Calculating symmetries in Newman-Tamburino metrics

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Abstract: In this paper I show that the Newman-Tamburino spherical metrics always admit a Killing vector, correcting a claim by Collinson and French, (1967 J. Math. Phys. 8 701) and also admit a homothety. A similar calculation is given for the limit of the Newman-Tamburino cylindrical metric.

1 Introduction

The Newman-Tamburino metrics are those vacuum solutions of the Einstein equations admitting hypersurface orthogonal geodesic rays with non-vanishing shear and divergence. In the Newman-Penrose formalism this implies that $\Psi_0 = \kappa = 0$, that $\rho$ is real and non-zero and $\sigma \neq 0$. In [1] Newman and Tamburino explicitly gave all such metrics and showed that they fall into two classes: the spherical, with $\rho^2 \neq \sigma \sigma$ and the cylindrical with $\rho^2 = \sigma \sigma$. In [2] Collinson and French claimed to have shown that the former metrics admit at most one Killing vector, and that happens only in a particular subcase. In fact, the spherical Newman-Tamburino metrics always admit a Killing vector and also always admit a homothety. This preprint is intended to show the full calculations and results when the homothetic equations of [3] are integrated for the Newman-Tamburino spherical metrics. The bulk of sections 3 and 4 come from Maple 9 worksheets, exported to \TeX and suitably tidied up for better readability.

Throughout I use the spin coefficient notation of [4]. For example I use $\kappa^\prime$, $\rho^\prime$, $\sigma^\prime$ and $\tau^\prime$ in place of the more traditional $-\nu$, $-\mu$, $-\pi$ and $-\lambda$.

2 Results

The contravariant form of the Newman-Tamburino spherical metric [1] (see also [5], equation (26.21)) is

\[
g^{22} = -\frac{2r^2(\zeta \bar{\zeta})^{1/2}}{R^2} + \frac{2rL}{A} + \frac{2r^3 A(\zeta^2 + \bar{\zeta}^2)}{R^4} - \frac{4r^2 A^2(\zeta \bar{\zeta})^{3/2}}{R^4}
\]

\[
g^{23} = 4A^2(\zeta \bar{\zeta})^{3/2}x \left[ \frac{L}{2a^3} - \frac{r - 2a}{2a^2 R^2} - \frac{r - a}{R^4} \right]
\]

\[
g^{24} = 4A^2(\zeta \bar{\zeta})^{3/2}y \left[ \frac{L}{2a^3} - \frac{r + 2a}{2a^2 R^2} - \frac{r + a}{R^4} \right]
\]

\[
g^{33} = -\frac{2(\zeta \bar{\zeta})^{3/2}}{(r + a)^2}
\]

\[
g^{44} = -\frac{2(\zeta \bar{\zeta})^{3/2}}{(r - a)^2}
\]

$g^{12} = 1$. 

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Here our coordinates are \( x^1 = u, x^2 = r, x^3 + ix^4 = x + iy = \zeta \) and

\[ A(u) = bu + c, \quad L = \frac{1}{2} \log \left( \frac{r + a}{r - a} \right), \quad a = A(\zeta \overline{\zeta})^{1/2}, \quad R^2 = r^2 - a^2. \]

Here \( b \) and \( c \) are real constants.

The Collinson and French result (also quoted in [5]) is that there is a Killing vector only in the case where \( A \) is constant — in this situation the Killing vector is the obvious \( \partial_u \). However, if \( b \neq 0 \) we can set \( c = 0 \) by a coordinate change and then the vector

\[ K^a = -u \partial_u + r \partial_r + 2x \partial_x + 2y \partial_y. \]

is a Killing vector, as will be shown in section 3. This can be checked directly: consider the flow of \( K^a \). This scales the coordinates by

\[ u \rightarrow \lambda^{-1} u, \quad r \rightarrow \lambda r, \quad \zeta \rightarrow \lambda^2 \zeta \]

for real parameter \( \lambda > 0 \). Under this scaling it is easy to check that all the contravariant components given above are homogeneous in \( \lambda \) (when \( A = bu \)), and all of the correct degree to make the flow isometric. For example, the \( g^{22} \) component is homogeneous of degree 2, and so the metric term \( g^{22} \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} \) is unchanged under the flow.

Also the vector

\[ H = r \partial_r + x \partial_x + y \partial_y \]

is a homothety, whatever \( A \) is (see section 3). Alternatively, the flow of \( H \) is

\[ u \rightarrow u, \quad r \rightarrow \lambda r, \quad \zeta \rightarrow \lambda \zeta, \]

and we again find that all the contravariant components given above are homogeneous in \( \lambda \), and all of the correct degree to make the flow homothetic. For example, the \( g^{23} \) component is homogeneous of degree 1, and so the metric term \( g^{23} \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} \) scales by \( \lambda^{-1} \): the same scaling applies to all the metric terms.

Newman and Tamburino [1] also give the following metric, which arises as a limit of the cylindrical case (see also [5] (26.23) for corrections to the \( du^2 \) coefficient):

\[ ds^2 = 2 \, du \, dr - x^{-2} \left[ b + \log(r^2 x^4) \right] du^2 + 4\frac{r}{x} du \, dx - r^2 \, dx^2 - x^2 \, dy^2, \]

with the same coordinates as used in the spherical case. The Killing vectors here are obvious (\( \partial_u \) and \( \partial_y \)) and as we shall see there is also a homothetic vector (see section 4)

\[ H_2 = 2r \partial_r - x \partial_x + 2y \partial_y. \]

One can use the flow of \( H_2 \) to check it is a homothety as well.
3 The Calculations (spherical case)

The basic information is taken from Collinson and French [2], and Newman and Tamburino [1]. See those papers for those spin coefficients that are not actually calculated here. I have checked in a separate calculation that their results are correct as quoted. I use as coordinates \( u, r, \zeta = x + iy \).

Collinson and French [2] wrote the conformal Killing equations in Newman-Penrose form and used that in their work, although there are a few minor typos in their paper. Here, I will use the formalism of [3], which generalised the ideas of [6] into a form suitable for this task. I will use the notation of [3] for the components of the homothety

\[
\xi_a = \xi_a^\ell \ell_a + \xi_{\pi} m_a - \xi_{\pi m_a},
\]

and its bivector, \( F_{ab} \), with anti-self dual

\[
- F_{ab} = 2\phi_{00} \ell_{[a} m_{b]} + 2\phi_{01} (\ell_{[a} m_{b]} - m_{[a} \bar{m}_{b]}) - 2\phi_{11} n_{[a} \bar{m}_{b]}.
\]

The tetrad is a standard tetrad (see [1]), based around the Debever-Penrose vector \( \ell_a = \partial_r \), see [1] and [2] for further detail. Since the tetrad is normalised, for the Penrose-Rindler spin coefficients used in [3] we have \( \gamma' = -\epsilon, \beta' = -\alpha \) etc.

