still isomorphic as objects in $\mathscr{K}(A)$. If $\mathsf{M}=(M^i,d^i_{\mathsf{M}})$ and $\mathsf{N}=(N^i,d^i_{\mathsf{N}})$ are objects in $\mathscr{K}(A)$ with Frobenius maps $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Then the restriction of the q-twist map $F_{(\mathsf{M},\mathsf{N})}$ defined in (3.2.1) is a q-twist map $\mathrm{Ht}(\mathsf{M},\mathsf{N})\to\mathrm{Ht}(\mathsf{M}^{[1]},\mathsf{N}^{[1]})$, and thus, induces a q-twist map

$$F^{\mathscr{K}}_{(\mathsf{M},\mathsf{N})}:\mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M},\mathsf{N})\to\mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M}^{[\mathcal{F}_1]},\mathsf{N}^{[\mathcal{F}_2]}).\tag{4.0.1}$$

A complex M in $\mathscr{C}(A)$ is called *contractible* if the identity map 1_{M} of M is homotopic to zero. It is equivalent to saying that M is a zero object in $\mathscr{K}(A)$. Obviously, a complex M is contractible if and only if so is its Frobenius twist $\mathsf{M}^{[1]}$. Further, each bounded complex M is decomposed into a direct sum $\mathsf{M}_0\oplus\mathsf{M}_{\bar{0}}$ such that M_0 is contractible and $\mathsf{M}_{\bar{0}}$ contains no non-zero contractible summands, and moreover, such a decomposition is unique up to isomorphism. Thus, if M and N are complexes in $\mathscr{C}^b(A)$ containing no non-zero contractible summands (i.e., $\mathsf{M}=\mathsf{M}_{\bar{0}}$, $\mathsf{N}=\mathsf{N}_{\bar{0}}$), then $\mathsf{M}\cong\mathsf{N}$ as complexes in $\mathscr{C}^b(A)$ if and only if $\mathsf{M}\cong\mathsf{N}$ as objects in $\mathscr{K}^b(A)$. In other words, we have isomorphism

$$\mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M},\mathsf{N})\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M}_{\bar{0}},\mathsf{N}_{\bar{0}});\ \bar{f}\mapsto\bar{f}_{\bar{0}},\tag{4.0.2}$$

where $f_{\bar{0}}=\pi_{\mathsf{N}_{\bar{0}}}\circ f\circ\iota_{\mathsf{M}_{\bar{0}}}$ for canonical injection $\iota_{\mathsf{M}_{\bar{0}}}=\binom{1_{\mathsf{M}_{\bar{0}}}}{0}:\mathsf{M}_{\bar{0}}\to\mathsf{M}$ and surjection $\pi_{\mathsf{N}_{\bar{0}}}=(1_{\mathsf{N}_{\bar{0}}},0):\mathsf{N}\to\mathsf{N}_{\bar{0}}$. Moreover, if we assume that the restriction of $\mathcal{F}_1$ to $\mathsf{M}_{\bar{0}}$ and $\mathcal{F}_2$ to $\mathsf{N}_{\bar{0}}$ are Frobenius morphisms on $\mathsf{M}_{\bar{0}}$ and $\mathsf{N}_{\bar{0}}$, respectively, then the following diagram is commutative,

$$\begin{array}{ccc}\mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M},\mathsf{N}) & \xrightarrow{F^{\mathscr{K}}} & \mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M}^{[\mathcal{F}_1]},\mathsf{N}^{[\mathcal{F}_2]})\\ \sim\downarrow & & \downarrow\sim\\ \mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M}_{\bar{0}},\mathsf{N}_{\bar{0}}) & \xrightarrow{F^{\mathscr{K}}_{\bar{0}}} & \mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M}_{\bar{0}}^{[\mathcal{F}_1]},\mathsf{N}_{\bar{0}}^{[\mathcal{F}_2]})\end{array}\tag{4.0.3}$$

where $F^{\mathscr{K}}=F^{\mathscr{K}}_{(\mathsf{M},\mathsf{N})}$ and $F^{\mathscr{K}}_{\bar{0}}=F^{\mathscr{K}}_{(\mathsf{M}_{\bar{0}},\mathsf{N}_{\bar{0}})}$.

We similarly define F-periodic and F-stable objects in $\mathscr{K}(A)$ by simply replacing "isomorphisms in $\mathscr{C}(A)$" by "isomorphisms in $\mathscr{K}(A)$" in Definition 3.3. Thus, we may introduce a function

$$p_{\mathscr{K}}:\mathrm{Ob}(\mathscr{K}(A))\to\mathbb{N}\cup\infty$$

with the property: if $p_{\mathscr{K}}(\mathsf{M})=r\in\mathbb{N}$, then $\mathsf{M}\cong\mathsf{M}^{[r]}$ in $\mathscr{K}(A)$ and r is minimal with this property. In this case, M is *F-periodic* in $\mathscr{K}(A)$. In particular, M is said to be *F-stable* in $\mathscr{K}(A)$ if $p_{\mathscr{K}}(\mathsf{M})=1$. Clearly, F-periodic (resp. F-stable) objects in $\mathscr{C}(A)$ are F-periodic (resp. F-stable) in $\mathscr{K}(A)$. Thus, each object X in $\mathscr{K}(A^F)$ gives an F-stable object $\mathsf{X}\otimes k$. By Lemma 3.5, all objects in $\mathscr{K}^b(A)$ are F-periodic.

For any two objects $\mathsf{X} = (X^i, d^i_{\mathsf{X}})$ and $\mathsf{Y} = (Y^i, d^i_{\mathsf{Y}})$ in $\mathscr{C}(A^F)$, we have an injection

$$\Phi(\mathsf{X},\mathsf{Y}) : \mathrm{Ht}(\mathsf{X},\mathsf{Y}) \longrightarrow \mathrm{Ht}(\mathsf{X}_k,\mathsf{Y}_k), f \longmapsto f\otimes 1.$$

Thus, the category embedding $\mathscr{C}(A^F) \to \mathscr{C}(A)$ induces a functor $\Phi : \mathscr{K}(A^F) \to \mathscr{K}(A)$.

Proposition 4.1. *The restriction of* $\Phi : \mathscr{K}^b(A^F) \to \mathscr{K}^b(A)$ *is a faithful functor.*

Proof. Let X and Y be bounded and let $\mathsf{X}_k = (M^i, d^i_{\mathsf{M}}) = \mathsf{M}$ and $\mathsf{Y}_k = (N^i, d^i_{\mathsf{N}}) = \mathsf{N}$. Then, for each $i\in\mathbb{Z}$, we have a *natural* Frobenius map $F_{1,i} = 1\otimes\mathfrak{f} : M^i = X^i\otimes k \to X^i\otimes k = M^i; x\otimes\lambda\mapsto x\otimes\lambda^q$. Similarly, we have a natural Frobenius map $F_{2,i} : N^i \to N^i$. It is clear that $\mathsf{M}^{[\mathcal{F}_1]} = \mathsf{M}$ and $\mathsf{N}^{[\mathcal{F}_2]} = \mathsf{N}$, where $\mathcal{F}_j = \{F_{j,i} \mid i\in\mathbb{Z}\}$ for $j = 1, 2$. Then the induced Frobenius map $F_{(\mathsf{M},\mathsf{N})}$ on $\mathrm{Hom}_{\mathscr{C}^b(A)}(\mathsf{M},\mathsf{N})$ restricts to a Frobenius map F on $\mathrm{Ht}(\mathsf{M},\mathsf{N})$ which fixes all $f\otimes 1$ for all $f\in\mathrm{Ht}(\mathsf{X},\mathsf{Y})$. Thus, $\Phi(\mathsf{X},\mathsf{Y})$ induces an injection $\phi : \mathrm{Ht}(\mathsf{X},\mathsf{Y}) \longrightarrow \mathrm{Ht}(\mathsf{M},\mathsf{N})^F$. We claim that ϕ is also surjective. Indeed, let $V = \oplus_{i\in\mathbb{Z}}\mathrm{Hom}_A(M^i, N^{i-1})$ (resp. $W = \oplus_{i\in\mathbb{Z}}\mathrm{Hom}_{A^F}(X^i, Y^{i-1})$). Then V (resp. W) is finite dimensional, and the Frobenius maps $F_{1,i}, F_{2,i}$, $i\in\mathbb{Z}$, induce a Frobenius map $F_V : V \to V, s = (s^i) \mapsto (F_{2,i-1}\circ s^i\circ F_{1,i}^{-1})_i$. Consider the surjective map

$$\varphi_V : V = \oplus_{i\in\mathbb{Z}}\mathrm{Hom}_A(M^i, N^{i-1}) \longrightarrow \mathrm{Ht}(\mathsf{M},\mathsf{N});\ s = (s^i) \longmapsto (s^{i+1}\circ d^i_{\mathsf{M}} + d^{i-1}_{\mathsf{N}}\circ s^i)_i.$$

It is clear that φ_V is compatible with Frobenius maps F_V and F, i.e., $F\circ\varphi_V = \varphi_V\circ F_V$. Hence φ_V induces a surjective map $\varphi_V^F : V^F = \oplus_{i\in\mathbb{Z}}\mathrm{Hom}_A(M^i, N^{i-1})^F \to \mathrm{Ht}(\mathsf{M},\mathsf{N})^F$. Now, consider the diagram

$$\begin{array}{ccc} V^F & \xrightarrow{\varphi_V^F} & \mathrm{Ht}(\mathsf{M},\mathsf{N})^F \\ \uparrow{\scriptstyle\psi} & & {\scriptstyle\phi}\uparrow \\ W & \xrightarrow{\varphi_W} & \mathrm{Ht}(\mathsf{X},\mathsf{Y}) \end{array}$$

where φ_W is defined similarly to φ_V with d_{M} and d_{N} replaced by d_{X} and d_{Y}, respectively, and $\psi = (\psi_i)$ with $\psi_i : \mathrm{Hom}_{A^F}(X^i, Y^{i-1}) \to \mathrm{Hom}_A(M^i, N^{i-1})^F, t^i \mapsto t^i\otimes 1$. Since, like φ_V, φ_W is surjective and ψ is an isomorphism, the commutativity of the diagram forces that ϕ is surjective, and so we obtain an isomorphism $\mathrm{Ht}(\mathsf{X},\mathsf{Y}) \cong \mathrm{Ht}(\mathsf{M},\mathsf{N})^F$. Finally, taking $F_{(\mathsf{M},\mathsf{N})}$-fixed points with the exact sequence

$$0 \longrightarrow \mathrm{Ht}(\mathsf{M},\mathsf{N}) \longrightarrow \mathrm{Hom}_{\mathscr{C}^b(A)}(\mathsf{M},\mathsf{N}) \longrightarrow \mathrm{Hom}_{\mathscr{K}^b(A)}(\mathsf{M},\mathsf{N}) \longrightarrow 0$$

yields the desired isomorphism

$$\mathrm{Hom}_{\mathscr{K}^b(A^F)}(\mathsf{X},\mathsf{Y}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{K}^b(A)}(\mathsf{M},\mathsf{N})^F \subseteq \mathrm{Hom}_{\mathscr{K}^b(A)}(\mathsf{M},\mathsf{N}). \tag{4.1.1}$$

□

Let $\mathsf{M} = (M^i, d^i)$ be an object in $\mathscr{K}^b(A)$ with $p_{\mathscr{K}}(\mathsf{M}) = r$. If M considered as a complex in $\mathscr{C}^b(A)$ contains no non-zero contractible summands, then M is an F-periodic complex in $\mathscr{C}^b(A)$ with $p_{\mathscr{C}}(\mathsf{M}) = r$. By Lemma 3.6, there is a Frobenius map $\mathcal{F}$ on M such that $\mathsf{M} = \mathsf{M}^{[\mathcal{F}^r]}$ as complexes. In general, $\mathsf{M} = \mathsf{M}_{\bar{0}}\oplus\mathsf{M}_0$, where M_0 is contractible and $\mathsf{M}_{\bar{0}}$ contains no contractible summands. We choose a Frobenius map $\mathcal{F}_{\bar{0}}$ on $\mathsf{M}_{\bar{0}}$ satisfying $\mathsf{M}_{\bar{0}}^{[\mathcal{F}_{\bar{0}}]} = \mathsf{M}_{\bar{0}}$ and *any* Frobenius map $\mathcal{F}_0$ on M_0. Then $\mathcal{F} = \mathcal{F}_{\bar{0}}\oplus\mathcal{F}_0$ is a Frobenius map on M. Clearly, $\mathsf{M}^{[\mathcal{F}]} = \mathsf{M}$ is not necessarily true. We shall write $\mathsf{M}^{[\mathcal{F}]} =_{\bar{0}} \mathsf{M}$ for an $\mathcal{F}$ chosen in this way. Thus, if $\mathsf{M}^{[\mathcal{F}_1]} =_{\bar{0}} \mathsf{M}$ and $\mathsf{N}^{[\mathcal{F}_2]} =_{\bar{0}} \mathsf{N}$, then, by (4.0.3), we may assume that $\mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M},\mathsf{N}) = \mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M}^{[\mathcal{F}_1]},\mathsf{N}^{[\mathcal{F}_2]})$ and that $F^{\mathscr{K}}_{(\mathsf{M},\mathsf{N})}$ is a Frobenius map on $\mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M},\mathsf{N})$.

We now define $\mathscr{K}^b(A)^F$ to be the category whose objects are F-stable complexes M in $\mathscr{K}^b(A)$ together with a Frobenius map $\mathfrak{F} = \mathfrak{F}_{\mathsf{M}}$ satisfying $\mathsf{M}^{[\mathfrak{F}]} =_{\bar{0}} \mathsf{M}$ and whose morphisms are defined as

$$\mathrm{Hom}_{\mathscr{K}(A)^F}(\mathsf{M},\mathsf{N}) := \mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M},\mathsf{N})^F \ \ (F = F^{\mathscr{K}})$$

consisting of morphisms in $\mathrm{Hom}_{\mathscr{K}^b(A)}(\mathsf{M},\mathsf{N})$ compatible with $\mathfrak{F}_{\mathsf{M}}$ and $\mathfrak{F}_{\mathsf{N}}$. Then $\mathscr{K}^b(A)^F$ becomes a triangulated category whose (distinguished) triangles are triangles in $\mathscr{K}^b(A)$ defined in $\mathscr{K}^b(A)^F$ (that is, are *F-stable triangles*).

