



Assessing mathematics using computer algebra

Chris Sangwin

Copyright © 2003 chris@sangwin.com

Last Revision Date: July 3, 2003

Table of Contents

1. Introduction
2. Assessing mathematics
3. Creating examples/instances
4. Mathematics and the Internet
5. Use of objective tests
6. Computer algebra marking
7. Alice interactive mathematics
8. Randomly generated questions
9. Creating examples/instances
10. Automatic feedback

11. Common misconceptions

12. Conclusion

1. Introduction

This talk is about mathematical computer aided assessment.

1. Assessing mathematics.
2. Computer algebra marking.
3. Students' examples of objects.
4. Automatic feedback to address misconceptions.

2. Assessing mathematics

In mathematics the majority of tasks may be classified:

1. Factual recall
2. Carry out a routine calculation or algorithm
3. Classify some mathematical object
4. Interpret situation or answer
5. Prove, show, justify – (general argument)
6. Extend a concept
7. Create an example/instance
8. Criticize a fallacy

Why does this matter?

Assessment strategies dictate learning styles.

hence

We may influence learning styles through choice of assessment strategies.

In contraposition

...

if we fail to ask students to practice some technique we can hardly complain afterwards that they have failed!

3. Creating examples/instances

Some questions ask for examples of objects.

They come in a number of forms:

- Find an x satisfying A and B but not C .
- Counter examples to...
- “Facilitator objects”.
Change of basis, substitution, Lyapunov functions.

They require higher level thinking.

Such questions are rare. (11.5 questions from 486 \approx 2.4%)

Pointon and Sangwin, 2003

Perhaps because they are time consuming to mark.

4. Mathematics and the Internet

Current difficulties using the web to assess maths:

- display of mathematics,
- commercial packages limited to ‘objective tests’
MC, numeric answer, “hot spot”, multiple response
etc.

We want to allow mathematical expressions as input:

- need a common syntax — easy to learn,
- answer provided needs to be checked *mathematically*.

Furthermore, we'd like to give
partial credit
feedback *etc.*

5. Use of objective tests

Consider the following question:

Example question 1

Determine the following integral:

$$\int \cos(x) \sin(2x) dx.$$

As a multiple choice question:

- $(2/3) \cos^3(x) + C$
- $-(2/3) \cos(x) + (2/3) \sin^3(x) + C$
- $-(2/3) \cos(x) + (1/3) \sin(x) \sin(2x) + C$
- *Don't know.*

How do we know the students don't differentiate the candidate solutions to check?

6. Computer algebra marking

Computer algebra systems can be used to *mark work*.

This checks for *algebraic equivalence*.

$$(x + 1)^2 \equiv x^2 + 2x + 1$$

Useful for marking many routine problems.

Checking for properties

CAS can also check for *properties*.

Example question 2

Give an example of a prime number.

The CAS checks whether a student's answer *is* prime.

To mark

Example question 3

Give an odd function.

1. calculate $f(x) + f(-x)$,
2. simplify,
3. check equality to zero.

*Being able to mark a question is not sufficient.
An assessment system is needed.*

7. Alice interactive mathematics

AIM is a computer algebra based assessment system.

- Working system used at many universities.
- Illustrates general computer algebra marking.

Original AIM was designed by Theodore Kolokolnikov.
New version by Neil Strickland.

Used in Birmingham since 2000.
Implementation with LDU grant.

System overview

The AIM system:

- internet based CAA system,
<http://www.mat.bham.ac.uk/aim/>
- uses
Maple (computer algebra), and
L^AT_EX (type setting)
- system itself is *free*, and open source,
- questions are (relatively) easy to author,
- becoming more widely used.

Technical details

- Only *one* copy of Maple needed on the server.
Site licence required by institution.
- Students do not need Maple.
- Maple checks for equivalence of answer and solution.
- Great sophistication possible.
- \vdots
- AIM is not suited to school level algebra.
- Maple syntax needs to be used,
eg “ $3*x$ ” not “ $3x$ ”.

AIM also allows...

- Multiple choice, multiple response, numerical and matrix entry.
- ... any Maple expression.
- Web allows links to resources.
- Use of simple HTML allows magnification.
- MathML upgrade would aid accessibility.

Demonstrating the AIM system

<http://www.mat.bham.ac.uk/aim/>

8. Randomly generated questions

Questions may be randomly generated and reverse engineered.

- Self testing and practice.
- Reduce plagiarism.
- Encourages group work.

Care needed when setting.

Answers generated from template.

Question space.

9. Creating examples/instances

AIM may mark some questions of this style.

Exemplar questions

Students' answers

Students show great variety in their answer, and method.

For example, 190 students were asked for two functions that satisfy $f'(1) = 0$.

Their answers were marked by AIM.

The students gave 93 ‘different’ answers.

1st Answer	Frequency	2nd Answer	Frequency
$x^2 - 2x$	46	$x^3 - 3x$	29
$1/2x^2 - x$	33	$x^2 - 2x$	11
$1/3x^3 - x$	11	$1/3x^3 - x$	9
$x^2 - 2x + 1$	7	$(x - 1)^2$	8
$x^2 - 2x + 3$	7	$1/4x^4 - x$	8
$(x - 1)^2$	5	$x^4 - 4x$	5
$2x^2 - 4x$	5	$1/2x^2 - x$	4
$x^3/3 - x^2/2$	5	$3/2x^2 - 3x$	4
0	4		

Data set.

Two strategies emerged:

JL: Ok, just take the parabola and shift it one.

...

B: I said, $x - 1 = 0$, then integrated it.

These problems can be used to generate discussion and debate

- sorting the data,
- methods used,
- ‘exotic’ examples.

$$f_1(x) = 0, \quad f_2(x) = |x|(x - 2), \quad f_3(x) = e^{\frac{-1}{(x-1)^2}}.$$

10. Automatic feedback

Sophisticated automatic feedback may be provided by computer algebra systems.

This

- is immediate,
- is based on properties of students' answers,
- could be positive and encouraging,
- may be based on common mistakes,
- may be based on common misconceptions.

11. Common misconceptions

Computer algebra can also test for a *type* of incorrect answer.

Misconceptions may be identified by

- educational research,
- previous teaching experience,
(setting MCQs)
- examining answers from previous students

Odd functions

On examining the odd functions given by students, the majority of coefficients ($\neq 1$) are *odd*,
eg

$$3x^5, \quad 5x^7, \quad 7x^5 - 3x.$$

Students' concept image of an odd function requires odd *coefficients*.

Furthermore, $f(x) = 0$ is odd, but was absent.

Functions that are odd *and* even.

When asked for a function that was both odd *and* even

35% gave the correct answer (eventually),

35% failed to answer the question.

Incorrect answers revealed that 24% of the students added an odd and even function.

Examples include

$$x + x^2, \quad x^2 + x^3, \quad x^5 - x^6.$$

The computer algebra system can test for these misconceptions.

12. Conclusion

The important features of AIM are

- Answers evaluated *algebraically*.
- Questions which encourage deeper learning possible.
- Sophisticated tailored feedback.

AIM is the *only* system which currently uses CAS in this way.

In the near future *all* computer aided assessment systems will link computer algebra and assessment to perform automatic marking.