In the Maple I use use \( z \) for \( \zeta \) and \( w \) for \( \bar{\zeta} \); \( H_1 \) for \( \xi_\ell \) etc. I typically add a \( b \) for a complex conjugate (\( \bar{H}_m \) is \( \xi_{\pi m} \)) and a \( 1 \) for a dash (\( \rho_1 \) is \( \rho' \)).

Firstly, define the terms \( a, a^0 \) (the latter is \( \alpha_0 \) in [2]).

\[
\begin{align*}
> & a := A(u) * z^{-1/2} * w^{1/2} : \\
> & a0 := 3/4 * w^{3/4} * z^{-1/4} ;
\end{align*}
\]

Rather than use the explicit definitions for \( L \) and \( R \) in [2] and [1], I will leave them as “unknown” functions and define a routine later that will substitute for their derivatives. I will also use \( Q(u, r, z, w) \) in place of \( 1/R^2 \) to make things more transparent. I define what these functions actually are so we can substitute for them more easily when that become useful. I also define dummy symbols to use in place of the full functional dependence of \( L \) and \( Q \) for ease of readability. I have also suppressed the functional dependence in the Maple output, replacing \( Q(u, r, z, w) \) with \( Q(x^a) \) for example.

\[
\begin{align*}
> & LL := L(u, r, z, w) : Lis := 1/2 * \log((r+a)/(r-a)) ;
\end{align*}
\]

\[
Lis := \frac{1}{2} \ln \left( \frac{r + A(u) \sqrt{z} \sqrt{w}}{r - A(u) \sqrt{z} \sqrt{w}} \right)
\]

\[
> & QQ := Q(u, r, z, w) : Qis := 1/(r^2 - a^2) ;
\end{align*}
\]

\[
Qis := (r^2 - (A(u))^2 zw)^{-1}
\]

Now we define the routine to simplify derivatives and products and also add a line to collect terms.
diffsbs := proc(XX)
  subs(diff(L(u,r,z,w),r)=-a*Q(u,r,z,w),XX):
  subs(diff(L(u,r,z,w),u)=r*Q(u,r,z,w)*diff(a,u),%):
  subs(diff(L(u,r,z,w),w)=r*Q(u,r,z,w)*diff(a,w),%):
  subs(diff(L(u,r,z,w),z)=r*Q(u,r,z,w)*diff(a,z),%):
  subs(diff(Q(u,r,z,w),r)=-2*r*Q(u,r,z,w)^2,%):
  subs(diff(Q(u,r,z,w),z)=2*Q(u,r,z,w)^2*a*diff(a,z),%):
  subs(diff(Q(u,r,z,w),u)=2*Q(u,r,z,w)^2*a*diff(a,u),%):
  subs(diff(Q(u,r,z,w),w)=2*Q(u,r,z,w)^2*a*diff(a,w),%):
  student[powsubs](r^2=a^2+1/Q(u,r,z,w),expand(%));
  collect(%,[L(u,r,z,w),Q(u,r,z,w),r,Hl(u),psi,z,w]);
end proc:

The terms \(S\) and \(S_b\) are \(\psi_1^0\) and its conjugate in [2].

\[
S := 2 A(u)^2 z^{3/4} w^{3/4} z; S_b := 2 A(u)^2 z^{3/4} w^{3/4} w:
\]

And the curvature component \(\Psi_1\) is given in [1].

\[
\Psi_1 := 2 A(u)^2 z^{7/4} w^{3/4} Q(x^a)^2
\]

Now from [2] we have \(\kappa = \epsilon = \tau' = \Psi_0 = 0\), and \(\rho\) and \(\sigma\) real — these can also be easily checked by Maple. So by [3](6a) \(D\xi = 0\). Using \(\tau = \overline{\tau} + \beta ([2])\), [3](6c) becomes

\[
\delta\xi_\ell = \tau\xi_\ell - \rho\xi_m - \sigma\xi_m - \phi_{11}.
\]

We next use equation [3](11), since \(\ell^a\) is a Debever-Penrose vector. Unfortunately, [3](11) contains an error — the right hand side is the complex conjugate of what it ought to be. With this correction, we have

\[
\phi_{11} = -\tau\xi_\ell + \rho\xi_m + \sigma\xi_m.
\]

Hence \(\delta\xi_\ell = 0\) and \(\xi_\ell = \xi_\ell(u)\), as found in [2].

Equation (10a) of [3] is

\[
D\phi_{11} = -\xi_\ell\Psi_1,
\]

and so integrates to give the \(r\) dependence of \(\phi_{11}\), here called \(p_{11}\). We ignore the factor independent of \(r\) when integrating:

\[
\int (-Qis^2, r);
\]

\[
\frac{r}{2 A(u)^2 z w (r^2 - A(u)^2 z w)} + \frac{1}{2} \arctanh \left( \frac{r}{A(u) \sqrt{z w}} \right) A(u)^{-3} z^{-1} w^{-1} \frac{1}{\sqrt{z w}}
\]

The \(\text{arctanh}\) term here is just \(L\) and we get

\[
p_{11} := S + Hl(u)/2/a^2*r*QQ - S + Hl(u)/2/a^3*LL + p_{110}(u,z,w);
\]

\[
p_{11} := \frac{z^{3/4} Hl(u) r Q(x^a)}{w^{1/4}} - \frac{z^{1/4} Hl(u) L(x^a)}{A(u) w^{3/4}} + p_{110}(u,z,w)
\]

The term \(S\) and \(S_b\) are \(\psi_1^0\) and its conjugate in [2].

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\[
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\]

\[
\frac{r}{2 A(u)^2 z w (r^2 - A(u)^2 z w)} + \frac{1}{2} \arctanh \left( \frac{r}{A(u) \sqrt{z w}} \right) A(u)^{-3} z^{-1} w^{-1} \frac{1}{\sqrt{z w}}
\]

The \(\text{arctanh}\) term here is just \(L\) and we get

\[
p_{11} := S + Hl(u)/2/a^2*r*QQ - S + Hl(u)/2/a^3*LL + p_{110}(u,z,w);
\]

\[
p_{11} := \frac{z^{3/4} Hl(u) r Q(x^a)}{w^{1/4}} - \frac{z^{1/4} Hl(u) L(x^a)}{A(u) w^{3/4}} + p_{110}(u,z,w)
\]
Here $p_{110}(u,z,w)$ is the integration constant. Now the spin coefficients — see [2].

Now a routine to take conjugates nicely, as we need conjugates to define $\tau$.

We need derivative operators $\delta$ and $\delta'$ to find $\rho'$. Firstly, the components of $m^a$ come from [2] and [1].