Corollary 4.2. *The faithful functor $\Phi : \mathscr{K}^b(A^F) \to \mathscr{K}^b(A)$ induces a triangulated category equivalence*

$$\mathscr{K}^b(A^F) \cong \mathscr{K}^b(A)^F.$$

Proof. If M is F-stable, then there is a Frobenius map $\mathfrak{F}$ on M such that $\mathsf{M} =_{\bar{0}} \mathsf{M}^{[\mathfrak{F}]}$. Thus, $\mathsf{M} \cong \mathsf{X}_k = \mathsf{X} \otimes k$ in $\mathscr{K}^b(A)$ where $\mathsf{X} = \mathsf{M}_{\bar{0}}^{\mathfrak{F}}$ is in $\mathscr{K}^b(A^F)$. This together with the faithfulness and fullness (by the isomorphism (4.1.1)) gives a category equivalence. Finally, the commutativity between the shift and Frobenius functors implies that Φ takes triangles to triangles. $\square$

Remarks 4.3. (a) There is also an intrinsic definition for the category $\mathscr{K}^b(A)^F$ (cf. Corollary 2.9 and Remark 3.8):

$$\begin{cases} \text{Objects:} \quad \mathsf{M} \text{ such that } \mathsf{M}^{[1]} \overset{\bar{\phi}_{\mathsf{M}}}{\cong} \mathsf{M} \text{ in } \mathscr{K}^b(A), \\ \text{Morphisms: } \mathrm{Hom}_{\mathscr{K}^b(A)^F}(\mathsf{M},\mathsf{N}) = \{\bar{f} \in \mathrm{Hom}_{\mathscr{K}^b(A)}(\mathsf{M},\mathsf{N}) \mid \bar{\phi}_{\mathsf{N}} \circ \bar{f}^{[1]} = \bar{f} \circ \bar{\phi}_{\mathsf{M}}\}. \end{cases}$$

To see the two definitions coincide, we follow the construction given in Corollary 2.9. The isomorphism $\bar{\phi}_{\mathsf{M}} : \mathsf{M}^{[1]} \xrightarrow{\sim} \mathsf{M}$ restricts to an isomorphism $\bar{\phi}_{\mathsf{M}} : \mathsf{M}_{\bar{0}}^{[1]} \xrightarrow{\sim} \mathsf{M}_{\bar{0}}$ which results in an isomorphism $\phi_{\mathsf{M}_{\bar{0}}} : \mathsf{M}_{\bar{0}}^{[1]} \xrightarrow{\sim} \mathsf{M}_{\bar{0}}$ in $\mathscr{C}(A)$. Thus, we obtain Frobenius maps $\mathfrak{F}_{\mathsf{M}_{\bar{0}}} := \phi_{\mathsf{M}_{\bar{0}}} \circ \tau_{\mathsf{M}_{\bar{0}}}$ on $\mathsf{M}_{\bar{0}}$ and $\mathfrak{F}_{\mathsf{N}_{\bar{0}}} = \phi_{\mathsf{N}_{\bar{0}}} \circ \tau_{\mathsf{N}_{\bar{0}}}$ on $\mathsf{N}_{\bar{0}}$. Extend them to obtain Frobenius maps $\mathfrak{F}_1$ in M and $\mathfrak{F}_2$ on N satisfying $\mathsf{M}^{[\mathfrak{F}_1]} =_{\bar{0}} \mathsf{M}$ and $\mathsf{N}^{[\mathfrak{F}_2]} =_{\bar{0}} \mathsf{N}$. Now, with the notation in (4.0.2) and (4.0.3), we have

$$F^{\mathscr{K}}_{(\mathsf{M},\mathsf{N})}(\bar{f}) = \bar{f} \iff F^{\mathscr{K}}_{(\mathsf{M}_{\bar{0}},\mathsf{N}_{\bar{0}})}(\bar{f}_{\bar{0}}) = \bar{f}_{\bar{0}} \iff \bar{\phi}_{\mathsf{N}_{\bar{0}}} \circ \bar{f}_{\bar{0}}^{[1]} = \bar{f}_{\bar{0}} \circ \bar{\phi}_{\mathsf{M}_{\bar{0}}} \iff \bar{\phi}_{\mathsf{N}} \circ \bar{f}^{[1]} = \bar{f} \circ \bar{\phi}_{\mathsf{M}},$$

Here the first equivalence follows from (4.0.3), the second from the definition (cf. Corollary 2.9), and the last from the one analogous to (4.0.3).

(b) Let M be an indecomposable object in $\mathscr{K}^b(A)$ such that M viewed as an object $\mathscr{C}^b(A)$ contains no non-zero contractible summands. Then $\mathrm{Ht}(\mathsf{M},\mathsf{M}) \subset \mathrm{Rad}\,(\mathrm{End}_{\mathscr{C}^b(A)}(\mathsf{M}))$ and, hence,

$$\mathrm{End}_{\mathscr{K}^b(A)}(\mathsf{M})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{K}^b(A)}(\mathsf{M})) \cong \mathrm{End}_{\mathscr{C}^b(A)}(\mathsf{M})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{C}^b(A)}(\mathsf{M})) \cong k,$$

that is, $\mathrm{End}_{\mathscr{K}^b(A)}(\mathsf{M})$ is a local algebra. Thus, $\mathscr{K}^b(A)$ is again a Krull-Schmidt category. Similarly, $\mathscr{K}^b(A^F)$ is a Krull-Schmidt category as well.

We end this section with the following result analogous to Theorem 3.9.

Theorem 4.4. *Let M be an indecomposable object in $\mathscr{K}^b(A)$ with $\mathscr{K}$-period r and $\tilde{\mathsf{M}}$ be the associated F-stable object. Then $\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}}$ is indecomposable in $\mathscr{K}^b(A^F)$ and*

$$\mathrm{End}_{\mathscr{K}^b(A^F)}(\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{K}^b(A^F)}(\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}})) \cong \mathrm{End}_{\mathscr{K}^b(A)^F}(\tilde{\mathsf{M}})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{K}^b(A)^F}(\tilde{\mathsf{M}})) \cong \mathbb{F}_{q^r}.$$

Moreover, every indecomposable object in $\mathscr{K}^b(A^F)$ is isomorphic to an object of the form $\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}}$ for some F-periodic indecomposable object M in $\mathscr{K}^b(A)$.

Proof. We may suppose that M has no non-zero contractible summands. Then all $\mathsf{M}^{[r]}$ have no non-zero contractible summands. Thus, both $\tilde{\mathsf{M}}$ and $\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}}$ have no non-zero contractible summands. Since $\mathrm{Ht}(\tilde{M}^{\tilde{\mathfrak{F}}}, \tilde{M}^{\tilde{\mathfrak{F}}}) \subseteq \mathrm{Rad}\,(\mathrm{End}_{\mathscr{C}^b(A^F)}(\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}}))$, by Theorem 3.9, we obtain

$$\mathrm{End}_{\mathscr{K}^b(A^F)}(\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{K}^b(A^F)}(\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}})) \cong \mathrm{End}_{\mathscr{C}^b(A^F)}(\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{C}^b(A^F)}(\tilde{\mathsf{M}}^{\tilde{\mathfrak{F}}})) \cong \mathbb{F}_{q^r}.$$

The rest of the proof is formal (see Theorem [4, 5.1]). □

Remark 4.5. For any full additive subcategory $\mathcal{A}$ of **mod**-A, we can define the category $\mathscr{C}(\mathcal{A})$ of complexes in $\mathcal{A}$, the triangulated category $\mathscr{K}(\mathcal{A})$, and their subcategories $\mathscr{C}^*(\mathcal{A})$, $\mathscr{K}^*(\mathcal{A})$ for $* = +, -, b$. Particular cases are $\mathcal{A} = \mathcal{P}(A)$ and $\mathcal{A} = \mathcal{I}(A)$, the full subcategories of **mod**-A consisting of all projective and injective A-modules, respectively. Since the Frobenius functor on **mod**-A induces category equivalence $(\ )^{[1]}$ on $\mathcal{P}(A)$ and $\mathcal{I}(A)$, respectively, it is easy to see that results such as Corollary 4.1 and Theorem 4.4 continue to hold for *full* subcategories $\mathscr{K}^b(\mathcal{P}(A))$ and $\mathscr{K}^b(\mathcal{I}(A))$. Note that both $\mathscr{K}^b(\mathcal{P}(A))$ and $\mathscr{K}^b(\mathcal{I}(A))$ are Krull-Schmidt categories as discussed in 4.3 (b).

5. Derived categories with Frobenius functors

A morphism $f : \mathsf{M} \to \mathsf{N}$ in $\mathscr{K}(A)$ is called *a quasi-isomorphism* if $H^i(f)$ is an isomorphism for each $i \in \mathbb{Z}$, where $H^i : \mathscr{K}(A) \to \textbf{mod}\text{-}A$ is the i-th cohomological functor. Let $\mathscr{D}(A)$ be the derived category of A, i.e., $\mathscr{D}(A)$ is the localization of $\mathscr{K}(A)$ at the class $\mathcal{S}$ of all quasi-isomorphisms. Similar to the homotopy case, we can define categories $\mathscr{D}^b(A)$, $\mathscr{D}^+(A)$, and $\mathscr{D}^-(A)$ which can be considered as full subcategories of $\mathscr{D}(A)$.

Clearly, f is a quasi-isomorphism if and only if so is $f^{[1]}$. Therefore, the Frobenius functor on $\mathscr{K}(A)$ induces a functor

$$(\ )^{[1]} = (\ )^{[1]}_{\mathscr{D}(A)} : \mathscr{D}(A) \to \mathscr{D}(A),$$

which is again an equivalence of triangulated categories. Note that if $\xi \in \mathrm{Hom}_{\mathscr{D}(A)}(\mathsf{M}, \mathsf{N})$ denotes the equivalence class $\overline{(\mathsf{L}, s, f)}$ of the triple (L, s, f), where L is an object in $\mathscr{D}(A)$, $s \in \mathrm{Hom}_{\mathscr{D}(A)}(\mathsf{L}, \mathsf{M})$ is a quasi-isomorphism, and $f \in \mathrm{Hom}_{\mathscr{D}(A)}(\mathsf{L}, \mathsf{N})$,[2] then $\xi^{[1]}$ is the equivalence class of $(\mathsf{L}^{[1]}, s^{[1]}, f^{[1]})$. Moreover, this Frobenius functor induces endofunctors on the full subcategories $\mathscr{D}^b(A)$, $\mathscr{D}^+(A)$, and $\mathscr{D}^-(A)$.

The shift functor on $\mathscr{K}(A)$ also induces the shift functor $\mathcal{T}$ on $\mathscr{D}(A)$ which commutes with the Frobenius functor. This fact will be used in §7.

As in the categories $\mathscr{C}(A)$ and $\mathscr{K}(A)$, an object M in $\mathscr{D}(A)$ is said to be *F-stable* if $\mathsf{M} \cong \mathsf{M}^{[1]}$ in $\mathscr{D}(A)$, and is said to be *F-periodic* if $\mathsf{M} \cong \mathsf{M}^{[r]}$ in $\mathscr{D}(A)$ for some integer $r \geqslant 1$. Call the minimal r with $\mathsf{M} \cong \mathsf{M}^{[r]}$, denoted by $p_{\mathscr{D}}(\mathsf{M})$, the *$\mathscr{D}$-period* of M. Clearly, $p_{\mathscr{C}}(\mathsf{M}) \geqslant p_{\mathscr{D}}(\mathsf{M})$. Thus, each object X_k with $\mathsf{X} \in \mathscr{D}(A^F)$ is F-stable in $\mathscr{D}(A)$, and each object in $\mathscr{D}^b(A)$ is F-periodic.

Lemma 5.1. *Let M be an F-stable object in $\mathscr{D}^b(A)$. Then $\mathsf{M} \cong \mathsf{X}_k$ for some $\mathsf{X} \in \mathscr{D}^b(A^F)$.*

Proof. Choose an object $\mathsf{P} = (P^i, d^i_{\mathsf{P}}) \in \mathscr{C}^-(\mathcal{P}(A))$ such that there is a quasi-isomorphism $\mathsf{P} \to \mathsf{M}$ in $\mathscr{K}^-(A)$. Since M is $\mathfrak{F}$-stable in $\mathscr{D}(A)$, we have $\mathsf{P} \cong \mathsf{P}^{[1]}$ in $\mathscr{K}^-(A)$. Here we have used the triangulated category equivalence $\mathscr{D}^-(A) \cong \mathscr{K}^-(\mathcal{P}(A))$. Let $n \in \mathbb{Z}$ be such that $H^i(\mathsf{P}) = 0$ for all $i < n$ and consider the truncated complex of P

$$\tau^{\geqslant n}(\mathsf{P}) : \cdots \longrightarrow 0 \longrightarrow \mathrm{Coker}\, d^{n-1}_{\mathsf{P}} \longrightarrow P^{n+1} \longrightarrow P^{n+2} \longrightarrow \cdots .$$

[2]The triple (L, s, f) is represented sometimes by the diagram $\mathsf{M} \overset{s}{\Longleftarrow} \mathsf{L} \overset{f}{\longrightarrow} \mathsf{N}$ (see, e.g., [13]).

Then $\tau^{\geqslant n}(\mathsf{P}) \cong \tau^{\geqslant n}(\mathsf{P}^{[1]}) \cong (\tau^{\geqslant n}(\mathsf{P}))^{[1]}$ in $\mathscr{K}^b(A)$. Thus, there is an object $\mathsf{X} \in \mathscr{C}^b(A^F)$ such that $\tau^{\geqslant n}(\mathsf{P}) \cong \mathsf{X}_k$ in $\mathscr{K}^b(A)$. Therefore, we have

$$\mathsf{M} \cong \mathsf{P} \cong \tau^{\geqslant n}(\mathsf{P}) \cong \mathsf{X}_k$$

in $\mathscr{D}^-(A)$, and hence, in $\mathscr{D}^b(A)$. □

For each object $\mathsf{M} = (M^i, d^i_{\mathsf{M}})$ in $\mathscr{D}(A)$ with a Frobenius map $\mathfrak{F}$, Lemma 3.2 implies that $\mathsf{M}^{[\mathfrak{F}]}$ and $\mathsf{M}^{[1]}$ are isomorphic as objects in $\mathscr{D}(A)$. Let $\mathsf{M} = (M^i, d^i_{\mathsf{M}})$ and $\mathsf{N} = (N^i, d^i_{\mathsf{N}})$ be objects in $\mathscr{K}(A)$ with Frobenius maps $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively. Then the q-twist map

$$F^{\mathscr{K}}_{(\mathsf{M},\mathsf{N})} : \mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M}, \mathsf{N}) \to \mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M}^{[\mathfrak{F}_1]}, \mathsf{N}^{[\mathfrak{F}_2]})$$

induces a q-twist map

(5.1.1)

$$F^{\mathscr{D}}_{(\mathsf{M},\mathsf{N})} : \mathrm{Hom}_{\mathscr{D}(A^F)}(\mathsf{M}, \mathsf{N}) \longrightarrow \mathrm{Hom}_{\mathscr{D}(A)}(\mathsf{M}^{[\mathfrak{F}_1]}, \mathsf{N}^{[\mathfrak{F}_2]});\ \overline{(\mathsf{L}, s, f)} \longmapsto \overline{(\mathsf{L}^{[\mathfrak{F}]}, s^{[\mathfrak{F}]}, f^{[\mathfrak{F}]})}.$$

Note that $\overline{(\mathsf{L}^{[\mathfrak{F}]}, s^{[\mathfrak{F}]}, f^{[\mathfrak{F}]})}$ does not depend on the choice of the Frobenius maps $\mathfrak{F}_0$ in L.

Similarly, if X, Y are objects in $\mathscr{D}(A^F)$, we have a map

$$\Psi(\mathsf{X}, \mathsf{Y}) : \mathrm{Hom}_{\mathscr{D}(A)}(\mathsf{X}, \mathsf{Y}) \longrightarrow \mathrm{Hom}_{\mathscr{D}(A)}(\mathsf{X}_k, \mathsf{Y}_k);\ \overline{(\mathsf{Z}, s, f)} \longmapsto \overline{(\mathsf{Z}_k, s \otimes 1, f \otimes 1)}.$$

Thus, the functor $\Phi : \mathscr{K}(A^F) \to \mathscr{K}(A)$ induces a functor $\Psi : \mathscr{D}(A^F) \to \mathscr{D}(A)$. The following is a derived counterpart of Lemma 4.1.

Lemma 5.2. *The restriction of* $\Psi : \mathscr{D}^b(A^F) \to \mathscr{D}^b(A)$ *is faithful.*

Proof. Recall from Remark 4.5 that $\mathcal{P}(A)$ (resp. $\mathcal{P}(A^F)$) is the full subcategory of **mod**-A (resp. **mod**-A^F) of all projective A-modules (resp. A^F-modules). For objects X, Y in $\mathscr{D}^b(A^F)$, choose an object P in $\mathscr{C}^-(\mathcal{P}(A^F))$ quasi-isomorphic to X (in $\mathscr{K}^-(A^F)$) and let n be an integer such that $Y^i = 0$ for all $i \leqslant n$. Then there are natural isomorphisms

$$\mathrm{Hom}_{\mathscr{D}^b(A^F)}(\mathsf{X}, \mathsf{Y}) \cong \mathrm{Hom}_{\mathscr{D}^-(A^F)}(\mathsf{P}, \mathsf{Y}) \cong \mathrm{Hom}_{\mathscr{K}^-(A^F)}(\mathsf{P}, \mathsf{Y}) \cong \mathrm{Hom}_{\mathscr{K}^b(A^F)}(\mathsf{P}_{\geqslant n}, \mathsf{Y})$$

where $\tau_{\geqslant n}\mathsf{P} = \mathsf{Z} = (Z^i, d^i_{\mathsf{Z}})$ is the "stupid" truncation of P defined by setting $Z^i = P^i$, $d^i_{\mathsf{Z}} = d^i_{\mathsf{Z}}$ if $i \geqslant n$, and $Z^i = 0$, $d^i_{\mathsf{Z}} = 0$ otherwise. (The middle isomorphism is classical; see, e.g., [20, (10.4.7)].)

On the other hand, we have natural isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{X}_k, \mathsf{Y}_k) \cong \mathrm{Hom}_{\mathscr{D}^-(A)}(\mathsf{P}_k, \mathsf{Y}_k) \\ \cong\ &\mathrm{Hom}_{\mathscr{K}^-(A)}(\mathsf{P}_k, \mathsf{Y}_k) \cong \mathrm{Hom}_{\mathscr{K}^b(A)}((\mathsf{P}_k)_{\geqslant n}, \mathsf{Y}_k). \end{aligned}$$

Note that $(\mathsf{P}_k)_{\geqslant n} = (\mathsf{P}_{\geqslant n}) \otimes k$. Since Φ is faithful, we have an injective map

$$\Phi(\mathsf{P}_{\geqslant n}, \mathsf{Y}) : \mathrm{Hom}_{\mathscr{K}^b(A^F)}(\mathsf{P}_{\geqslant n}, \mathsf{Y}) \to \mathrm{Hom}_{\mathscr{K}^b(A)}((\mathsf{P}_k)_{\geqslant n}, \mathsf{Y}_k). \tag{5.2.1}$$

This implies that Ψ is faithful. □

The last isomorphism in the proof shows that $\mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{X}_k, \mathsf{Y}_k)$ is finite dimensional. Applying Lemma 5.1, we obtain

Corollary 5.3. *If* M, N *are* F*-stable objects in* $\mathscr{D}^b(A)$*, then the* q*-twist map* $F^{\mathscr{D}}_{(\mathsf{M},\mathsf{N})}$ *defined in (5.1.1) is a Frobenius map on* $\mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{M}, \mathsf{N})$*.*

This allows us, similar to $\mathscr{K}^b(A)^F$, to define the category $\mathscr{D}^b(A)^F$ consisting of F-stable objects in $\mathscr{D}^b(A)$ and Hom-spaces

$$\mathrm{Hom}_{\mathscr{D}^b(A)^F}(\mathsf{M}, \mathsf{N}) := \mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{M}, \mathsf{N})^F,$$

where $F = F^{\mathscr{D}}_{(\mathsf{M},\mathsf{N})}$ is defined in (5.1.1) with respect to the natural Frobenius maps on M and N via the isomorphisms given in Lemma 5.1.

Theorem 5.4. *The functor* Ψ *induces a triangulated category equivalence*

$$\mathscr{D}^b(A^F) \cong \mathscr{D}^b(A)^F.$$

Proof. Since the map given in (5.2.1) induces an isomorphism onto $\mathrm{Hom}_{\mathscr{K}^b(A)}((\mathsf{P}_k)_{\geqslant n}, \mathsf{Y}_k)^F$; see (4.1.1), the functor Ψ is faithful and full and takes triangles to triangles. These together with Lemma 5.1 gives the required equivalence. □

Remark 5.5. We may define the category $\mathscr{D}^b(A)^F$ in an intrinsic way as done for $\mathscr{K}^b(A)$ in Remark 4.3 (a) (cf. 2.9 and 3.8):

$$\begin{cases} \text{Objects:} \quad \mathsf{M} \text{ such that } \mathsf{M}^{[1]} \overset{\phi^{\mathscr{D}}_{\mathsf{M}}}{\cong} \mathsf{M} \text{ in } \mathscr{D}(A), \\ \text{Morphisms: } \mathrm{Hom}_{\mathscr{D}^b(A)^F}(\mathsf{M},\mathsf{N}) = \{\xi \in \mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{M},\mathsf{N}) \mid \phi^{\mathscr{D}}_{\mathsf{N}} \circ \xi^{[1]} = \xi \circ \phi^{\mathscr{D}}_{\mathsf{M}}\}. \end{cases}$$

We need to show that $\mathrm{Hom}_{\mathscr{D}^b(A)^F}(\mathsf{M},\mathsf{N}) \cong \mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{M},\mathsf{N})^F$. To see this, choose P in $\mathscr{K}^-(\mathcal{P}(A))$ quasi-isomorphic to M and choose integer n for which $N^i = 0$ for all $i \leqslant n$. Then $\phi^{\mathscr{D}}_{\mathsf{M}}$ induces an isomorphism $\mathsf{P}^{[1]} \overset{\phi^{\mathscr{K}}_{\mathsf{P}}}{\cong} \mathsf{P}$ in $\mathscr{K}^-(\mathcal{P}(A))$, and hence, isomorphism $\mathsf{P}^{[1]}_{\geqslant n} \overset{\phi^{\mathscr{K}}_{\mathsf{P}}}{\cong} \mathsf{P}_{\geqslant n}$ in $\mathscr{K}^b(\mathcal{P}(A))$. By the proof of Lemma 5.2, $\xi : \mathsf{M} \to \mathsf{N}$ in $\mathscr{D}^b(A)$ determines $\tilde{\xi} : \mathsf{P}_{\geqslant n} \to \mathsf{N}$. Now our assertion follows from the following natural relations:

$$F^{\mathscr{D}}_{(\mathsf{M},\mathsf{N})}(\xi) = \xi \iff F^{\mathscr{K}}_{(\mathsf{P}_{\geqslant n},\mathsf{N})}(\tilde{\xi}) = \tilde{\xi} \iff \phi^{\mathscr{K}}_{\mathsf{N}} \circ \tilde{\xi}^{[1]} = \tilde{\xi} \circ \phi^{\mathscr{K}}_{\mathsf{P}_{\geqslant n}} \iff \phi^{\mathscr{D}}_{\mathsf{N}} \circ \xi^{[1]} = \xi \circ \phi^{\mathscr{D}}_{\mathsf{M}}.$$

Let $\mathsf{M} \in \mathscr{D}^b(A)$ be F-periodic with $\mathscr{D}$-period r. Then

$$\tilde{\mathsf{M}} := \mathsf{M} \oplus \mathsf{M}^{[1]} \oplus \cdots \oplus \mathsf{M}^{[r-1]}$$

is F-stable. By Lemma 5.1, there is a complex $\mathsf{X} \in \mathscr{C}^b(A^F)$ such that $\tilde{\mathsf{M}} \cong \mathsf{X}_k$ in $\mathscr{D}^b(A)$. In general, we do not know whether X is indecomposable in $\mathscr{D}^b(A^F)$ even if we know M is indecomposable. However, this is the case if A has finite global dimension.

Theorem 5.6. *Suppose that A has a* finite global dimension. *Let* $\mathsf{M} \in \mathscr{D}^b(A)$ *be indecomposable with $\mathscr{D}$-period r and* $\tilde{\mathsf{M}} = \oplus_{i=0}^{r-1}\mathsf{M}^{[i]}$. *Let further* $\mathsf{X} \in \mathscr{D}^b(A^F)$ *be such that* $\tilde{\mathsf{M}} \cong \mathsf{X}_k$ *in* $\mathscr{D}^b(A)$. *Then*

$$\mathrm{End}_{\mathscr{D}^b(A^F)}(\mathsf{X})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{D}^b(A^F)}(\mathsf{X})) \cong \mathrm{End}_{\mathscr{D}^b(A)^F}(\tilde{\mathsf{M}})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{D}^b(A)^F}(\tilde{\mathsf{M}})) \cong \mathbb{F}_{q^r}.$$

Hence X *is indecomposable. Moreover, every indecomposable object in* $\mathscr{D}^b(A^F)$ *can be obtained in this way.*

Proof. Let $\mathfrak{Q} : \underline{\mathscr{K}(A) \to \mathscr{D}(A)}$ be the natural functor defined by $\mathfrak{Q}(\mathsf{M}) = \mathsf{M}$ for each object M in $\mathscr{K}(A)$, and $\mathfrak{Q}(f) = \overline{(\mathsf{M}, 1_{\mathsf{M}}, f)}$ for $f \in \mathrm{Hom}_{\mathscr{K}(A)}(\mathsf{M},\mathsf{N})$. It is clear to see that

$$\mathfrak{Q} \circ (\ )^{[1]}_{\mathscr{K}(A)} = (\ )^{[1]}_{\mathscr{D}(A)} \circ \mathfrak{Q}. \tag{5.6.1}$$

Since A has a finite global dimension, we have that the composition Θ of the inclusion $\mathscr{K}^b(\mathcal{P}(A)) \to \mathscr{K}^b(A)$ and the natural functor $\mathfrak{Q} : \mathscr{K}^b(A) \to \mathscr{D}^b(A)$ is an equivalence of triangulated categories. Moreover, (5.6.1) implies that Θ is compatible with the Frobenius functors on $\mathscr{K}^b(\mathcal{P}(A))$ and $\mathscr{D}^b(A)$, i.e., $\Theta(\mathsf{P}^{[1]}) = (\Theta(\mathsf{P}))^{[1]}$ for all $\mathsf{P} \in \mathscr{K}^b(\mathcal{P}(A))$. Then the theorem follows from Theorem 4.4 and Remark 4.5. □

Remarks 5.7. (a) If A has a finite global dimension, then the equivalence $\Theta : \mathscr{K}^b(\mathcal{P}(A)) \to \mathscr{D}^b(A)$ implies particularly that $\mathscr{D}^b(A)$ is a Krull-Schmidt category since so is $\mathscr{K}^b(\mathcal{P}(A))$.

(b) If we further assume that A is hereditary, then Theorem 5.6 follows immediately from [11, 4.1] and [4, 5.1]. In this case, M is isomorphic to a stalk complex $\mathcal{T}^m M^\bullet$ for some indecomposable A-module M (see 3.4(a)), and hence, $\tilde{\mathsf{M}}$ is isomorphic to $\mathcal{T}^m(\tilde{M}^\bullet)$. Now, apply [4, 5.1] to obtain

$$\mathrm{End}_{\mathscr{D}^b(A)^F}(\tilde{\mathsf{M}})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{D}^b(A)^F}(\tilde{\mathsf{M}})) \cong \mathrm{End}_{\mathbf{mod}^F\text{-}A}(\tilde{M})/\mathrm{Rad}\,(\mathrm{End}_{\mathbf{mod}^F\text{-}A}(\tilde{M})) \cong \mathbb{F}_{q^r}.$$

6. Auslander-Reiten triangles and F-stable triangles

Let us first recall from [12] the definition of Auslander-Reiten triangles in a triangulated category. Let k be an arbitrary field and $\mathcal{C}$ an additive k-category. Let further $\mathcal{C}$ be a triangulated category with the translation functor $\mathcal{T}:\mathcal{C}\to\mathcal{C}$. We assume that, for every $X,Y\in\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a finite dimensional k-vector space, and that the endomorphism ring of an indecomposable object in $\mathcal{C}$ is local.

A morphism $f:X\to Y$ in $\mathcal{C}$ is called a *section* (resp. *retraction*) if there exists $g:Y\to X$ such that $gf=1_X$ (resp. $fg=1_Y$).

Definition 6.1. A distinguished triangle in $\mathcal{C}$

$$X\xrightarrow{f}Y\xrightarrow{g}Z\xrightarrow{h}\mathcal{T}(X)$$

is called an *Auslander-Reiten triangle* if the following conditions are satisfied:
(a) X,Z are indecomposable,
(b) $h\neq 0$,
(c) any morphism $W\to Z$ which is not a retraction factors through g.

Remarks 6.2. (1) $h\neq 0\iff f$ is not a section $\iff g$ is not a retraction.

(2) Let $X\stackrel{f}{\to}Y\stackrel{g}{\to}Z\stackrel{h}{\to}\mathcal{T}(X)$ be an Auslander-Reiten triangle. Then any morphism $X\to W$ which is not a section factors through f.

By definition, a morphism $f:X\to Y$ in $\mathcal{C}$ is called a *source map* if the following conditions are satisfied:
(a) f is not a section,
(b) any morphism $X\to W$ which is not a section factors through f,
(c) any morphism $\varphi:Y\to Y$ satisfying $f=\varphi f$ is an isomorphism.
Dually, a morphism $g:Y\to Z$ in $\mathcal{C}$ is called a *sink map* if the following conditions are satisfied:
(a) f is not a retraction,
(b) any morphism $W\to Z$ which is not a retraction factors through g,
(c) any morphism $\psi:Y\to Y$ satisfying $g=g\psi$ is an isomorphism.

Note that if $f:X\to Y$ in $\mathcal{C}$ is a source (resp. sink) map, then X (resp. Y) is indecomposable. Moreover, a distinguished triangle $X\stackrel{f}{\to}Y\stackrel{g}{\to}Z\stackrel{h}{\to}\mathcal{T}(X)$ is an Auslander-Reiten triangle if and only if f is a source map (or if and only if g is a sink map).

Remark 6.3. Analogous to source and sink maps in a module category,[3] both source and sink maps in $\mathcal{C}$ admit a functorial description. Let $\mathbf{Fun}(\mathcal{C})$ denote the category whose objects are the covariant additive functors from $\mathcal{C}$ to the category $\mathbf{mod}$-k of finite dimensional k-vector spaces, and whose morphisms are the natural transformations of functors. Thus, each object M in $\mathcal{C}$ defines a functor $\mathrm{Hom}_{\mathcal{C}}(M,-)$ in $\mathbf{Fun}(\mathcal{C})$ and the quotient functor $\mathcal{K}_M$ by its subfunctor $\mathrm{Rad}_{\mathcal{C}}(M,-)$, where $\mathrm{Rad}_{\mathcal{C}}(-,-)$ denotes the radical of $\mathcal{C}$ (see [10, 3.2]). It follows from the definition above that for an indecomposable object X, a morphism $f:X\to Y$ in $\mathcal{C}$ is a source map if and only if the induced sequence

$$\mathrm{Hom}_{\mathcal{C}}(Y,-)\xrightarrow{f_*}\mathrm{Hom}_{\mathcal{C}}(X,-)\xrightarrow{\xi_X}\mathcal{K}_X\longrightarrow 0$$

is a minimal projective resolution of $\mathcal{K}_X$ in $\mathbf{Fun}(\mathcal{C})$, where ξ_X denotes the canonical projection. Dually, one has a functorial description of a sink map in $\mathcal{C}$.