We check some curvature equations next before we go on.
\[\text{Psi}_2 := -\text{diffsbs}(\text{diff}(\rho_1, r) - \rho_1 \rho - \text{sigma} \text{sgma}_1) \]  
\[\Psi_2 := -4A(u)^2 z^{5/2}L(x^a)^2 \sqrt{w} - (2A(u) z^2 r + 4A(u)^2 z^{3/2}w^{3/2}) Q(x^a)^2\]

This expression for \(\Psi_2\) agrees with [1].

\[\text{diffsbs}(\text{diff}(\gamma, r) - \beta \text{conj}(\tau) - \alpha \tau - \Psi_2); \]  
\[\text{diffsbs}(\text{del}(\rho) - \text{del}(\sigma) - \rho (\text{conj}(\alpha) + \beta) + \sigma (3 \alpha - \text{conj}(\beta)) + \Psi_1); \]  
\[\text{diffsbs}(\text{del}(\beta) - \text{del}(\alpha) - \rho \rho_1 + \sigma \sigma_1 + \alpha \text{conj}(\alpha) + \beta \text{conj}(\beta) - 2 \alpha \beta - \Psi_2); \]

Integrating [3] (6g) and using [3](11) (corrected, see above):
\[\text{Hm1} := -r \text{p110}(u, z, w) + Hl(u) S/2/a^3 r \text{LL} + \text{Hm0}; \]
\[\text{Hmb1} := -r \text{p110b}(u, z, w) + Hl(u) Sb/2/a^3 r \text{LL} + \text{Hmb0}; \]

By [3] (11) the following ought to be zero.
\[\text{diffsbs} (\text{p11} + \tau Hl(u) - \rho \text{Hm1} - \sigma \text{Hmb1}); \]
\[\text{collect}(%/\text{QQ}, r); \]

\[\text{expand(solve(coeff(%/r, 1), \text{Hm0}));}, \text{expand(solve(coeff(%/r, 0), \text{Hmb0});} \]

So we get
\[\text{Hm} := -r \text{p110}(u, z, w) - a \text{p110b}(u, z, w) + Hl(u) \text{expand}(S/2/a^3 (r \text{LL} - a)); \]
Hmb:=conj(Hm);

These agree with the components in [2] (their $V_3$ and $V_4$). Now we use [3] (10b) to get $\phi_{01}$.

diffsbs(Psi1*Hm/2/sigma-beta*p11/sigma+del(p11)/2/sigma):

$p01:=collect(%,[Hl(u),Q(u,r,z,w),L(u,r,z,w),r]);$

$$p01 = \left( \frac{1}{2} \sqrt{z} \sqrt{wr} + \left( 2 \frac{z^{3/2}r}{\sqrt{w}} + A(u) zw \right) L(x^a) \right) Q(x^a) - \frac{z \left( L(x^a) \right)^2}{A(u) w} - \frac{L(x^a)}{2A(u)} Hl(u)$$

$$+ \frac{z^{3/4}L(x^a)p110(u,z,w)}{w^{1/4}} + \frac{3w^{3/4}p110(u,z,w)}{4z^{1/4}}$$

$$- \left( A(u) \frac{z^5}{4} w^{1/4} r \right) p110(u,z,w) + A(u)^2 z^{7/4} w^{3/4} p110b(u,z,w) Q(x^a)$$

$$+ \left( \frac{3p110(u,z,w)}{4A(u) w^{3/4}} + \frac{w^{1/4} \frac{\partial}{\partial w} p110(u,z,w)}{A(u)} \right) z^{1/4} r - \frac{z^{3/4} w^{3/4} \frac{\partial}{\partial z} p110(u,z,w)}{4A(u) w^{3/4}}$$

Now [3](10c) and (8a) will give us information on the $w$ (that is, $\zeta$) dependence of $\phi_{01}$.

diffsbs(Psi1*Hmb-Psi2*Hl(u)+2*rho*p01+2*alpha*p11-del1(p11)); # [3]10c

$$p110(u,z,w) = \frac{F1(u,z)}{w^{3/4}}$$

$$\text{dsolve}(\%=0,p110(u,z,w));$$

Both giving the same result. Now we turn to $\xi_n$ and [3](6i), which we solve for $\sigma\xi_\ell$.

diffsbs(diff(p01,r)+del1(p11)-2*rho*p01-2*alpha*p11); # [3] (8a)

$$p110(u,z,w) = \frac{F1(u,z)}{w^{3/4}}$$

$\text{dsolve}(\%=0,p110(u,z,w));$

Both giving the same result. Now we turn to $\xi_n$ and [3](6i), which we solve for $\sigma\xi_\ell$.

rhs6i:=diffsbs((-del(Hm)-conj(sigma1)*Hl(u)-Hm*(conj(alpha)-beta))):

The imaginary part ought to be zero as $\xi_n$ is real, so using results from [3] (10c) and (8a), we find the imaginary part divide out a common non-zero factor and call what’s left $X$.  

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\[ \text{Imrhs6i} := \% - \text{conj}(\%) ; \]
\[ \text{subs}(\text{p110}(u,z,w) = F(u,z)/w^{3/4}, \text{p110b}(u,z,w) = Fb(u,w)/z^{3/4}, \%) ; \]
\[ X := \text{expand}(\% / r/\text{QQ}/\text{sqrt}(z)/\text{sqrt}(w)) ; \]

\[
X := -4 z^{3/4} A(u) \frac{\partial}{\partial z} F(u,z) + 4 w^{3/4} A(u) \frac{\partial}{\partial w} Fb(u,w) + 3 \frac{A(u) Fb(u,w)}{w^{1/4}} - 3 \frac{A(u) F(u,z)}{z^{1/4}}
\]

Assuming \( F \) is differentiable in \( z \) we can split this
\[ \text{subs}(Fb=0, X) ; \%
\]

This is a (real) function of \( u \) and \( w \). We choose the shape of the separation function to simplify the solution to the differential equation slightly.
\[ \text{dsolve}(\% = -4 * A(u) * G(u), F(u,z)) ; \]

Check this out:
\[ \text{subs}(F(u,z) = G(u) * z^{1/4} + H(u) / z^{3/4}, Fb(u,w) = G(u) * w^{1/4} + Hb(u) / w^{3/4}, X) ; \]
\[ \text{expand}(\%) ; \]

So we define a simplification routine for \( \phi_{11}^0 \).
\[ \text{P110sbs1} := \text{proc}(XX) \]
\[ \text{subs}(\text{p110}(u,z,w) = F(u,z)/w^{3/4}, \text{p110b}(u,z,w) = Fb(u,w)/z^{3/4}, XX) ; \]
\[ \text{subs}(F(u,z) = G(u) * z^{1/4} + H(u) / z^{3/4}, Fb(u,w) = G(u) * w^{1/4} + Hb(u) / w^{3/4}, \%) ; \]
\[ \text{expand}(\%) ; \]
\[ \text{end proc} : \]
And check it works
\[ \text{P110sbs1(Imrhs6i)} ; \]

0

Turning to \([3](6b)\) next,
\[ \text{eqn6b} := Hl(u) * (\text{gamma} + \text{conj}(\text{gamma})) - \text{conj}(\tau) * Hm - \tau * \text{conj}(Hm) - p01 - \text{conj}(p01) + \psi ; \]
\[ \text{P110sbs1(diffsbs(\%))} ; \]

\[ \psi - G(u) - 3 \frac{H(u)}{2z} - 3 \frac{Hb(u)}{2w} \]