The existence of Auslander-Reiten triangles in an arbitrary given triangulated category $\mathcal{C}$ is not necessarily true. However, we have the following due to Happel [12, 4.6].

[3] They are also called minimal left resp. right almost split morphisms.

Theorem 6.4. *Let A be a finite dimensional k-algebra of finite global dimension. Then the bounded derived category $\mathscr{D}^b(A)$ has Auslander-Reiten triangles.*

We now turn to the situation where the algebra A is equipped with a Frobenius morphism F. We further assume that A has a finite global dimension. Then A^F has also a finite global dimension. By the theorem above, both $\mathscr{D}^b(A)$ and $\mathscr{D}^b(A^F)$ have Auslander-Reiten triangles. A triangle in $\mathscr{D}^b(A)$ is called an *F-stable triangle* if it is defined in $\mathscr{D}^b(A)^F$. In the rest of the section, we aim at the construction of all Auslander-Reiten triangles in $\mathscr{D}^b(A^F)$ by the derived equivalence given in Theorem 5.4 and F-stable triangles.

Lemma 6.5. *A morphism $f : \mathsf{M} \to \mathsf{N}$ in $\mathscr{D}^b(A)$ is a source (resp. sink) map if and only if so is $f^{[1]} : \mathsf{M}^{[1]} \to \mathsf{N}^{[1]}$. And a distinguished triangle $\mathsf{N} \xrightarrow{f} \mathsf{L} \xrightarrow{g} \mathsf{M} \xrightarrow{h} \mathcal{T}(\mathsf{N})$ is an Auslander-Reiten triangle if and only if so is $\mathsf{N}^{[1]} \xrightarrow{f^{[1]}} \mathsf{L}^{[1]} \xrightarrow{g^{[1]}} \mathsf{M}^{[1]} \xrightarrow{h^{[1]}} \mathcal{T}(\mathsf{N}^{[1]})$.*

Since the global dimensions of A and A^F are finite, it is known that $\mathscr{D}^b(A)$ (resp. $\mathscr{D}^b(A^F)$) is equivalent to $\mathscr{K}^b(\mathcal{P}(A))$ (resp. $\mathscr{K}^b(\mathcal{P}(A^F))$) and to $\mathscr{K}^b(\mathcal{I}(A))$ (resp. $\mathscr{K}^b(\mathcal{I}(A^F))$) as triangulated categories.

In the following we work with the triangulated categories $\mathscr{K}^b(\mathcal{P}(A))$, $\mathscr{K}^b(\mathcal{P}(A^F))$ instead of $\mathscr{D}^b(A)$, $\mathscr{D}^b(A^F)$ and show that Auslander-Reiten triangles in $\mathscr{K}^b(\mathcal{P}(A^F))$ can be obtained by folding those in $\mathscr{K}^b(\mathcal{P}(A))$.

Let $f : \mathsf{P} \to \mathsf{T}$ be a source map in $\mathscr{K}^b(\mathcal{P}(A))$. Then for each integer $s \geqslant 1$, $\varphi^{[s]} : \mathsf{P}^{[s]} \to \mathsf{T}^{[s]}$ is also a source map. Let r be the $\mathscr{K}$-period of P. This implies $\mathsf{T}^{[r]} \cong \mathsf{T}$. We assume that both P and T as complexes in $\mathscr{C}^b(A)$ contain no non-zero contractible summands. Then there are Frobenius maps $\mathfrak{F}_1 = \{\mathfrak{F}_{1,i} : P^i \to P^i \mid i \in \mathbb{Z}\}$ and $\mathfrak{F}_2 = \{\mathfrak{F}_{2,i} : T^i \to T^i \mid i \in \mathbb{Z}\}$ such that $\mathsf{P}^{[\mathfrak{F}_1^r]} = \mathsf{P}$ and $\mathsf{T}^{[\mathfrak{F}_2^r]} = \mathsf{T}$ as complexes in $\mathscr{C}^b(A)$ (see Remark 4.3). Thus, both $\tilde{\mathsf{P}} = \oplus_{i=0}^{r-1}\mathsf{P}^{[\mathfrak{F}_1^i]}$ and $\tilde{\mathsf{T}} = \oplus_{i=0}^{r-1}\mathsf{T}^{[\mathfrak{F}_2^i]}$ are $\mathfrak{F}$-stable with respect to $\tilde{\mathfrak{F}}_1$ and $\tilde{\mathfrak{F}}_2$ as defined in (3.8.1), respectively. Since both $f : \mathsf{P} \to \mathsf{T}$ and $f^{[\mathfrak{F}^r]} : \mathsf{P} = \mathsf{P}^{[\mathfrak{F}_1^r]} \to \mathsf{T}^{[\mathfrak{F}_2^r]} = \mathsf{T}$ are source maps, there is a $\varphi \in \mathrm{Aut}_{\mathscr{K}^b(\mathcal{P}(A))}(\mathsf{T})$ such that $f^{[\mathfrak{F}^r]} = \varphi \circ f$. Let $F = F_{(\mathsf{T},\mathsf{T})}$ and $F_{(\mathsf{P},\mathsf{T})}$ be, respectively, the induced Frobenius maps on $\mathrm{Hom}_{\mathscr{K}^b(\mathcal{P}(A))}(\mathsf{T},\mathsf{T})$ and $\mathrm{Hom}_{\mathscr{K}^b(\mathcal{P}(A))}(\mathsf{P},\mathsf{T})$ (see (4.0.1)). Then $F_{(\mathsf{P},\mathsf{T})}(h \circ g) = F(h) \circ F_{(\mathsf{P},\mathsf{T})}(g)$ for all $h \in \mathrm{End}_{\mathscr{K}^b(\mathcal{P}(A))}(\mathsf{T})$ and $g \in \mathrm{Hom}_{\mathscr{K}^b(\mathcal{P}(A))}(\mathsf{P},\mathsf{T})$. By restricting F to the connected algebraic group $\mathrm{Aut}_{\mathscr{K}^b(\mathcal{P}(A))}(\mathsf{T})$ and applying Lang-Steinberg's theorem, there is an element $\psi \in \mathrm{Aut}_{\mathscr{K}^b(\mathcal{P}(A))}(\mathsf{T})$ satisfying $\varphi^{-1} = \psi^{-1} \circ F^r(\psi)$. Substituting this in $f^{[\mathfrak{F}^r]} = \varphi \circ f$ yields

$$\psi \circ f = F^r(\psi) \circ f^{[\mathfrak{F}^r]} = F^r(\psi) \circ F_{(\mathsf{P},\mathsf{T})}(f) = F^r_{(\mathsf{P},\mathsf{T})}(\psi \circ f).$$

Thus, $\psi \circ f : \mathsf{P} \to \mathsf{T}$ is a source map satisfying $(\psi \circ f)^{[\mathfrak{F}^r]} = \psi \circ f$. Replacing f by $\psi \circ f$, we may assume that f is chosen to satisfy $f^{[\mathfrak{F}^r]} = f$. Now we define

$$\tilde{f} := \mathrm{diag}(f, f^{[\mathfrak{F}]}, \cdots, f^{[\mathfrak{F}^{r-1}]}) : \tilde{\mathsf{P}} = \oplus_{i=0}^{r-1}\mathsf{P}^{[\mathfrak{F}_1^i]} \to \oplus_{i=0}^{r-1}\mathsf{T}^{[\mathfrak{F}_2^i]} = \tilde{\mathsf{T}}.$$

The equality $f^{[\mathfrak{F}^r]} = f$ implies that $\tilde{f}$ is compatible with the Frobenius morphisms $\mathfrak{F}_i$ (i.e., it is a morphism in $\mathscr{K}^b(\mathcal{P}(A))^F$; cf. Remark 4.5), and hence, induces a morphism $f_0 : \tilde{\mathsf{P}}^{\tilde{\mathfrak{F}}} \to \tilde{\mathsf{T}}^{\tilde{\mathfrak{F}}}$ in $\mathscr{K}^b(\mathcal{P}(A^F))$.

Since $f : \mathsf{P} \to \mathsf{T}$ is a source map, we have by [12, 4.5] that the distinguished triangle

$$\mathsf{P} \xrightarrow{f} \mathsf{T} \xrightarrow{\alpha(f)} C(f) \xrightarrow{\beta(f)} \mathcal{T}(\mathsf{P})$$

is an Auslander-Reiten triangle, where $C(f)$ is the mapping cone of f (cf., e.g., [16, 1.4]). Then, for each $1 \leqslant s \leqslant r-1$, we obtain an Auslander-Reiten triangle

$$\mathsf{P}^{[\mathfrak{F}_1^s]} \xrightarrow{f^{[\mathfrak{F}^s]}} \mathsf{T}^{[\mathfrak{F}_2^s]} \xrightarrow{\alpha(f^{[\mathfrak{F}^s]})} C(f^{[\mathfrak{F}^s]}) \xrightarrow{\beta(f^{[\mathfrak{F}^s]})} \mathcal{T}(\mathsf{P}^{[\mathfrak{F}_1^s]}).$$

Summing up, we get a distinguished F-stable triangle in $\mathscr{K}^b(\mathcal{P}(A))^F$

$$\tilde{\mathsf{P}} \xrightarrow{\tilde{f}} \tilde{\mathsf{T}} \xrightarrow{\alpha(\tilde{f})} C(\tilde{f}) \xrightarrow{\beta(\tilde{f})} \mathcal{T}(\tilde{\mathsf{P}}).$$

This gives a distinguished triangle in $\mathscr{K}^b(\mathcal{P}(A^F))$

$$\tilde{\mathsf{P}}^{\tilde{\mathfrak{F}}} \xrightarrow{f_0} \tilde{\mathsf{T}}^{\tilde{\mathfrak{F}}} \xrightarrow{\alpha(f_0)} C(f_0) \xrightarrow{\beta(f_0)} \mathcal{T}(\tilde{\mathsf{P}}^{\tilde{\mathfrak{F}}}).$$

Theorem 6.6. *Let $f : \mathsf{P} \to \mathsf{T}$ be a source map in $\mathscr{K}^b(\mathcal{P}(A))$. Then $f_0 : \tilde{\mathsf{P}}^{\tilde{\mathfrak{F}}} \to \tilde{\mathsf{T}}^{\tilde{\mathfrak{F}}}$ is a source map in $\mathscr{K}^b(\mathcal{P}(A^F))$. In particular,*

$$\tilde{\mathsf{P}}^{\tilde{\mathfrak{F}}} \xrightarrow{f_0} \tilde{\mathsf{T}}^{\tilde{\mathfrak{F}}} \xrightarrow{\alpha(f_0)} C(f_0) \xrightarrow{\beta(f_0)} \mathcal{T}(\tilde{\mathsf{P}}^{\tilde{\mathfrak{F}}})$$

is an Auslander-Reiten triangle in $\mathscr{K}^b(\mathcal{P}(A^F))$. Moreover, every Auslander-Reiten triangle in $\mathscr{K}^b(\mathcal{P}(A^F))$ can be constructed in this way.

Proof. For simplicity, we write $\mathsf{X} = \tilde{\mathsf{P}}^{\tilde{\mathfrak{F}}}$ and $\mathsf{Y} = \tilde{\mathsf{T}}^{\tilde{\mathfrak{F}}}$. By Remark 6.3, the source map $f : \mathsf{P} \to \mathsf{T}$ gives rise to a minimal projective resolution of $\mathcal{K}_{\mathsf{P}}$ in $\mathbf{Fun}(\mathscr{K}^b(\mathcal{P}(A)))$

$$\mathrm{Hom}_{\mathscr{K}^b(\mathcal{P}(A))}(\mathsf{T}, -) \xrightarrow{f_*} \mathrm{Hom}_{\mathscr{K}^b(\mathcal{P}(A))}(\mathsf{P}, -) \xrightarrow{\xi_{\mathsf{P}}} \mathcal{K}_{\mathsf{P}} \longrightarrow 0.$$

Using a similar argument as in the proof of [4, Theorem 7.4], we obtain a minimal projective resolution of $\mathcal{K}_{\tilde{\mathsf{P}}}$

$$\mathrm{Hom}_{\mathscr{K}^b(\mathcal{P}(A))}(\tilde{\mathsf{T}}, -) \xrightarrow{\tilde{f}_*} \mathrm{Hom}_{\mathscr{K}^b(\mathcal{P}(A))}(\tilde{\mathsf{P}}, -) \xrightarrow{\xi_{\tilde{\mathsf{P}}}} \mathcal{K}_{\tilde{\mathsf{P}}} \longrightarrow 0,$$

which induces a minimal projective resolution of $\mathcal{K}_{\mathsf{X}}$ in $\mathbf{Fun}(\mathscr{K}^b(\mathcal{P}(A^F)))$

$$\mathrm{Hom}_{\mathscr{K}^b(\mathcal{P}(A^F))}(\mathsf{Y}, -) \xrightarrow{(f_0)_*} \mathrm{Hom}_{\mathscr{K}^b(\mathcal{P}(A^F))}(\mathsf{X}, -) \xrightarrow{\xi_{\mathsf{X}}} \mathcal{K}_{\mathsf{X}} \longrightarrow 0.$$

Hence, again by Remark 6.3, $f_0 : \mathsf{X} \to \mathsf{Y}$ is a source map in $\mathscr{K}^b(\mathcal{P}(A^F))$. □

Remark 6.7. Since $\mathscr{D}^b(A)$ is equivalent to $\mathscr{K}^b(\mathcal{P}(A))$ as triangulated categories, the above theorem yields in fact a construction of Auslander-Reiten triangles in $\mathscr{D}^b(A^F)$ in terms of F-stable triangles in $\mathscr{D}^b(A)$.

7. Folding the Auslander-Reiten quiver of $\mathscr{D}^b(A)$

We begin with the definition of the Auslander-Reiten quiver of $\mathscr{D}^b(A)$ for a finite dimensional algebra A over an *arbitrary* field k, where A is assumed to have a finite global dimension. For any two objects $\mathsf{M} = \oplus_i \mathsf{M}_i$ and $\mathsf{N} = \oplus_j \mathsf{N}_j$ in $\mathscr{D}^b(A)$, where M_i and N_j are indecomposable, we define the *radical* of $\mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{M}, \mathsf{N})$ by

$$\mathrm{Rad}_{\mathscr{D}^b(A)}(\mathsf{M}, \mathsf{N}) = \{(f_{ji}) : \mathsf{M} \to \mathsf{N} \mid f_{ji} : \mathsf{M}_i \to \mathsf{N}_j \text{ is not an isomorphism for all } i, j\}.$$

Then $\mathrm{Rad}_{\mathscr{D}^b(A)}(-,-)$ is an ideal of $\mathscr{D}^b(A)$. Inductively, for each $n > 1$, the n-th power of the radical is defined to be

$$\mathrm{Rad}^n_{\mathscr{D}^b(A)}(\mathsf{M}, \mathsf{N}) = \sum_{\mathsf{L}} \mathrm{Rad}^{n-1}_{\mathscr{D}^b(A)}(\mathsf{L}, \mathsf{N}) \circ \mathrm{Rad}_{\mathscr{D}^b(A)}(\mathsf{M}, \mathsf{L}).$$

For an object M in $\mathscr{D}^b(A)$, let D_{M} denote the k-algebra

$$D_{\mathsf{M}} := \mathrm{End}_{\mathscr{D}^b(A)}(\mathsf{M})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{D}^b(A)}(\mathsf{M})).$$

This is a division algebra if M is indecomposable. By definition, the *Auslander-Reiten quiver* (or *AR-quiver* for short) of $\mathscr{D}^b(A)$ is a (simple) k-modulated quiver $\mathcal{Q}(\mathscr{D}^b(A))$ consisting of a valued graph $\Gamma = \Gamma(\mathscr{D}^b(A))$ and a k-modulation $\mathbb{M} = \mathbb{M}(\mathscr{D}^b(A))$ defined on Γ. Here, the vertices of Γ are isoclasses $[\mathsf{M}]$ of indecomposable objects in $\mathscr{D}^b(A)$ and the arrows $[\mathsf{M}] \to [\mathsf{N}]$ for indecomposable objects M and N are defined by the condition $\mathrm{Irr}_{\mathscr{D}^b(A)}(\mathsf{M},\mathsf{N}) \neq 0$, where

$$\mathrm{Irr}_{\mathscr{D}^b(A)}(\mathsf{M},\mathsf{N}) := \mathrm{Rad}_{\mathscr{D}^b(A)}(\mathsf{M},\mathsf{N})/\mathrm{Rad}^2_{\mathscr{D}^b(A)}(\mathsf{M},\mathsf{N})$$

is the space of irreducible homomorphisms from M to N. Each arrow $[\mathsf{M}] \to [\mathsf{N}]$ has the valuation $(d_{\mathsf{MN}}, d'_{\mathsf{MN}})$ with d_{MN} and d'_{MN} being the dimensions of $\mathrm{Irr}_{\mathscr{D}^b(A)}(\mathsf{M},\mathsf{N})$ considered as left D_{N}-space and right D_{M}-space, respectively. The k-modulation $\mathbb{M}$ is given by division algebras D_{M} for vertices $[\mathsf{M}]$ and (non-zero) D_{N}-D_{M}-bimodules $\mathrm{Irr}_{\mathscr{D}^b(A)}(\mathsf{M},\mathsf{N})$ for arrows $[\mathsf{M}] \to [\mathsf{N}]$.

If k is algebraically closed, then $D_{\mathsf{M}} \cong k$ and $d_{\mathsf{MN}} = d'_{\mathsf{MN}} = \dim_k \mathrm{Irr}_{\mathscr{D}^b(A)}(\mathsf{M},\mathsf{N})$ for all indecomposable objects M and N. So the modulation in the AR-quiver $\mathcal{Q}(\mathscr{D}^b(A))$ consists of k-spaces which can be represented by drawing d_{MN} arrows from $[\mathsf{M}]$ to $[\mathsf{N}]$. In this way, we turn the modulated quiver $\mathcal{Q}(\mathscr{D}^b(A))$ to an ordinary quiver.

Remark 7.1. Using a similar argument as in [1, p.255], the valuation $(d = d_{\mathsf{MN}}, d' = d'_{\mathsf{MN}})$ of an arrow $[\mathsf{M}] \to [\mathsf{N}]$ in $\mathcal{Q}(\mathscr{D}^b(A))$ can be described as follows. Let $\mathsf{M} \to \mathsf{L}$ be the source map in $\mathscr{D}^b(A)$ starting at M. Then $\mathsf{L} \cong d\mathsf{N} \oplus \mathsf{L}_1$ such that L_1 admits no summand isomorphic to N. Dually, let $\mathsf{K} \to \mathsf{N}$ be the sink map in $\mathscr{D}^b(A)$ ending at N. Then $\mathsf{K} \cong d'\mathsf{M} \oplus \mathsf{K}_1$ such that K_1 has no summand isomorphic to M.

We now turn to our typical situation where A is a finite dimensional algebra over $k = \overline{\mathbb{F}}_q$ together with a Frobenius morphism F on A. We assume that A has a finite global dimension. Thus, A^F has a finite global dimension, too. By Theorem 6.4, both $\mathscr{D}^b(A)$ and $\mathscr{D}^b(A^F)$ have Auslander-Reiten triangles. In the following we are going to relate the Auslander-Reiten quiver of $\mathscr{D}^b(A)$ to that of $\mathscr{D}^b(A^F)$.

By viewing the AR-quiver $\mathcal{Q} = \mathcal{Q}(\mathscr{D}^b(A))$ of $\mathscr{D}^b(A)$ as an ordinary quiver, we see that the Frobenius functor on $\mathscr{D}^b(A)$ induces an automorphism δ of $\mathcal{Q}$. For each vertex $[\mathsf{M}] \in \mathcal{Q}$, $\delta([\mathsf{M}])$ is defined to be $[\mathsf{M}^{[1]}]$. If M and N are indecomposable objects with $\mathscr{D}$-period r_1 and r_2, respectively, then there are n_{st} arrows $\gamma^{(m)}_{s,t}$ from $[\mathsf{M}^{[s]}]$ to $[\mathsf{N}^{[t]}]$ in $\mathcal{Q}$, where $0 \leqslant s \leqslant r_1 - 1$, $0 \leqslant t \leqslant r_2 - 1$, $n_{st} = \dim_k \mathrm{Irr}_{\mathscr{D}^b(A)}(\mathsf{M}^{[s]}, \mathsf{N}^{[t]})$ and $1 \leqslant m \leqslant n_{st}$. Note that $n_{st} = n_{s+1,t+1}$ for all s, t, where subscripts are considered as integers modulo r_1 and r_2, respectively. We now define

$$\delta(\gamma^{(m)}_{s,t}) = \gamma^{(m)}_{s+1,t+1} \text{ for all } 0 \leqslant s \leqslant r_1 - 1 \text{ and } 0 \leqslant t \leqslant r_2 - 1.$$

Recall from [4, §8] (see also the Appendix at the end of the paper) that associated to $(\mathcal{Q}, \delta)$ we may define a modulated quiver $\mathfrak{M}_{\mathcal{Q},\delta}$ as follows. Let $\mathcal{A} = k\mathcal{Q}$ denote the path algebra of $\mathcal{Q}$ and $F = F_{\mathcal{Q},\delta;q}$ be the Frobenius morphism of $\mathcal{A}$ induced by the automorphism δ. For each vertex $\mathbf{i}(\mathsf{M})$ (i.e., the δ-orbit of $[\mathsf{M}]$) and each arrow $\boldsymbol{\rho}$ (i.e., a δ-orbit of arrows in $\mathcal{Q}$) in $\Gamma(\mathcal{Q}, \delta)$, we define subspaces of $\mathcal{A}$

$$\mathcal{A}_{\mathbf{i}(\mathsf{M})} = \bigoplus_{s=0}^{r-1} k e_{[\mathsf{M}^{[s]}]} \text{ and } \mathcal{A}_{\boldsymbol{\rho}} = \bigoplus_{\rho \in \boldsymbol{\rho}} k\rho,$$

where r is the $\mathscr{D}$-period of M. By definition, the $\mathbb{F}_q$-modulation $\mathbb{M}(\mathcal{Q}, \delta)$ is given by $(\mathcal{A}_{\mathbf{i}(M)})^F$ and $(\mathcal{A}_{\boldsymbol{\rho}})^F$ for all vertices $\mathbf{i}(\mathsf{M})$ and arrows $\boldsymbol{\rho}$ in $\Gamma(\mathcal{Q}, \delta)$.

Theorem 7.2. *The modulated quiver $\mathfrak{M}_{\mathcal{Q},\delta}$ associated to the AR-quiver $(\mathcal{Q}, \delta)$ of $\mathscr{D}^b(A)$ defined above is isomorphic to the AR-quiver $\mathcal{Q}(\mathscr{D}^b(A^F))$ of $\mathscr{D}^b(A^F)$.*

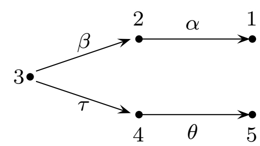
Proof. By Theorem 5.6, the correspondence $\mathbf{i}(\mathsf{M}) \to [\mathsf{X}]$ gives a bijection between vertices of $\mathfrak{M}_{\mathcal{Q},\delta}$ and those of $\mathcal{Q}(\mathscr{D}^b(A^F))$, where M are indecomposable objects in $\mathscr{D}^b(A)$ and $\mathsf{X} \in \mathscr{D}^b(A^F)$ are such that $\tilde{\mathsf{M}} \cong \mathsf{X}_k$. Further, by using a similar argument as in the proof of [4, Theorem 8.3], we have natural $\mathbb{F}_q$-algebra isomorphisms $(\mathcal{A}_{\mathbf{i}(\mathsf{M})})^F \cong D_{\mathsf{X}}$, where $F = F_{\mathcal{Q},\delta;q}$, and under these isomorphisms, for indecomposable objects M and N in $\mathscr{D}^b(A)$ with $\tilde{\mathsf{M}} \cong \mathsf{X}_k$ and $\tilde{\mathsf{N}} \cong \mathsf{Y}_k$, there is a D_{Y}-D_{X}-bimodule isomorphism

$$\bigoplus_{\boldsymbol{\rho}:\mathbf{i}(\mathsf{M})\to\mathbf{i}(\mathsf{N})} (\mathcal{A}_{\boldsymbol{\rho}})^F \cong \mathrm{Irr}_{\mathscr{D}^b(A^F)}(\mathsf{X}, \mathsf{Y}).$$

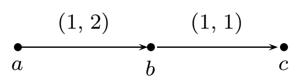
That is, the modulated quivers $\mathfrak{M}_{\mathcal{Q},\delta}$ and $\mathcal{Q}(\mathscr{D}^b(A^F))$ are isomorphic (in the sense of [4, 6.2]). $\square$

We end this section with an example explaining this folding process.

Example 7.3. Let Q be the following quiver



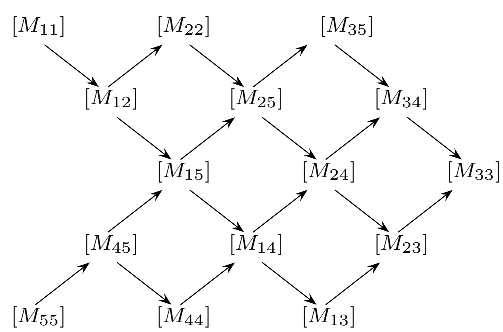
with an automorphism σ fixing 3, exchanging 1 and 5, 2 and 4. Then the corresponding valued quiver $\Gamma(Q,\sigma)$ has the form



with $(\varepsilon_a, \varepsilon_b, \varepsilon_c) = (1, 2, 2)$. Here $\varepsilon_{\mathbf{i}} = \#\{\text{vertices in the orbit } \mathbf{i}\}$. Then for each pair (i, j) with $1 \leqslant i \leqslant j \leqslant 5$, there is a unique (up to isomorphism) indecomposable representation of Q over k with support $\{s \mid i \leqslant s \leqslant j\}$; and they furnish a complete set of indecomposable representations of Q. Further, we have

$$M_{ij}^{[1]} \cong \begin{cases} M_{\sigma(i),\sigma(j)} & \text{if } \sigma(i) \leqslant \sigma(j) \\ M_{\sigma(j),\sigma(i)} & \text{if } \sigma(i) > \sigma(j). \end{cases}$$

It is well known that the Auslander-Reiten quiver of $A = kQ$ has the form (cf. [1, Chap.VII])



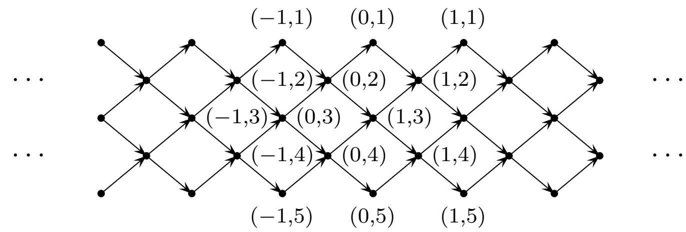
By [11, Corollary 4.5], the AR-quiver $\mathcal{Q} = \mathcal{Q}(\mathscr{D}^b(A))$ of $\mathscr{D}^b(A)$ is isomorphic to the quiver $\mathbb{Z}Q$, where

$$(\mathbb{Z}Q)_0 = \{(s, i) \mid s \in \mathbb{Z}, i \in Q_0\}$$

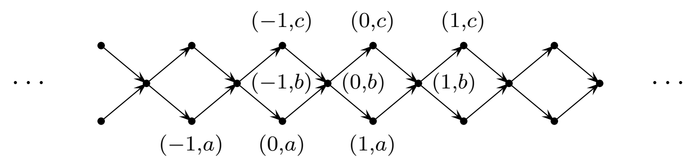
and

$$(\mathbb{Z}Q)_1 = \{(s, \rho) : (s, i) \longrightarrow (s, j), (s, j) \longrightarrow (s + 1, i) \mid s \in \mathbb{Z}, \rho : i \to j \in Q_1\}.$$

In other words, $\mathcal{Q} \cong \mathbb{Z}Q$ is the following quiver



It is easy to see that the automorphism δ of $\mathcal{Q} = \mathbb{Z}Q$ induced by the Frobenius functor on $\mathscr{D}^b(A)$ is such that δ fixes vertices $(s, 3)$, and exchanges $(s, 1)$ and $(s, 5)$, $(s, 2)$ and $(s, 4)$ for all $s \in \mathbb{Z}$. By the theorem above, the AR-quiver of $\mathscr{D}^b(A^F)$ is isomorphic to the modulated quiver $\mathfrak{M}_{\mathcal{Q},\delta}$ associated with $(\mathcal{Q}, \delta)$, whose underlying valued quiver is as follows



The valuations for arrows $(s, b) \longrightarrow (s, c)$ and $(s, c) \longrightarrow (s + 1, b)$ are $(1, 1)$, and the valuations for arrows $(s, a) \longrightarrow (s, b)$ and $(s, b) \longrightarrow (s + 1, a)$ are $(1, 2)$ and $(2, 1)$, respectively, for all $s \in \mathbb{Z}$. The $\mathbb{F}_q$-modulation of $\mathfrak{M}_{\mathcal{Q},\delta}$ is given in an obvious way.

8. Folding root categories

For the moment, let A be a finite dimensional algebra over an arbitrary field. Following [11, 5.1], the *root category* $\mathscr{R}(A)$ of A is the quotient category $\mathscr{D}^b(A)/(\mathcal{T}^2)$ of $\mathscr{D}^b(A)$ by the automorphism $\mathcal{T}^2$, where $\mathcal{T}$ is the shift functor on $\mathscr{D}^b(A)$ (cf. Lemma 3.1). Thus, by definition, the objects in $\mathscr{R}(A)$ are $\mathcal{T}^2$-orbits of objects in $\mathscr{D}^b(A)$, i. e., $\mathcal{O}_{\mathsf{M}} = \{\mathcal{T}^{2i}\mathsf{M} \mid i \in \mathbb{Z}\}$, $\mathsf{M} \in \mathscr{D}^b(A)$. A morphism $f = (f_{ji}) : \mathcal{O}_{\mathsf{M}} \to \mathcal{O}_{\mathsf{N}}$ is given by morphisms $f_{ji} : \mathcal{T}^{2i}\mathsf{M} \to \mathcal{T}^{2j}\mathsf{N}$ in $\mathscr{D}^b(A)$ satisfying

(1) $\mathcal{T}(f_{ji}) = f_{j+1,i+1}$ for all $i, j \in \mathbf{Z}$,
(2) For each fixed $i \in \mathbb{Z}$, all, but finitely many f_{ji}, are zero.