This ought to be \( \dot{\xi}_\ell \), a function of \( u \) only, so \( H = 0 \) and we define a new simplification routine and test it out:
\[ \text{P110sbs2} := \text{proc}(XX) ; \]
\[ \text{expand( subs( p110(u,z,w) = (psi-diff(Hl(u),u)) * z^{1/4} / w^{3/4} , p110b(u,z,w) = (psi-diff(Hl(u),u)) * w^{1/4} / z^{3/4} , XX)) ;} \]
\[ \text{collect(\%, [L(u,r,z,w), Q(u,r,z,w), r, Hl(u), z, w]) ; \text{end proc} ;} \]
\[ \text{P110sbs2(diffsbs(eqn6b))} ; \]
This is as it should be. Now we can define $\xi$. 

$$\frac{d}{du} H_l(u)$$

We check this against the $[2]$ version, called $V_2$ there. It is clear from the shape of $\xi_m (= V_3$ of $[2]$) that $a_0$ in $[2]$ is my $\phi_{11}$.

$$ay_0 := p_{110}(u, z, w); ay_0b := p_{110b}(u, z, w):$$

$$V_2 := rL^2 \left( -S^2 - Sb^2 \right) \frac{H_l(u)}{4/a^5} - LL^2 S Sb \frac{H_l(u)}{4/a^4} - \left( -2a^3 \frac{ay_0 S + ay_0b Sb}{2/a} + r \left( 2a H_l(u) \frac{d}{du} a - ay_0 Sb - ay_0b S \right) \frac{2}{a^2} + \right) \frac{-2a^3 \frac{ay_0 S + ay_0b Sb}{2/a} + r \left( -2a^3 \frac{ay_0 S + ay_0b Sb}{2/a} \right) \frac{1}{QQ} \left( -3ay_0 S - 3ay_0b Sb + H_l(u) S Sb \right) \frac{2}{a^3} }{8/a^3}:$$

$$\text{expand}(H_n - P_{110sbs2}(V_2));$$

$$\text{simplify(subs(psi=0,diff(H_l(u),u)=0,%)); 0}$$

So our $\xi_n$ agrees with $[2]$ in the case of their Killing vector ($\psi = 0$ and $\xi_\ell \text{ constant}$). However, if $\xi_\ell$ is not constant, the terms differ:

$$\text{simplify(subs(psi=0,%%));}$$

$$-\frac{2}{z^{3/2}w^{3/2}} \frac{dH_l(u)}{du} \left( \right) \frac{A(u)}{A(u)} + L(x^a) r \left( z^{5/2} w^{5/2} + 2 z^{3/2} w^2 L(x^a) \right) \frac{A(u)}{A(u)} + L(x^a) r \left( z^{5/2} w^{5/2} + 2 z^{3/2} w^2 L(x^a) \right) \frac{A(u)}{A(u)}$$

Next, we put our $\xi_n$ into $[3](6d)$.

$$\text{eqn6d := diffsbs(diff(Hn,r)-p_{01}-conj(p_{01})-psi):}$$

$$\text{P110sbs2(%)};$$

$$\frac{H_l(u)}{A(u)} \frac{d}{du} A(u) - \frac{d}{du} H_l(u)$$

$$\text{dsolve(%,H_l(u));}$$

$$H_l(u) = _C1 A(u)$$

So next a routine to replace $\xi_\ell(u)$ with a multiple of $A(u)$, and also to kill off the second derivative of $A(u)$.

$$\text{Hlsbs := proc(XX) \text{sub}(H_l(u)=C*A(u),XX);sub(diff(A(u),u,u)=0,%);sub(diff(H_l(u),u,u)=0,%);%; \text{end proc};}$$

We now try $[3](6j)$.

$$\text{del1(Hm)+conj(rho1)*H_l(u)+rho*Hn+(conj(beta)-alpha)*Hm-p_{01}+conj(p_{01})+psi:}$$

$$\text{Hlsbs(P110sbs2(diffssbs(%)));}$$

0
So that is satisfied. Now for $\phi_{00}$, which we get from the conjugate of [3](6f).

\[ eqn6f:=-\text{del}1(Hn)-(\text{conj}(\beta)+\alpha)Hn-\text{conj}(\rho1)\text{Hmb}-\sigma1\text{Hm}: \]

I’ve suppressed this component as it’s very long, but we check the result with [3] (8d).

\[ p00:=\text{diffsbs}(\text{P110sbs2}(\text{diffsbs}(\%))): \]

To go any further we need to get the components of $n^a = (1, U, X^3, X^4)$ and to define $D'$. Taking the metric terms from [1] and [2]:

\[ g_{up22}:= -2r^2\sqrt{w}\sqrt{z}\text{QQ} + 2r\text{LL}/A(u) + \text{QQ}^2(2r^3A(u)\text{(w}^2+z^2) - 4r^2A(u)2w^3/2z^{3/2})/2): \]

These next two terms are the components of $m^a$.

\[ g_{up33}:= -2z^3/4w^3/4r^2\text{QQ}; g_{up44}:= -2z^3/4w^3/4r^2\text{QQ}; \]

These next two terms are the components of $m^a$.

\[ g_{up23}:=4A(u)2z^3/2w^3/2(r+z)/2(\text{LL}/a^3-(r-2a)*\text{QQ}/2/a^2) \]

\[ g_{up44}:=4A(u)2z^3/2w^3/2(r-z)/2\text{I}(\text{LL}/a^3-(r+2a)*\text{QQ}/2/a^2) \]

These next two terms are the components of $m^a$.

\[ g_{up23}:=\text{omega}2\text{conj}(\text{omega}); \]

\[ U:=\sqrt{z}\sqrt{w}(x^a)^2 + ((w^2+z^2)A(u)r - 2A(u)^2z^{3/2}w^{3/2})Q(x^a) - \sqrt{z}\sqrt{w} + \frac{r}{A(u)}L(x^a) \]

\[ g_{up23}:=4A(u)2z^3/2w^3/2(r+z)/2(\text{LL}/a^3-(r-2a)*\text{QQ}/2/a^2) \]

\[ g_{up44}:=4A(u)2z^3/2w^3/2(r-z)/2\text{I}(\text{LL}/a^3-(r+2a)*\text{QQ}/2/a^2) \]

These next two terms are the components of $m^a$.  

\[ \text{x3}:=P*(r-a)*\text{QQ}; \text{x4}:=I*P*(r+a)*\text{QQ}; \]