The composition $gf = (h_{ji})$ of morphisms $f : \mathcal{O}_{\mathsf{M}} \to \mathcal{O}_{\mathsf{N}}$ and $g : \mathcal{O}_{\mathsf{N}} \to \mathcal{O}_{\mathsf{L}}$ is such that $h_{ji} = \sum_{s\in\mathbb{Z}} g_{js} f_{si}$.

Remark 8.1. Let $Q = (Q_0, Q_1)$ be a quiver without oriented cycles. Then the path algebra $A = kQ$ is finite dimensional and hereditary. For each A-module M, the dimension vector of M will be denoted by $\mathbf{dim}\, M \in \mathbb{Z}Q_0$. Then we define the dimension vector of an object $\mathsf{M} = (M^i, d^i) \in \mathscr{D}^b(A)$ by

$$\mathbf{dim}\, \mathsf{M} = \sum_{i\in\mathbb{Z}} (-1)^i \mathbf{dim}\, M^i \in \mathbb{Z}Q_0.$$

This is well defined since M is bounded. By [11, Lemma 4.1] and [15], the correspondence $\mathsf{M} \mapsto \mathbf{dim}\, \mathsf{M}$ induces a surjective map $\mathbf{dim}$ from the set of isoclasses of indecomposable objects in $\mathscr{D}^b(A)$ to the set of all (positive and negative) roots of the Kac-Moody Lie algebra $\mathfrak{g}(Q)$ associated to Q. Since $\mathbf{dim}\, \mathsf{M} = \mathbf{dim}\, \mathcal{T}^2\mathsf{M}$, $\mathbf{dim}$ also induces a surjective map from the set of isoclasses of indecomposable objects in $\mathscr{R}(A)$ to the set of all roots of $\mathfrak{g}(Q)$. In particular, if Q is a Dynkin quiver, $\mathbf{dim}$ induces a bijection between the set of isoclasses of indecomposable objects of $\mathscr{R}(A)$ to the set of all roots of $\mathfrak{g}(Q)$ (see [11, Corollary 4.7]). This is the reason why $\mathscr{R}(A)$ is called the root category.

There is a canonical functor $\Pi = \Pi_A : \mathscr{D}^b(A) \to \mathscr{R}(A)$ such that for each object $\mathsf{M} \in \mathscr{D}^b(A)$, $\Pi(\mathsf{M}) = \mathcal{O}_{\mathsf{M}}$, and for each morphism $f : \mathsf{M} \to \mathsf{N}$, $\Pi(f) = (f_{ji}) : \mathcal{O}_{\mathsf{M}} \to \mathcal{O}_{\mathsf{N}}$ is given by $f_{ii} = \mathcal{T}^{2i}(f)$

and $f_{ji} = 0$ for $i \neq j$. Then for M and N in $\mathscr{D}^b(A)$, Π induces isomorphisms

$$\begin{array}{c} \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{D}^b(A)}(\mathcal{T}^{2i}\mathsf{M},\mathsf{N}) \cong \mathrm{Hom}_{\mathscr{R}}(\mathcal{O}_{\mathsf{M}},\mathcal{O}_{\mathsf{N}}) \\ \text{and} \\ \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{M},\mathcal{T}^{2j}\mathsf{N}) \cong \mathrm{Hom}_{\mathscr{R}}(\mathcal{O}_{\mathsf{M}},\mathcal{O}_{\mathsf{N}}). \end{array} \tag{8.1.1}$$

This means that Π is a covering functor with Galois group $\langle \mathcal{T}^2 \rangle$ in the sense of [9].

We now turn to the case where A is a finite dimensional algebra over $k = \overline{\mathbb{F}}_q$ together with a Frobenius morphism F on A. We first observe that the faithful functor $\Psi : \mathscr{D}^b(A^F) \to \mathscr{D}^b(A)$ in Lemma 5.2 commutes with the shift functors on $\mathscr{D}^b(A^F)$ and $\mathscr{D}^b(A)$, i. e., we have the following commutative square

$$\begin{array}{ccc} \mathscr{D}^b(A^F) & \xrightarrow{\quad\Psi\quad} & \mathscr{D}^b(A) \\ {\scriptstyle\mathcal{T}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{T}} \\ \mathscr{D}^b(A^F) & \xrightarrow{\quad\Psi\quad} & \mathscr{D}^b(A) \end{array}$$

Then Ψ induces a faithful functor $\Psi_{\mathscr{R}} : \mathscr{R}(A^F) \to \mathscr{R}(A)$ with the following commutative diagram

$$\begin{array}{ccc} \mathscr{D}^b(A^F) & \xrightarrow{\quad\Psi\quad} & \mathscr{D}^b(A) \\ {\scriptstyle\Pi_{A^F}}\big\downarrow & & \big\downarrow{\scriptstyle\Pi_A} \\ \mathscr{R}(A^F) & \xrightarrow{\quad\Psi_{\mathscr{R}}\quad} & \mathscr{R}(A) \end{array}$$

By the construction, the Frobenius functor $(\ )^{[1]}_{\mathscr{D}(A)}$ commutes with the shift functor $\mathcal{T}$, i. e., we have the commutative diagram

$$\begin{array}{ccc} \mathscr{D}^b(A) & \xrightarrow{\quad(\ )^{[1]}_{\mathscr{D}(A)}\quad} & \mathscr{D}^b(A) \\ {\scriptstyle\mathcal{T}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{T}} \\ \mathscr{D}^b(A) & \xrightarrow{\quad(\ )^{[1]}_{\mathscr{D}(A)}\quad} & \mathscr{D}^b(A) \end{array}$$

Hence, $(\ )^{[1]}_{\mathscr{D}(A)}$ induces a functor

$$(\ )^{[1]}_{\mathscr{R}(A)} : \mathscr{R}(A) = \mathscr{D}^b(A)/(\mathcal{T}^2) \longrightarrow \mathscr{D}^b(A)/(\mathcal{T}^2) = \mathscr{R}(A).$$

Note that $(\mathcal{O}_{\mathsf{M}})^{[1]} = \mathcal{O}_{\mathsf{M}^{[1]}}$ for each M in $\mathscr{D}^b(A)$ and $f^{[1]} = (f^{[1]}_{ji})$ for a morphism $f = (f_{ji})$ in $\mathscr{R}(A)$.

As in the previous sections, an object $\mathcal{O}_{\mathsf{M}}$ in $\mathscr{R}(A)$ is said to be F-*stable* if $\mathcal{O}_{\mathsf{M}} \cong (\mathcal{O}_{\mathsf{M}})^{[1]}$, and is said to be F-*periodic* if $\mathcal{O}_{\mathsf{M}} \cong (\mathcal{O}_{\mathsf{M}})^{[r]}$ for some integer $r \geqslant 1$. Call the minimal r with $\mathcal{O}_{\mathsf{M}} \cong (\mathcal{O}_{\mathsf{M}})^{[r]}$ the $\mathscr{R}$-period, denoted by $p_{\mathscr{R}}(\mathcal{O}_{\mathsf{M}})$.

We are now ready to describe the image of the functor $\Psi_{\mathscr{R}}$ for an A with a finite global dimension.

Lemma 8.2. *If A has a finite global dimension, then each F-stable object $\mathcal{O}_{\mathsf{M}}$ in $\mathscr{R}(A)$ is isomorphic to some $\mathcal{O}_{\mathsf{X}_k}$ for some X in $\mathscr{D}^b(A^F)$.*

Proof. Let M be non-zero. Since $\mathcal{O}_{\mathsf{M}} \cong (\mathcal{O}_{\mathsf{M}})^{[1]} \cong \mathcal{O}_{\mathsf{M}^{[1]}}$ in $\mathscr{R}(A)$, there is an $i \in \mathbb{Z}$ such that $\mathsf{M}^{[1]} \cong \mathcal{T}^{2i}\mathsf{M}$ in $\mathscr{D}^b(A)$. The commutativity between the shift and Frobenius functors implies $\mathsf{M}^{[n]} \cong \mathcal{T}^{2ni}\mathsf{M}$ for arbitrary $n \geqslant 1$. Suppose $i \neq 0$. Since A is of finite global dimension, we have $\mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{M}^{[n]}, \mathcal{T}^{2ni}\mathsf{M}) = 0$ for n large enough. This contradicts the fact that $\mathsf{M} \neq 0$. Hence, $i = 0$ and $\mathsf{M}^{[1]} \cong \mathsf{M}$, that is, M is F-stable in $\mathscr{D}^b(A)$. By Lemma 5.1, there is an X in $\mathscr{D}^b(A^F)$ such that $\mathsf{M} \cong \mathsf{X}_k$. □

For each object M in $\mathscr{D}^b(A)$ with a Frobenius map $\mathfrak{F}$, we define the $\mathfrak{F}$-*twist* of $\mathcal{O}_{\mathsf{M}}$ by setting $(\mathcal{O}_{\mathsf{M}})^{[\mathfrak{F}]} = \mathcal{O}_{\mathsf{M}^{[\mathfrak{F}]}}$. Clearly, we always have $(\mathcal{O}_{\mathsf{M}})^{[\mathfrak{F}]} \cong (\mathcal{O}_{\mathsf{M}})^{[1]}$ since $\mathsf{M}^{[\mathfrak{F}]} \cong \mathsf{M}^{[1]}$. Let M and N be objects in $\mathscr{D}^b(A)$ with Frobenius maps $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively. Then the Frobenius map $F^{\mathscr{D}}_{(\mathsf{M},\mathsf{N})} : \mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{M}, \mathsf{N}) \to \mathrm{Hom}_{\mathscr{D}^b(A)}(\mathsf{M}^{[\mathfrak{F}_1]}, \mathsf{N}^{[\mathfrak{F}_2]})$ induces a q-twist map

$$\begin{aligned} F^{\mathscr{R}}_{(\mathsf{M},\mathsf{N})} : \mathrm{Hom}_{\mathscr{R}(A)}(\mathcal{O}_{\mathsf{M}}, \mathcal{O}_{\mathsf{N}}) &\longrightarrow \mathrm{Hom}_{\mathscr{R}(A)}(\mathcal{O}_{\mathsf{M}^{[\mathfrak{F}_1]}}, \mathcal{O}_{\mathsf{N}^{[\mathfrak{F}_2]}}) \\ f = (f_{ji}) &\longmapsto f^{[\mathfrak{F}]} = (F^{\mathscr{D}}_{(\mathcal{T}^i\mathsf{M}, \mathcal{T}^j\mathsf{N})}(f_{ji})). \end{aligned} \tag{8.2.1}$$

Lemma 8.3. *Assume that A has a finite global dimension. Let X and Y be objects in $\mathscr{D}^b(A^F)$ and $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be the natural Frobenius morphisms on $\mathsf{M} = \mathsf{X}_k$ and $\mathsf{N} = \mathsf{Y}_k$ defined by X and Y, respectively. Then $F^{\mathscr{R}}_{(\mathsf{M},\mathsf{N})}$ is a Frobenius map.*

Proof. By Lemma 2.2, it suffices to show that $\mathrm{Hom}_{\mathscr{R}(A)}(\mathcal{O}_{\mathsf{M}}, \mathcal{O}_{\mathsf{N}})$ is finite dimensional. By 8.1.1, we have

$$\mathrm{Hom}_{\mathscr{R}}(\mathcal{O}_{\mathsf{M}}, \mathcal{O}_{\mathsf{N}}) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{D}^b(A)}(\mathcal{T}^{2i}\mathsf{M}, \mathsf{N}).$$

Since the global dimension of A is finite, $\mathrm{Hom}_{\mathscr{D}^b(A)}(\mathcal{T}^{2i}\mathsf{M}, \mathsf{N}) = 0$ for i large or small enough. Hence, $\mathrm{Hom}_{\mathscr{R}(A)}(\mathcal{O}_{\mathsf{M}}, \mathcal{O}_{\mathsf{N}})$ is finite dimensional. □

Thus, for an algebra A with a finite global dimension. We define the category $\mathscr{R}(A)^F$ which consists of F-stable orbits $\mathcal{O}_{\mathsf{M}}$ in $\mathscr{R}(A)$ and Hom-spaces

$$\mathrm{Hom}_{\mathscr{R}(A)^F}(\mathcal{O}_{\mathsf{M}}, \mathcal{O}_{\mathsf{N}}) := \mathrm{Hom}_{\mathscr{R}(A)}(\mathcal{O}_{\mathsf{M}}, \mathcal{O}_{\mathsf{N}})^F,$$

where $F = F^{\mathscr{R}}_{(\mathsf{M},\mathsf{N})}$ is defined with respect to the natural Frobenius maps on M and N via the isomorphisms given in Lemma 8.2.

Remark 8.4. As seen in Remark 5.5, we may alternatively define $\mathscr{R}(A)^F$ to be the category consisting of

$$\begin{cases} \text{Objects:} \quad \mathcal{O}_{\mathsf{M}} \text{ such that } \mathcal{O}^{[1]}_{\mathsf{M}} \overset{\phi^{\mathscr{R}}_{\mathsf{M}}}{\cong} \mathcal{O}_{\mathsf{M}} \text{ in } \mathscr{R}(A), \\ \text{Morphisms: } \mathrm{Hom}_{\mathscr{R}(A)^F}(\mathsf{M}, \mathsf{N}) = \{\xi \in \mathrm{Hom}_{\mathscr{R}(A)}(\mathsf{M}, \mathsf{N}) \mid \phi^{\mathscr{R}}_{\mathsf{N}} \circ \xi^{[1]} = \xi \circ \phi^{\mathscr{R}}_{\mathsf{M}}\}. \end{cases}$$

We leave the proof to the reader.

Theorem 8.5. *Let A be of a finite global dimension. Then $\Psi_{\mathscr{R}}$ induces a category equivalence*

$$\mathscr{R}(A^F) \cong \mathscr{R}(A)^F.$$

Proof. Clearly, $\Psi_{\mathscr{R}}$ induces a faithful functor $\bar{\Psi}_{\mathscr{R}} : \mathscr{R}(A^F) \to \mathscr{R}(A)^F$ which is dense (in the sense of Lemma 8.2). Theorem 5.4 together with (8.1.1) implies that $\bar{\Psi}_{\mathscr{R}}$ is full. □

Finally, the indecomposable objects in $\mathscr{R}(A^F)$ can also be constructed in a way similar to those given in Theorems 4.4 and 5.6.

Theorem 8.6. *Suppose that A has a finite global dimension. Let $\mathsf{M} \in \mathscr{D}^b(A)$ be indecomposable with $\mathscr{D}$-period r and $\tilde{\mathsf{M}} = \oplus_{i=0}^{r-1}\mathsf{M}^{[i]}$. Let further $\mathsf{X} \in \mathscr{D}^b(A^F)$ be such that $\tilde{\mathsf{M}} \cong \mathsf{X}_k$ in $\mathscr{D}^b(A)$. Then $\mathcal{O}_\mathsf{X}$ is indecomposable in $\mathscr{R}(A^F)$ and*

$$\mathrm{End}_{\mathscr{R}(A^F)}(\mathcal{O}_\mathsf{X})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{R}(A^F)}(\mathcal{O}_\mathsf{X})) \cong \mathbb{F}_{q^r}.$$

Moreover, every indecomposable object in $\mathscr{R}(A^F)$ can be obtained in this way.