These next two terms are the components of $m^a$.
\[ X3 := \left( \left( -z^{3/2} \sqrt{w} - \sqrt{w}^{3/2} \right) r + A(u)zw^2 + A(u)z^2w \right) Q(x^a) + \left( w A(u) - \frac{z}{A(u)} \right) L(x^a) \]
\[ + \left( -z^{3/2} \sqrt{w} - \sqrt{w}^{3/2} \right) r + A(u)zw^2 + A(u)z^2w \right) Q(x^a) \]

\[ X4 := \left( i \left( \left( -z^{3/2} \sqrt{w} + \sqrt{w}^{3/2} \right) r + zw^2 A(u) - z^2 w A(u) \right) Q(x^a) + \left( w A(u) - \frac{z}{A(u)} \right) L(x^a) \right) \]
\[ + \left( -i \sqrt{w}^{3/2} + iz^{3/2} \sqrt{w} \right) r + iz^2 w A(u) - izw^2 A(u) \right) Q(x^a) \]

As a double check we firstly define the (contravariant) tetrad and then check against the metric terms.

\[ \ell := \langle 0,1,0,0 \rangle; \quad e_n := \langle 1, U, X3, X4 \rangle; \]
\[ e_m := \langle 0, \omega, \xi_3, \xi_4 \rangle; \quad e_{mb} := \text{map}(\text{conj}, e_m); \]
\[ \ell.\text{Transpose}(e_n) - e_m.\text{Transpose}(e_{mb}); \]
\[ g := \text{map}(\text{diffsbs}, \%); \]
\[ \text{diffsbs}(g[2,2] - g_{\text{up}22}); \]
\[ \text{diffsbs}(g[2,3] - g_{\text{up}23}); \]
\[ \text{diffsbs}(g[2,4] - g_{\text{up}24}); \]
\[ \text{diffsbs}(\text{simplify}(g[3,3] - g_{\text{up}33})); \]
\[ \text{diffsbs}(\text{simplify}(g[4,4] - g_{\text{up}44})); \]

For a second check we apply two of the commutators [4] (4.11.11) to \( r \) and check what we get.

\[ \text{diff}(U,r) + \gamma + \text{conj}(\gamma) - \tau*\text{conj}(\omega) - \text{conj}(\tau)*\omega; \]
\[ \text{diffsbs}(\%); \]
\[ \text{diff}(X3,r) - \tau*\text{conj}(\xi_3) - \text{conj}(\tau)*\xi_3; \]
\[ \text{diffsbs}(\%); \]
Since all this checks out we go ahead and define $D'$.

> $D1:=\text{proc}(\text{XX})$
> $\text{diff}(\text{XX},u)+\text{diff}(\text{XX},r)*U+(X3+I*X4)*\text{diff}(\text{XX},z)+(X3-I*X4)*\text{diff}(\text{XX},w);$  
> $\text{P110sbs2}(\text{diffsbs}(\%));$
> $\text{end proc};$

We make use of $D'$ firstly to find the last spin coefficient, $\kappa'$, using [4](4.11.12).

> $D1(\text{beta})-\text{del}(\text{gamma})-\tau*\rho1-\alpha*\text{conj}(\sigma1)-\beta*(\rho1+\gamma-\text{conj}(\gamma))$
> $+\gamma*(\tau-\text{beta}-\text{conj}(\alpha))$: # should be $-\kappa'*\sigma$
> $\text{diffsbs}(\%);$  
> $\kappa1:=-\text{diffsbs}(\text{P110sbs2}(\%/\sigma));$

\[
\kappa1 := -\left( \frac{z^{5/4}w^{1/4}r - \frac{z^{11/4}}{w^{1/4}} A(u)}{w^{1/4}} \right) Q(x^a) L(x^a) + \right.
\left. \left( -2 \frac{z^{9/4}}{w^{3/4}} + 2 z^{1/4} w^{1/4} \right) r - 3 z^{7/4} w^{3/4} A(u) \right) Q(x^a) L(x^a) + \right.
\left. \left( \frac{dA}{du} z^{1/4} w^{5/4} - 2 z^{5/4} w^{1/4} \right) r - A(u) \left( z^{7/4} w^{3/4} \frac{dA}{du} + 4 z^{3/4} w^{7/4} \right) \right) Q(x^a) L(x^a)
\]

Now to look at the equations that involve $D'$. Firstly [3] (6h):

> $\text{eqn6h}:=$ $\text{D1}(Hm)+\text{conj}(\kappa1)*Hl(u)+\tau*Hn+(\text{conj}(\gamma)+\gamma)*Hm-\text{conj}(p00);$  
> $\text{P110sbs2}(\text{diffsbs}(\%)); \text{factor}(\text{Hlsbs}(\%));$

0

Then we look at [3] (6e):

> $\text{eqn6e}:=$ $\text{D1}(Hn)+(\gamma+\text{conj}(\gamma))*Hn+\kappa1*Hm+\text{conj}(\kappa1)*Hmb;$  
> $\text{P110sbs2}(\text{diffsbs}(\%));$
> $\text{factor}(\text{Hlsbs}(\%));$

0

and [3] (8c)

> $\text{eq8c}:=$ $\text{diffsbs}(\text{del}(p01)+\text{D1}(p11)-\sigma*p00-2*\tau*p01-(\rho1+2*\gamma)*p11);$  
> $\text{factor}(\text{Hlsbs}(\text{P110sbs2}(\%)));$

0

and finally [3] (10d).

12
Next we consider what happens if we have a Killing vector ($\psi = 0$) with $\xi_\ell$ zero ($C = 0$)

\[
\begin{align*}
&\text{subs}(C=0,\psi=0, Hlsbs(P110sbs2(Hm))) ; \\
&\text{subs}(C=0,\psi=0, Hlsbs(P110sbs2(Hn))); \\
\end{align*}
\]

Hence we cannot have both $\psi$ and $C$ zero. This is the Collinson and French result: only one Killing vector at most. We have a look at the homothety.

\[
\begin{align*}
&\frac{C A(u)}{-r C \frac{d A}{d u} + 2 r \psi} \quad \frac{2 \psi x - 2 C \frac{d A}{d u} x}{2 \psi y - 2 C \frac{d A}{d u} y} \\
\end{align*}
\]

So the obvious Killing vector if $A$ is constant:

\[
\begin{align*}
&\text{KK:=subs(psi=0,C=1/B,diff(A(u),u)=0,A(u)=B,K);} ; \\
&\begin{pmatrix}
1 \\
0 \\
0 \\
\end{pmatrix}
\end{align*}
\]

The new Killing vector in the other case:

\[
\begin{align*}
&\text{KK2:=subs(psi=0,A(u)=u*B,C=1/B,K):map(simplify,\%);} ; \\
&\begin{pmatrix}
u \\
-r \\
-2x \\
-2y \\
\end{pmatrix}
\end{align*}
\]

And the proper homothety for both cases:

\[
\begin{align*}
&\text{HH:=subs(C=0,psi=1,K);} ;
\end{align*}
\]
We now turn to the remaining curvature equations and Bianchi identities. To make life easy, we define the weighted derivative operators, [4] section 4.14.

\[
\begin{pmatrix}
0 \\
2r \\
2x \\
2y
\end{pmatrix}
\]