Proof. By the definition of $\mathscr{R}(A^F)$, we have

$$\mathrm{End}_{\mathscr{R}(A^F)}(\mathcal{O}_\mathsf{X})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{R}(A^F)}(\mathcal{O}_\mathsf{X})) \cong \mathrm{End}_{\mathscr{D}^b(A^F)}(\mathsf{X})/\mathrm{Rad}\,(\mathrm{End}_{\mathscr{D}^b(A^F)}(\mathsf{X})),$$

which is by Theorem 5.6 isomorphic to $\mathbb{F}_{q^r}$. Conversely, if $\mathcal{O}_\mathsf{X}$ is an indecomposable object in $\mathscr{R}(A^F)$, then $\mathcal{O}_{\mathsf{X}_k} = \mathcal{O}_{\mathsf{M}_1} \oplus \cdots \oplus \mathcal{O}_{\mathsf{M}_s}$, where $\mathsf{X}_k = \mathsf{M}_1 \oplus \cdots \oplus \mathsf{M}_s$ with M_i indecomposable in $\mathscr{D}^b(A)$, $1 \leqslant i \leqslant s$. Applying once again Theorem 5.6, we must have that

$$\mathsf{X}_k = \mathsf{M}_1 \oplus \cdots \oplus \mathsf{M}_s \cong \tilde{\mathsf{M}}$$

for some indecomposable object M in $\mathscr{D}^b(A)$. □

Let A be a hereditary (basis) algebra with a Frobenius map F. Then both A and A^F are related by an ad-quiver[4] (Q, σ) (see [4, §6], or more generally, the Appendix below) such that A can be identified as the path algebra of Q and A^F as the path (or tensor) algebra of the folded (or modulated) quiver $(\Gamma, \mathbb{M})$ of Q via σ. The quiver automorphism σ extends linearly to a group automorphism σ on $\mathbb{Z}Q_0$ defined by $\sigma(\sum_{i\in Q_0} a_i i) = \sum_{i\in Q_0} a_i\sigma(i)$. This induces an automorphism

$$\hat{\sigma} : (\mathbb{Z}Q_0)^\sigma \to \mathbb{Z}\Gamma_0.$$

Note that, if $\Delta(Q)$ (resp. $\Delta(\Gamma)$) denotes the root system associated to Q (resp. Γ), then $\hat{\sigma}$ restricts to a bijection $\Delta(Q) \cap (\mathbb{Z}Q_0)^\sigma \to \Delta(\Gamma)$. Now, [11, 4.1] implies immediately the following (cf. 5.7(b)).

Corollary 8.7. *Maintain the notation above and assume that A is* hereditary. *If $\mathcal{O}_\mathsf{M}$ is an indecomposable objects in $\mathscr{R}(A)$, then $\hat{\sigma}(\mathbf{dim}\,\mathcal{O}_{\tilde{\mathsf{M}}}) = \mathbf{dim}\,\mathcal{O}_\mathsf{X}$, where $\mathsf{X} \in \mathscr{D}^b(A^F)$ with $\mathsf{M} \cong \mathsf{X}_k$ in $\mathscr{D}^b(A)$, and*

$$\Delta(\Gamma) = \{\hat{\sigma}(\mathbf{dim}\,\mathcal{O}_{\tilde{\mathsf{M}}}) \mid \mathsf{M} \text{ indecomposable in } \mathscr{D}^b(A)\}.$$

Hence, in this case, the relation over the indecomposable objects given in Theorem 8.6 results in the same relation on the associated root systems as the folding relation induced from σ.

9. Appendix: Quivers with automorphisms and relations

In this Appendix, we shall show that every finite dimensional basic $\mathbb{F}_q$-algebra is isomorphic to the fixed point algebra $(kQ/I)^F$, where Q is a quiver with an automorphism σ, $F = F_{Q,\sigma;q}$ is the Frobenius morphism on the path algebra kQ induced by σ, and I is an F-stable admissible ideal I (the relation ideal) of kQ. Thus, we generalize the relation between hereditary algebras and quivers with automorphisms (see [4, Th.6.5]) to the general case.

The most important application of Frobenius morphisms is to the theory of quivers with automorphisms. In fact, we shall see below that every quiver automorphism induces a Frobenius morphism on the path algebra of the quiver, and thus, gives rise to an $\mathbb{F}_q$-modulated quiver (i.e., an $\mathbb{F}_q$-species in [8, 5]) whose tensor (or path) algebra is the associated fixed point algebra.

Let $Q = (Q_0, Q_1)$ be a finite quiver, where Q_0 (resp. Q_1) denotes the set of vertices (resp. arrows) of Q. For each arrow ρ in Q_1, we denote by $h\rho$ and $t\rho$ the head and the tail of ρ, respectively. Let σ be an automorphism of Q, that is, σ is a permutation on the vertices of Q and on the arrows of Q satisfying the compatibility conditions: $\sigma(h\rho) = h\sigma(\rho)$ and $\sigma(t\rho) = t\sigma(\rho)$ for any $\rho \in Q_1$.

[4] This is a quiver Q without oriented cycles and with an automorphism σ such that each σ-orbit of vertices is an independent (or isolated) set.

Let $A := kQ$ be the path algebra of Q over $k = \overline{\mathbb{F}}_q$ which has the identity $1 = \sum_{i\in Q_0} e_i$ where e_i is the idempotent (as a length zero path) corresponding to the vertex i. Then σ induces a Frobenius morphism

$$F_{Q,\sigma} = F_{Q,\sigma;q} : A \to A;\ \sum_s x_s p_s \longmapsto \sum_s x_s^q \sigma(p_s), \tag{9.0.1}$$

where $\sum_s x_s p_s$ is a k-linear combination of paths p_s, and $\sigma(p_s) = \sigma(\rho_t)\cdots\sigma(\rho_1)$ if $p_s = \rho_t\cdots\rho_1$ for arrows $\rho_1, \ldots, \rho_t$ in Q_1.

We now construct an $\mathbb{F}_q$-modulated quiver from a quiver Q with an automorphism σ. Let Γ_0 and Γ_1 denote the set of σ-orbits in Q_0 and Q_1, respectively. Thus, we obtain a new quiver $\Gamma = (\Gamma_0, \Gamma_1)$. For each arrow $\boldsymbol{\rho} : \mathbf{i} \longrightarrow \mathbf{j}$ in Γ, define

$$\varepsilon_{\boldsymbol{\rho}} = \#\{\text{arrows in } \boldsymbol{\rho}\},\ d_{\boldsymbol{\rho}} = \varepsilon_{\boldsymbol{\rho}}/\varepsilon_{\mathbf{j}},\ \text{ and }\ d'_{\boldsymbol{\rho}} = \varepsilon_{\boldsymbol{\rho}}/\varepsilon_{\mathbf{i}}, \tag{9.0.2}$$

where $\varepsilon_{\mathbf{k}} = \#\{\text{vertices in } \sigma\text{-orbit } \mathbf{k}\}$ for $\mathbf{k} \in \Gamma_0$. The quiver Γ together with the valuation $\{\varepsilon_{\mathbf{d}}\}_{\mathbf{d}\in\Gamma_0}$, $\{(d_{\boldsymbol{\rho}}, d'_{\boldsymbol{\rho}})\}_{\boldsymbol{\rho}\in\Gamma_1}$ defines a valued quiver $\Gamma = \Gamma(Q, \sigma)$. Clearly, each valued quiver can be obtained in this way from a quiver with an automorphism.

Using the Frobenius morphism $F = F_{Q,\sigma}$ on A defined above, we can attach naturally to Γ an $\mathbb{F}_q$-modulation to obtain an $\mathbb{F}_q$-modulated quiver as follows: for each vertex $\mathbf{i} \in \Gamma_0$ and each arrow $\boldsymbol{\rho} \in \Gamma_1$, we fix $i_0 \in \mathbf{i}$, $\rho_0 \in \boldsymbol{\rho}$, and consider the F-stable subspaces of A

$$A_{\mathbf{i}} = \bigoplus_{i\in\mathbf{i}} ke_i = \bigoplus_{s=0}^{\varepsilon_{\mathbf{i}}-1} ke_{\sigma^s(i_0)}\ \text{ and }\ A_{\boldsymbol{\rho}} = \bigoplus_{\rho\in\boldsymbol{\rho}} k\rho = \bigoplus_{t=0}^{\varepsilon_{\boldsymbol{\rho}}-1} k\sigma^t(\rho_0),$$

where e_i denotes the idempotent corresponding to the vertex i. Then

(9.0.3)

$$A_{\mathbf{i}}^F = \{\sum_{s=0}^{\varepsilon_{\mathbf{i}}-1} x^{q^s} e_{\sigma^s(i_0)} \mid x \in k, x^{q^{\varepsilon_{\mathbf{i}}}} = x\} \text{ and } A_{\boldsymbol{\rho}}^F = \{\sum_{t=0}^{\varepsilon_{\boldsymbol{\rho}}-1} x^{q^t}\sigma^t(\rho_0) \mid x \in k, x^{q^{\varepsilon_{\boldsymbol{\rho}}}} = x\}.$$

Further, the algebra structure of A induces an $A_{\mathbf{j}}^F$-$A_{\mathbf{i}}^F$-bimodule structure on $A_{\boldsymbol{\rho}}^F$ for each arrow $\boldsymbol{\rho} : \mathbf{i} \longrightarrow \mathbf{j}$ in Γ. Thus, we obtain an $\mathbb{F}_q$-modulation $\mathbb{M} = \mathbb{M}(Q, \sigma) := (\{A_{\mathbf{i}}^F\}_{\mathbf{i}}, \{A_{\boldsymbol{\rho}}^F\}_{\boldsymbol{\rho}})$ over the valued quiver Γ. The $\mathbb{F}_q$-modulated quiver $(\Gamma, \mathbb{M})$ defined above will be denoted by $\mathfrak{M}_{Q,\sigma} = \mathfrak{M}_{Q,\sigma;q} = (\Gamma, \mathbb{M})$.

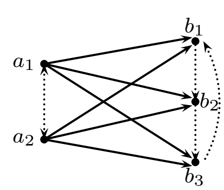
Example 9.1. For $r, s \geqslant 1$, we define $Q = Q^{(r,s)}$ to be the *complete bipartite quiver* with

$$Q_0 = \{a_1, \ldots, a_r; b_1, \ldots, b_s\}, \quad Q_1 = \{\tau_{ji} : a_i \longrightarrow b_j \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}.$$

Clearly, Q admits an automorphism σ such that

$$\sigma(a_i) = a_{i+1}\ \text{ and }\ \sigma(b_j) = b_{j+1},$$

where $a_{r+1} = a_1$ and $b_{s+1} = b_1$. Thus, $\sigma(\tau_{ji}) = \tau_{j+1,i+1}$. For example, if $r = 2$ and $s = 3$, the quiver $Q = Q^{(2,3)}$ and the automorphism σ are illustrated by



Let d and m denote, respectively, the greatest common divisor and the least common multiple of r and s. We denote by $\rho_1, \rho_2, \dots, \rho_d$ the σ-orbits of $\tau_{11}, \tau_{21}, \dots, \tau_{d1}$ in Q_1, respectively. Then, it is easy to see that $\rho_1, \rho_2, \dots, \rho_d$ are distinct and form all the σ-orbits in Q_1. Let further 1 and 2 denote the σ-orbits $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_s\}$, respectively. Then the valued quiver $\Gamma = \Gamma(Q, \sigma)$ associated to (Q, σ) has two vertices $1, 2$ and d arrows $\rho_1, \dots, \rho_d$ from 1 to 2. The valuation is given by $\varepsilon_1 = r, \varepsilon_2 = s$ and $(d_{\rho_i}, d'_{\rho_i}) = (\frac{m}{s}, \frac{m}{r})$ for all $1 \leqslant i \leqslant d$. By definition, we have

$$A_1 = \bigoplus_{i=1}^{r} ke_i \ \text{ and } \ A_2 = \bigoplus_{j=1}^{s} kf_j,$$

where e_i, f_j denote the idempotents corresponding to vertices a_i, b_j, respectively. This implies that

$$A_1^F = \{\sum_{i=1}^{r} x^{q^{i-1}} e_i \mid x \in \mathbb{F}_{q^r}\} \ \text{ and } \ A_2^F = \{\sum_{j=1}^{s} y^{q^{j-1}} f_j \mid y \in \mathbb{F}_{q^s}\}.$$

Then we have isomorphisms

$$A_1^F \xrightarrow{\sim} \mathbb{F}_{q^r}, \text{ via } \textstyle\sum_{i=1}^{r} x^{q^{i-1}} e_i \mapsto x,$$
$$A_2^F \xrightarrow{\sim} \mathbb{F}_{q^s}, \text{ via } \textstyle\sum_{j=1}^{s} y^{q^{j-1}} f_j \mapsto y.$$

Further, for each $1 \leqslant i \leqslant d$, we have

$$A_{\rho_i}^F = \{\sum_{t=0}^{m-1} z^{q^t} \sigma^t(\tau_{i1}) \mid z \in \mathbb{F}_{q^m}\}.$$

By identifying $A_{\rho_i}^F$ with $\mathbb{F}_{q^m}$ via $\sum_{t=0}^{m-1} z^{q^t} \sigma^t(\tau_{i1}) \mapsto z$, the A_2^F-A_1^F-bimodule $A_{\rho_i}^F$ is identified with the $\mathbb{F}_{q^s}$-$\mathbb{F}_{q^r}$-bimodule $\Theta_i^{r,s}$, where $\Theta_i^{r,s} = \mathbb{F}_{q^m}$ as an $\mathbb{F}_q$-vector space, and its $\mathbb{F}_{q^s}$-$\mathbb{F}_{q^r}$-bimodule structure is defined by

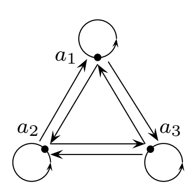
$$y \cdot z \cdot x = y^{q^{i-1}} zx \ \text{ for all } x \in \mathbb{F}_{q^r}, y \in \mathbb{F}_{q^s}, z \in \mathbb{F}_{q^m}.$$

Finally, the $\mathbb{F}_q$-modulation $\mathbb{M}(Q, \sigma; q)$ can be identified with $(\{\mathbb{F}_{q^r}, \mathbb{F}_{q^s}\}, \{\Theta_i^{r,s}\}_{i=1}^d)$.

If $r = s$, by identifying vertices a_i as b_i in $Q^{(r,s)}$ for $1 \leqslant i \leqslant r = s$, we obtain the *complete quiver* $Q^{(r)}$ on r vertices, i. e.,

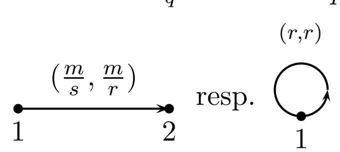
$$Q_0^{(r)} = \{a_1, a_2, \dots, a_r\} \quad \text{ and } \quad Q_1^{(r)} = \{\tau_{ji} : a_i \longrightarrow a_j \mid 1 \leqslant i, j \leqslant r\}.$$

For example, when $r = 3$, $Q^{(3)}$ has the form



In this case, the associated valued quiver $\Gamma(Q^{(r)}, \sigma)$ contains one vertex 1 and r loops $\rho_1, \dots, \rho_r$; and the $\mathbb{F}_q$-modulation $\mathbb{M}(Q^{(r)}, \sigma; q)$ can be identified with $(\{\mathbb{F}_{q^r}\}, \{\Theta_i^{r,r}\}_{i=1}^r)$.