As a check on the calculations, we can run through the curvature equations, [4] (4.12.32), some of which we’ve used already, some of which will give us \(\Psi_3\) and \(\Psi_4\). The only ones that do not give zero are \((b')\) and \((c')\), the first of which gives us \(\Psi_4\):

\[
\begin{align*}
\text{thorn1}(\sigma_1,-3,1) - & \text{edth1}(\kappa_1,-3,-1) - \sigma_1(\rho_1 + \text{conj}(\rho_1)) \\
& + \kappa_1(\text{conj}(\tau)) \\
\Psi_4 := & -8 L(x^a)^3 z^4 Q(x^a)^2 A(u)^2 \\
& + \left( -12 \frac{r z^{7/2} A(u)}{w} - 32 A(u)^2 z^3 w \right) Q(x^a)^2 - 8 z^2 Q(x^a) L(x^a)^2 \\
& + \left[ -32 A(u)^2 z^2 w^2 - 8 A(u)^2 z^3 w \frac{dA}{du} - 24 r z^{5/2} \sqrt{w} A(u) \right] Q(x^a)^2 \\
& + \left( -8 z w - 8 \frac{dA}{du} z^2 \right) Q(x^a) L(x^a) - 8 \frac{dA}{du} z w Q(x^a) \\
& + \left( -12 r A(u) z^{3/2} w^{3/2} - 8 z w^3 A(u)^2 - 8 A(u)^2 z^2 w^2 \frac{dA}{du} A(u) \right) Q(x^a)^2
\end{align*}
\]

And \((c')\) gives \(\Psi_3\) (as do several others):
\[
\Psi_3 := 6 Q (x^a)^2 L (x^a)^2 A(u)^2 z^{13/4} w^{1/4} + 14 (A(u)^2 z^{9/4} w^{5/4} + 6 \frac{A(u)}{w^{1/4}}) Q (x^a)^2 + 2 Q (x^a)^2 z^{5/4} w^{1/4} \right) L (x^a)
+ \left( 2 A(u)^2 z^{5/4} w^{1/4} \frac{dA}{du} + 6 z^{5/4} w^{9/4} A(u)^2 + 6 A(u) z^{7/4} w^{3/4} \right) Q (x^a)^2
+ 2 \frac{dA}{du} z^{5/4} w^{1/4} Q (x^a)
\]

Now we check the leading terms (in inverse powers of \( r \)) of our \( \Psi_3 \) and \( \Psi_4 \) and compare to [1].

\[
\text{subs}(L(u,r,z,w)=Lis,Q(u,r,z,w)=Qis,Hlsbs(Psi4)):
\]

\[
T4:=\text{subs}(r=1/R,%):
\]

\[
\text{series}(T4,R=0,3) \text{ assuming } R::\text{positive};
\]

\[
-8 \frac{dA}{du} z w R^2 + O \left( R^3 \right)
\]

Here the leading term agrees with [1]. Next \( \Psi_3 \)

\[
\text{subs}(L(u,r,z,w)=Lis,Q(u,r,z,w)=Qis,Hlsbs(Psi3)):
\]

\[
T3:=\text{subs}(r=1/R,%):
\]

\[
\text{series}(T3,R=0,4) \text{ assuming } R::\text{positive};
\]

\[
2 \frac{dA}{du} z^{5/4} w^{1/4} R^2 + 8 z^{7/4} w^{3/4} A(u) R^3 + O \left( R^4 \right)
\]

We find that the leading term agrees with [1], but in the second term the powers of \( z = \zeta \) and \( w = \bar{\zeta} \) are wrong in [1]. We can also check that the Bianchi identities, [4] (4.12.36-39) are satisfied (and they are).

Finally, we turn to the remaining integrability conditions, [3] (10e) to (10h).

\[
\text{P110sbs2}(\text{diffsbs}(\text{Psi2*Hmb-diff(p00,r)}-\text{Psi3*Hl(u)})); \quad \# [3] 10e
\]

\[
0
\]

\[
\text{P110sbs2}(\text{diffsbs}(\text{Psi3*Hm-Psi2*Hn-2*rho1*p01+2*beta*p00+del(p00)})); \quad \# [3] 10f
\]

\[
0
\]

\[
\text{P110sbs2}(\text{diffsbs}(\text{Psi3*Hm-Psi4*Hl(u)}+2*sigma1*p01-2*alpha*p00-del1(p00)))); \quad \# [3] 10g
\]

\[
0
\]

\[
\text{P110sbs2}(\text{diffsbs}(\text{Psi4*Hm-Psi3*Hn-2*kappa1*p01+2*gamma*p00+D1(p00)})); \quad \# [3] 10h
\]
So we see that all the homothetic and Killing equations are satisfied and we have shown that there is always a Killing vector in these metrics and also always a homothety.

4 The Calculations (limit cylindrical case)

Since neither [1] not [2] give the spin coefficients for the limit cylindrical metric, we will need to calculate them using Maple’s tensor package. Note that we use the corrected version of this metric, see [5] equation (26.23)

\[
\text{with(tensor):}
\]
\[
\text{coord:=[u,r,x,y]:g_c:=array(1..4,1..4,symmetric,sparse):}
\]
\[
g_c[1,1]:=-\text{expand(simplify((b+log(r^2*x^4))/x^2/2)}
\]
\[
\text{assuming r::positive,x::positive);}
\]
\[
g_{c,1,1} := -\frac{b}{2x^2} - \frac{\ln (r)}{x^2} - 2 \frac{\ln (x)}{x^2}
\]

\[
g_c[1,2]:=1:g_c[3,3]:=-2*r^2:g_c[4,4]:=-2*x^2:g_c[1,3]:=2*r/x:
\]

Next we calculate all the relevant tensors.

\[
\text{tensorsGR(coord,g,gup,'detg', 'C1','C2','Rm','Rc', 'R','G','C');}
\]

To calculate the spin coefficients, I use a set of routines available on my website http://www.maths.unsw.edu.au/~jds/papers.html

\[
\text{read ‘PRcoeff‘;}
\]

Now we define the tetrad, with the choice of \(m^a\) dictated by the need for the tetrad to be right-handed, so the anti-self duality used in the definition of the homothetic bivector (see [3]) is satisfied.

\[
\text{md:=create([-1],vector([0,0,r,-I*x])):mup:=raise(gup,md,1):}
\]
\[
\text{mbd:=create([-1],vector([0,0,r,I*x])):mbup:=raise(gup,mbd,1):}
\]
\[
\text{ld:=create([-1],vector([1,0,0,0])):lup:=raise(gup,ld,1):}
\]
\[
\text{nd:=create([-1],vector([g_c[1,1]/2,1,2*r/x,0])):nup:=raise(gup,nd,1):}
\]
\[
\text{his:=linalg[stackmatrix]}(\text{ld[compts]},\text{nd[compts]},\text{md[compts]},\text{mbd[compts]}):
\]
\[
\text{h:=create([1,-1],op(his)):
\]

Using the routines PRspin and PRcrv from the PRcoeff file we calculate the spin coefficients, and curvature components.

\[
\text{spins:=PRspin(g,h,C2,coord):}
\]
\[
\text{crv:=PRcurve(g,h,C,Rc,coord);}
\]
From these two calculations we find that the non-zero spin coefficients are

\[ \tau = \beta = \tau' = -\frac{1}{2rx}, \quad \rho = \sigma = -\frac{1}{2r}, \quad \gamma = \frac{1}{4rx^2}, \quad \rho' = \sigma' = \frac{b + \log(r^2x^4)}{8rx^2}; \]

and the non-zero curvature components are

\[ \Psi_1 = \frac{1}{2r^2x}; \quad \Psi_2 = \frac{1}{2r^2x}; \quad \Psi_3 = \frac{b + \log(r^2x^4)}{8rx^2}. \]

Now we define the derivative operators \( \mathcal{D}, \mathcal{D}', \delta \) and \( \delta' \)

\[ \mathcal{D} := \text{XX} -> \text{add}(\text{lup}[\text{compts}][i] \cdot \text{diff}(\text{XX}, \text{coord}[i]), i=1..4): \]
\[ \mathcal{D}1 := \text{XX} -> \text{add}(\text{nup}[\text{compts}][i] \cdot \text{diff}(\text{XX}, \text{coord}[i]), i=1..4): \]
\[ \delta := \text{XX} -> \text{add}(\text{mup}[\text{compts}][i] \cdot \text{diff}(\text{XX}, \text{coord}[i]), i=1..4): \]
\[ \delta1 := \text{XX} -> \text{add}(\text{mbup}[\text{compts}][i] \cdot \text{diff}(\text{XX}, \text{coord}[i]), i=1..4): \]

Now to find the Killing vectors. Using [3] (6a),(6c) and (11) gives \( \xi_\ell = \xi_\ell(u) \). Then from [3] (10a) we get \( \phi_{11} \), and find that \( \phi_{11}^0(u,x,y) \), the integration constant, is real by [3] (6g), which also gives \( \xi_m \). So

\[ \text{Hm} := -r \cdot p110(u,x,y) + \text{I} \cdot \text{Hm0}(u,x,y); \text{Hmb} := -r \cdot p110(u,x,y) - \text{I} \cdot \text{Hm0}(u,x,y); \]
\[ p11 := \text{Hl}(u)/2x/r + p110(u,x,y); \quad \# \text{note that } p11 \text{ is real} \]

We also solve [3] (10b) for \( \phi_{01} \).

\[ \text{crv[Psil]}*\text{Hm}-2*\text{spins[sigma]}*p01-(\text{spins[beta]}-\text{spins[alpha1]})*p11+\delta(p11): \]
\[ p01 := \text{expand(solve(\%,p01))}; \]

\[ p01 := -\frac{p110(u,x,y)}{2x} - i \cdot \frac{\text{Hm0}(u,x,y)}{2xr} - \frac{\text{Hl}(u)}{4x^2r} + \frac{1}{2} \frac{\partial}{\partial x} p110(u,x,y) - i \frac{r}{2x} \frac{\partial}{\partial y} p110(u,x,y) \]

Now looking at [3] (8a), using the fact that \( \kappa = 0 \):

\[ \text{diff}(p01,r)+\delta1(p11)-2*\text{spins[rho]}*p01- \]
\[ (\text{spins[taul]}+\text{spins[alpha]}-\text{spins[betal]})*p11: \]
\[ \text{expand(\%)}; \]

\[ -\frac{3i}{2x} \frac{\partial}{\partial y} p110(u,x,y) \]

So \( \phi_{11}^0 \) is independent of \( y \). Now looking at [3] (6b):

\[ \text{D1(Hl(u))+2*spins[psilon1]}*Hl(u)+\text{spins[tau]}*\text{Hm}+\text{spins[tau]}*\text{Hmb}+p01 \]
\[ +\text{subs(}\text{I=-I},p01)-\text{psi}; \]
\[ \text{collect(\%,r)}; \]
\[ \frac{d}{du} H_l(u) + \frac{\partial}{\partial x} p_{110}(u, x, y) - \psi \]

So we solve this for \( \phi_{11}^0 \), recalling that \( \phi_{11}^0 \) is independent of \( y \), and use it to redefine \( \phi_{11}, \phi_{01} \) and \( \xi_m \).

\[
p_{11} := \frac{H_l(u)}{2x/r} + (\psi - \text{diff}(H_l(u), u))x + p_0(u): \quad \text{# note that } p_0 \text{ is real}
\]

\[
p_{01} := \text{expand}(\text{subs}(p_{110}(u, x, y) = (\psi - \text{diff}(H_l(u), u))x + p_0(u), p_{01}));
\]

\[
H_m := \text{expand}(\text{subs}(p_{110}(u, x, y) = (\psi - \text{diff}(H_l(u), u))x + p_0(u), H_m));
\]

\[
H_{mb} := \text{expand}(\text{subs}(p_{110}(u, x, y) = (\psi - \text{diff}(H_l(u), u))x + p_0(u), H_{mb}));
\]

Turning to \([3](10c)\)

\[
\text{crv}[\Psi_2]*H_l(u) - \text{crv}[\Psi_1]*H_{mb} - 2*\text{spins}[\rho]*p_{01} - (\text{spins}[\alpha] - \text{spins}[\beta_1])*p_{11} + \text{del1}(p_{11});
\]

\[
\text{expand}(\%);
\]

\[
0
\]

Now the right hand side of \([3](6d)\) is

\[
-2*\text{spins}[\epsilon]*H_n - \text{spins}[\tau_1]*H_m - \text{spins}[\tau_1]*H_{mb} + p_{01} + \text{subs}(I = -I, p_{01}) + \psi;
\]

\[
\text{expand}(\%);
\]

\[
\frac{d}{du} H_l(u) - 2 \frac{p_0(u)}{x} - \frac{H_l(u)}{2x^2 r}
\]

This is \( D\xi_n \), so we integrate

\[
\int(\%, r);
\]

\[
r \frac{d}{du} H_l(u) - 2 \frac{r}{x} p_0(u) - \frac{H_l(u) \ln(r)}{2x^2}
\]

\[
H_n := \% + H_n^0(u, x, y);
\]

Turning to \([3](6i)\),

\[
\text{del}(H_m) + \text{spins}[\sigma_1]*H_l(u) + \text{spins}[\sigma_1]*H_n + (\text{spins}[\alpha_1] + \text{spins}[\alpha])\times H_m;
\]

The coefficients of \( r \) are independent, so we collect the terms.