Lemma 9.2. *For $r, s \geqslant 1$, let $\mathfrak{M}^\natural$ denote the $\mathbb{F}_q$-modulated quiver with underlying valued quiver*



and $\mathbb{F}_q$-modulation $(\{\mathbb{F}_{q^r},\mathbb{F}_{q^s}\},\{\mathbb{F}_{q^s}\otimes_{\mathbb{F}_q}\mathbb{F}_{q^r}\})$ (resp. $(\{\mathbb{F}_{q^r}\},\{\mathbb{F}_{q^r}\otimes_{\mathbb{F}_q}\mathbb{F}_{q^r}\})$), where m is the least common multiple of r and s, and $\mathbb{F}_{q^s}\otimes_{\mathbb{F}_q}\mathbb{F}_{q^r}$ (resp. $\mathbb{F}_{q^r}\otimes_{\mathbb{F}_q}\mathbb{F}_{q^r}$) is the natural $\mathbb{F}_{q^s}$-$\mathbb{F}_{q^r}$ (resp. $\mathbb{F}_{q^r}$-$\mathbb{F}_{q^r}$)-bimodule. Then, $\mathfrak{M}^\natural$ is isomorphic to the modulated quiver $\mathfrak{M}_{Q^{(r,s)},\sigma}$ (resp. $\mathfrak{M}_{Q^{(r)},\sigma}$) defined in Example 9.1.

Proof. Let d denote the greatest common divisor of r and s. By viewing $\mathbb{F}_{q^r}$ and $\mathbb{F}_{q^s}$ as $\mathbb{F}_q$-algebras and as subfields of $\mathbb{F}_{q^m}$, we have an $\mathbb{F}_q$-algebra isomorphism

$$\mathbb{F}_{q^s}\otimes_{\mathbb{F}_q}\mathbb{F}_{q^r}\xrightarrow{\sim}\underbrace{\mathbb{F}_{q^m}\oplus\cdots\oplus\mathbb{F}_{q^m}}_{d};\quad x\otimes y\mapsto(yx,y^qx,\ldots,y^{q^{d-1}}x).$$

Clearly, as an $\mathbb{F}_{q^s}$-$\mathbb{F}_{q^r}$-bimodule, this induces a bimodule isomorphism

$$\mathbb{F}_{q^s}\otimes_{\mathbb{F}_q}\mathbb{F}_{q^r}\cong\Theta_1^{r,s}\oplus\Theta_2^{r,s}\oplus\cdots\oplus\Theta_d^{r,s}.$$

Therefore, the two $\mathbb{F}_q$-modulated quivers are isomorphic. □

Theorem 9.3. *Let (Q,σ) be a finite quiver with an automorphism σ, and let $\mathfrak{M}_{Q,\sigma}$ be the $\mathbb{F}_q$-modulated quiver associated to (Q,σ).*

1. *If $A=kQ$ is the path algebra of Q, and $F=F_{Q,\sigma}$ the Frobenius morphism on A induced by σ, then the fixed point algebra A^F is isomorphic to the tensor algebra $T(\mathfrak{M}_{Q,\sigma})$ of $\mathfrak{M}_{Q,\sigma}$.*
2. *For any given $\mathbb{F}_q$-modulated quiver $\mathfrak{M}$, there is a quiver Q and an admissible automorphism σ such that the tensor algebra $T(\mathfrak{M})$ of $\mathfrak{M}$ is isomorphic to $(kQ)^{F_{Q,\sigma}}$.*

Proof. The proof of (1) is the same as given in [4, Proposition 6.3]. Here we provide a proof for (2) by using Lemma 9.2 (cf. the proof of [4, Theorem 6.5]).

Let $\mathfrak{M}=(\Gamma,\mathbb{M})$ be an $\mathbb{F}_q$-modulated quiver given by a valued quiver $\Gamma=(\Gamma_0,\Gamma_1)$ with valuation $(\{\varepsilon_x\}_{x\in\Gamma_0},\{(d_\rho,d'_\rho)\}_{\rho\in\Gamma_1})$ and an $\mathbb{F}_q$-modulation $\mathbb{M}=(\{D_x\}_{x\in\Gamma_0},\{M_\rho\}_{\rho\in\Gamma_1})$ of Γ. In view of [4, §6], we may suppose that Γ is a simple valued quiver (i. e., no parallel edges, but allowing simple loops). We shall look at the case where Γ is connected and consists of one arrow only. The general case can be constructed by induction on the number of arrows.

Let $\rho:x\longrightarrow y$ be the unique arrow of Γ. (If $x=y$, then ρ is a loop.) For simplicity, set $r=\varepsilon_x, s=\varepsilon_y$ and let d and m denote respectively the greatest common divisor and the least common multiple of r and s. Further, we may identify D_x and D_y with $\mathbb{F}_{q^r}$ and $\mathbb{F}_{q^s}$, respectively, then M_ρ is an $\mathbb{F}_{q^s}$-$\mathbb{F}_{q^r}$-bimodule which can be decomposed into a sum of simple bimodules

$$M_\rho=M_1\oplus M_2\oplus\cdots\oplus M_t.$$

Since each M_i is a simple $\mathbb{F}_{q^s}$-$\mathbb{F}_{q^r}$-bimodule, there exists $1\leqslant n_i\leqslant d$ such that $M_i\cong\Theta_{n_i}^{r,s}$. Let $Q=(Q_0,Q_1)$ denote the quiver with

$$Q_0=\{a_1,\ldots,a_r,b_1,\ldots,b_s\}\ \text{ and }\ Q_1=\{\tau_j^{(i)}:a_j\longrightarrow b_{j+n_i-1}\mid 1\leqslant j\leqslant m,1\leqslant i\leqslant t\},$$

where the subscripts of a_j and b_{j+n_i-1} are considered as positive integers modulo r and s, respectively. (Note that if $x=y$, then $a_i=b_i$ for $1\leqslant i\leqslant r=s$.) We define an automorphism σ of Q via

$$a_1\mapsto a_2\mapsto\cdots\mapsto a_r\mapsto a_1,\ \ b_1\mapsto b_2\mapsto\cdots\mapsto b_s\mapsto b_1,$$
$$\tau_1^{(i)}\mapsto\tau_2^{(i)}\mapsto\cdots\mapsto\tau_m^{(i)}\mapsto\tau_1^{(i)}\ \text{ for } 1\leqslant i\leqslant t.$$

From the construction given before Example 9.1, we see that the valued quiver $\Gamma(Q,\sigma)$ consists of two vertices 1 and 2 (i. e., the σ-orbits $\{a_1,a_2,\ldots,a_r\}$ and $\{b_1,b_2,\ldots,b_s\}$, respectively) and of t arrows ρ_i (= the σ-orbit $\{\tau_1^{(i)},\tau_2^{(i)},\ldots,\tau_m^{(i)}\}$) for $1\leqslant i\leqslant t$. Let $A=kQ$ and $F=F_{Q,\sigma;q}$. Then $A_1^F\cong\mathbb{F}_{q^r}$ and $A_2^F\cong\mathbb{F}_{q^s}$. Under these isomorphisms, we see from Example 9.1 that the

A_2^F-A_1^F-bimodule $A_{\rho_i}^F$ is isomorphic to $\Theta_{n_i}^{r,s} \cong M_i$ for $1 \leqslant i \leqslant t$. Thus, the $\mathbb{F}_q$-modulated quiver $\mathfrak{M}_{Q,\sigma}$ associated with the pair (Q,σ) is isomorphic to $\mathfrak{M}$. By (1), we obtain that

$$(kQ)^{F_{Q,\sigma}} \cong T(\mathfrak{M}_{Q,\sigma}) \cong T(\mathfrak{M}).$$

□

Remark 9.4. This theorem is a generalized version of [4, 6.3], where σ is assumed to be admissible, and of [4, 6.5], where only finite dimensional hereditary basic algebras are considered.

The next result shows that, up to Morita equivalence, the representation theory of quivers with automorphisms covers that of all finite dimensional $\mathbb{F}_q$-algebras.

Theorem 9.5. (1) *Let B be a finite dimensional basic $\mathbb{F}_q$-algebra. Then there exists a finite quiver Q together with an automorphism σ and an $F_{Q,\sigma}$-stable admissible ideal I of kQ such that*

$$B \cong (kQ/I)^{F_{Q,\sigma}}.$$

(2) *If A is a finite dimensional k-algebra with Frobenius morphism F. Then there exists a basic algebra A' with a Frobenius morphism F' such that both pairs (A, A') and $(A^F, A'^{F'})$ are Morita equivalent.*

Proof. (1) Since B is hereditary and basic, there is an $\mathbb{F}_q$-modulated quiver $\mathfrak{M} = (\Gamma, \mathbb{M})$, the Ext-quiver of B, such that $B \cong T(\mathfrak{M})/\mathcal{I}$, where $T(\mathfrak{M})$ is the tensor algebra of $\mathfrak{M}$ and $\mathcal{I}$ is the relation ideal of B (see [6, Theorem 8.5.2] or [2, p.104]). By Theorem 9.3(2), there exist a quiver Q and an automorphism σ on Q such that $T(\mathfrak{M}) = (kQ)^{F_{Q,\sigma}}$. However, $F_{Q,\sigma}$ is given by $F_{Q,\sigma}(x \otimes \lambda) = x \otimes \lambda^q$ for $x \otimes \lambda \in kQ = (kQ)^{F_{Q,\sigma}} \otimes_{\mathbb{F}_q} k$. This means that the ideal $I := \mathcal{I} \otimes k$ is $F_{Q,\sigma}$-stable, and $\mathcal{I} = I^{F_{Q,\sigma}}$. Therefore,

$$B \cong T(\mathfrak{M})/\mathcal{I} \cong kQ^{F_{Q,\sigma}}/I^{F_{Q,\sigma}} \cong (kQ/I)^{F_{Q,\sigma}}.$$

(2) Since $B = A^F$ is a finite dimensional $\mathbb{F}_q$-algebra, there is a finite dimensional basic $\mathbb{F}_q$-algebra C which is Morita equivalent to A^F. Then $A' = C \otimes_{\mathbb{F}_q} k$ is clearly a finite dimensional basic k-algebra, and $C = A'^{F'}$ where F' is the Frobenius morphism given by $F'(x \otimes \lambda) = x \otimes \lambda^q$ for all $x \in C$ and $\lambda \in k$. It remains to prove that A and A' are Morita equivalent.

By definition, there are bimodules ${}_BX_C$ and ${}_CY_B$ and surjective maps $\phi : X \otimes_C Y \longrightarrow B$ of B-B-bimodules and $\psi : Y \otimes_B X \longrightarrow C$ of C-C-bimodules satisfying

$$x\psi(y \otimes z) = \phi(x \otimes y)z \text{ and } y\phi(z \otimes w) = \psi(y \otimes z)w \text{ for all } x, z \in X \text{ and } y, w \in Y.$$

Set $X_k = X \otimes_{\mathbb{F}_q} k$ and $Y_k = Y \otimes_{\mathbb{F}_q} k$. Then X_k becomes an A-A-bimodule and Y_k becomes an A'-A'-bimodule. Moreover, the maps

$$\tilde{\phi} : X_k \otimes_{A'} Y_k \longrightarrow A,\ (x \otimes \lambda) \otimes (y \otimes \mu) \longmapsto \phi(x, y) \otimes \lambda\mu,\ \ x \in X, y \in Y, \lambda, \mu \in k$$

and

$$\tilde{\psi} : Y_k \otimes_A X_k \longrightarrow A',\ (z \otimes \lambda) \otimes (w \otimes \mu) \longmapsto \psi(z, w) \otimes \lambda\mu,\ \ z \in Y, w \in X, \lambda, \mu \in k$$

satisfy the similar conditions as ϕ and ψ. Thus A and A' are Morita equivalent. □

Remarks 9.6. (a) Let F and F' be Frobenius morphisms on A and A', respectively. Then, in general, that A and A' are Morita equivalent does not necessarily imply that A^F and $A'^{F'}$ are Morita equivalent. For example, let Q be the quiver with three vertices $1, 2, 3$ and two arrows $\alpha : 1 \longrightarrow 2$ and $\beta : 1 \longrightarrow 3$. Let σ_1 be the identity automorphism of Q and σ_2 the automorphism exchanging arrows α and β. Then $(kQ)^{F_{Q,\sigma_1}}$ is the path algebra $\mathbb{F}_qQ$ and

$$(kQ)^{F_{Q,\sigma_2}} \cong \begin{bmatrix} \mathbb{F}_q & 0 \\ \mathbb{F}_{q^2} & \mathbb{F}_{q^2} \end{bmatrix}.$$

It is obvious that $(kQ)^{F_{Q,\sigma_1}}$ and $(kQ)^{F_{Q,\sigma_2}}$ are not Morita equivalent.

(b) From Theorem 9.5(2), we see that the study of finite dimensional k-algebras with Frobenius morphism can be reduced to that of basic k-algebras. Further, each Frobenius morphism on a finite dimensional basic k-algebra is induced from an automorphism of its Ext-quiver.

F-inherited properties and theories. A Frobenius morphism F links an algebra A over $k = \overline{\mathbb{F}}_q$ with an algebra A^F over $\mathbb{F}_q$. It would be interesting to know which theories on A or properties of A are inherited by A^F. More precisely, we say that a theory on A is *F-inherited* if the same theory holds on A^F; while a property of A is called *F-invariant* if A^F has the property. For example, by Theorem 2.10, the number of irreducible modules is *not* F-invariant. However, we do have the following F-invariants.

Proposition 9.7. *Let A be a finite dimensional algebra with a Frobenius morphism F. The following properties of A are F-invariants.*

(P1) *An algebra is of finite representation-type;*
(P2) *An algebra is hereditary;*
(P3) *An algebra has a finite global dimension;*
(P4) *An algebra is a basic algebra;*
(P5) *An algebra is self-injective;*
(P6) *An algebra is preprojective.*

Proof. The first two statements follow from Theorems 9.3 and 2.10. (P3) follows from the fact that, for two F-stable A-modules (M, F_M) and (N, F_N), there is a natural isomorphism

$$\text{Ext}\,^n_A(M, N) \cong \text{Ext}\,^n_{A^F}(M^F, N^F) \otimes_{\mathbb{F}_q} k; \quad \text{for each } n \geqslant 0, \tag{9.7.1}$$

see [3, 8.16]. The rest can be checked directly by definition. □

In general, if a structural property on A is not compatible with the Frobenius morphism F on A, then this property is not F-invariant. For example, the following properties seem not F-invariant.

(a) An algebra is quasi-hereditary;
(b) An algebra is cellular;
(c) An algebra is symmetric.

For F-inherited theories, we have the following list:

(T1) The Morita equivalence theory is F-inherited (Theorem 9.5(2)).
(T2) The Auslander-Reiten theory is F-inherited ([4, 8.3]).
(T3) The Kac theory is F-inherited ([4, 10.3]).
(T4) The derived category theory is F-inherited (Theorem 5.4).

If one regards Kac's conjecture [15] as part of the Kac theory, it would be natural to expect that this conjecture is F-inherited, that is, the same conjecture is true for all symmetrizable (generalized) Cartan matrices.

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