\[
\text{collect(\%), r});
\]

\[
-\psi + \frac{1}{2} \frac{d}{du} H_l(u) - \frac{p_0(u)}{2x} - \frac{1}{2x} \frac{\partial}{\partial y} H_m \theta (u, x, y)
\]

\[
- \left( \frac{1}{2} \frac{\partial}{\partial x} H_m \theta (u, x, y) + \frac{H_l(u)b}{8x^2} + \frac{H_l(u)\ln(x)}{2x^2} - \frac{i H_m \theta (u, x, y)}{2x} + \frac{1}{2} H_n \theta (u, x, y) \right) r^{-1}
\]

The imaginary part of the \( r^{-1} \) term implies \( \xi_m^0 = xf(u, y) \), for some function \( f(u, y) \), so:
> X:=collect(subs(Hm0(u,x,y)=x*f(u,y),Hm),r);
> Y:=solve(op(5,X),Hn0(u,x,y));

\[ Y := -Hl(u) \frac{b + 4 \ln(x)}{4x^2} \]

> expand(subs(Hn0(u,x,y)=Y,X));

\[-\psi + \frac{1}{2} \frac{d}{du} Hl(u) - \frac{p\theta(u)}{2x} - \frac{1}{2} \frac{\partial}{\partial y} f(u,y)\]

> XX:=rhs(dsolve(%,f(u,y)))*x;

\[ XX := \left( -2yr + y^2 + \frac{dy}{du} Hl(u) + \frac{yp\theta(u)}{x} + F1(u) \right) x \]

Where \( F1 \) is an arbitrary function. This \( XX \) is \( \xi^0_{m} \). So

> Hm:=collect(subs(Hm0(u,x,y)=XX,Hm),[r,x,y]);
> Hmb:=collect(subs(Hm0(u,x,y)=XX,Hmb),[r,x,y]):

\[ Hm := \left[ \left( -\psi + \frac{d}{du} Hl(u) \right) x - p\theta(u) \right] r + \left[ i \left( -2\psi + \frac{d}{du} Hl(u) \right) y + iF1(u) \right] x - iy p\theta(u) \]

> Hn:=subs(Hn0(u,x,y)=Y,Hn);

\[ Hn := \left( \frac{d}{du} Hl(u) \right) r - 2 \frac{p\theta(u)}{x} r + Hl(u) \frac{2ln(r) + b + 4\ln(x)}{4x^2} \]

> p01:=expand(subs(Hm0(u,x,y)=XX,p01));

\[ p01 := \frac{p\theta(u)}{2x} + \frac{iy\psi}{r} - \frac{iy}{2r} \frac{d}{du} Hl(u) + \frac{iy}{2r} p\theta(u) - \frac{i}{2r} F1(u) - \frac{Hl(u)}{4x^2} \]

Returning to the integrability conditions, we look at \[3\] (10d)

> eq10d:=crv[\Psi2]*Hm-crv[\Psi1]*Hn-2*spins[\tau]*p01-2*spins[\gamma]*p11+D1(p11):
> expand(\%);

\[ \frac{p\theta(u)}{2x^2r} + \frac{d}{du} p\theta(u) - x \frac{d^2}{du^2} Hl(u) \]

So by comparing coefficients we have

> p0(u):=0;Hl:=x->k0*x+k1;

\[ p\theta(u) := 0 \]
\[ Hl := x \mapsto k0x + k1 \]
And a quick check shows that eqn10d ([3] (10d)) is satisfied. Next, the conjugate of [3] (6h) will give us \( \phi_{00} \).

\[ D1(Hmb) + \text{spins}[\tau]*Hn + (\text{spins}[\gamma] + \text{spins}[\epsilon1])*Hmb: \]

\[ p00 := \text{collect}(\text{expand}(\%),[\psi,k1,k0]); \]

\[
p00 := \left( -\frac{b}{4} - \ln\left(\frac{r}{x}\right) - \ln\left(\frac{x}{r}\right) \right) \psi + \left( \frac{\ln(r)}{4rx^3} + \frac{\ln(x)}{2rx^3} + \frac{b}{8rx^3} \right) k1 + \left( \frac{ub}{8rx^3} + \frac{b}{4} - \frac{1}{2x} + \frac{\ln(r)}{2x} + \frac{\ln(x)}{x} + \frac{u \ln(x)}{2rx^3} + \frac{\ln(r) u}{4rx^3} \right) k0 - i \left( \frac{d}{du} F1(u) \right) \]

We next check some further integrability conditions, [3] (10e) first.

\[ \text{crv}[\Psi3]*Hl(u) - \text{crv}[\Psi2]*Hmb - 2*\text{spins}[\tau1]*p01 + 2*\text{spins}[\epsilon]*p00 + \text{diff}(p00,r): \]

\[ \text{expand}(\%); \]

\[ 0 \]

And then [3] (8d).

\[ e8d := \text{diff}(p00,r) + \text{del1}(p01) - \text{spins}[\rho]*p00 - 2*\text{spins}[\tau1]*p01 - 2*\text{spins}[\epsilon]*p00 - 2*\text{spins}[\tau1]*p01 - \text{spins}[\sigma1]*p11: \]

\[ \text{expand}(\%); \]

\[
-\frac{ix}{2r} \left( \frac{d}{du} F1(u) \right) \]

So the integrability function \( _F1 \) is constant:

\[ _F1(u) := k3; \text{expand}(e8d); \]

\[ _F1(u) := k3 \]

0

Also [3] (6e) is

\[ \text{eqn6e} := \text{expand}(\text{D1}(Hn) + 2*\text{spins}[\gamma]*Hn); \]

\[ \text{eqn6e} := -\frac{k0}{2x^2} \]

Thus \( k0 = 0 \), and the coefficients simplify as follows:

\[ Hl(u); Hn; Hm; \]

\[
-\frac{\ln(r) k1}{2x^2} - \frac{k1 (b + 4 \ln(x))}{4x^2} - rx\psi + (-2i \psi + ik3)x \]
\[
\left( -\frac{b}{4x} - \frac{\ln(r)}{2x} - \frac{\ln(x)}{x} \right) \psi + \left( \frac{\ln(r)}{rx^3} + \frac{\ln(x)}{2rx^3} + \frac{b}{8rx^3} \right) k1 \\
\frac{iy\psi}{r} - \frac{ik3}{2r} - \frac{k1}{4x^2r} \\
\frac{k1}{2rx} + x\psi
\]

All the remaining homothetic equations and integrability equations are satisfied, and we are left with the general homothetic vector:

\[
> \text{lin_com}(Hl(u), nup, Hn, lup, -Hm, mbup, -Hmb, mup);
\]

\[
\text{TABLE}([\text{index\_char} = [1], \text{compts} = \text{vector}([k1, 2r\psi, -x\psi, 2y\psi - k3])])
\]

That is,

\[
k_1 \partial_u + k_3 \partial_y + \psi (2r \partial_r - x \partial_x + 2y \partial_y).
\]

5 Acknowledgments

Maple is a registered trademark of Waterloo Maple Inc.

6 References


[3] Steele J D 2002 Class. Quantum Grav. 19 259


[6] Fayos F and Sopuerta C F 2001 Class. Quantum Grav. 18 353