On Rational Representation of Stationary Axisymmetric Vacuum Metrics I: Cubic and Quartic Coordinates

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1. Statement of the problem

There is an abundance of literature (for a recent review see [1]) in the field of finding exact solutions of the Ernst equation (see (1.2) below), which is the cornerstone of the study of stationary axisymmetric vacuum metrics in General Relativity. However, an overview of the subject can be obtained by considering the vital rôle played by the static axisymmetric vacuum solutions (the so-called Weyl solutions) and, in particular, examining the techniques used to generalise such solutions to the corresponding stationary cases. The aim of this paper is to investigate the links between various Weyl solutions as well as between the Weyl solutions and their stationary counterparts. We will build up a picture of these links in the Weyl case, and attempt to carry this structure across to the stationary case. Furthermore, we will point out some open problems in these structures both for the Weyl metrics and also their stationary counterparts.

Despite the great success and power of (i) the symmetry techniques introduced by Geroch [2], Kinnersley and Chitre [3], and also Hauser and Ernst [4]; (ii) the inverse scattering method (see, e.g. [5]); and (iii) various Bäcklund transformations techniques [6] in generating solutions of the stationary axisymmetric vacuum equations, it is in general difficult to have control over the solution generated from a given seed solution by applying anyone of these methods. We hope that our viewpoint can help to ameliorate some of the deficiencies in these generating techniques.

In Weyl's canonical coordinates, the stationary axisymmetric metric is expressed in the form

$$ds^2 = e^{-2U}[e^{2V}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - e^{2U}(dt + Ad\varphi)^2$$ (1.1)

where $U$, $V$ and $A$ are functions of $\rho$ and $z$ only. The Ernst equation is written as

$$(E + \overline{E})(E_{\rho\rho} + \rho^{-1}E_{\rho} + E_{zz}) = 2(E_{\rho}^2 + E_z^2)$$ (1.2)

where the Ernst potential is

$$E = e^{2U} + i\omega$$ (1.3a)

The functions $V$ and $A$ are determined by

$$V_\zeta = \sqrt{2}\rho \frac{E_\zeta \overline{E}_\zeta}{(E + \overline{E})^2}, \quad A_\zeta = 2\rho \frac{(E - \overline{E})_\zeta}{(E + \overline{E})^2}$$ (1.3b)

where $\sqrt{2}\partial_\zeta = \partial_\rho - i\partial_z$.

The Ernst equation can be reformulated by introducing a new potential $\xi$ via

$$\xi = (1 - E)/(1 + E)$$ (1.4)
Then $\xi$ satisfies the differential equation

\[
(\bar{\xi} \xi - 1) (\xi_{\rho \rho} + \rho^{-1} \xi_{\rho} + \xi_{zz}) = 2 \bar{\xi} (\xi_{\rho}^2 + \xi_{z}^2)
\] 

(1.2')

In prolate spheroidal coordinates (see (2.1) below) the potential $\xi$ has proved essential in the generation of the Tomimatsu-Sato-Cosgrove [7] class of solutions.

It is straightforward to show that a stationary axisymmetric vacuum solution, which is analytic at spatial infinity (i.e. when the reciprocal of the radial coordinate $R := \sqrt{\rho^2 + z^2}$ equals zero), is asymptotically flat if the Ernst potential satisfies

\[
Re E = 1 - \frac{2M}{R} + O\left(\frac{1}{R^2}\right), \quad Im E = -\frac{2J}{R^2} \cos \theta + O\left(\frac{1}{R^3}\right)
\]

(1.5)

where $\tan \theta := \rho/z$. The real constants $M$ and $J$ are the total mass and the total angular momentum of the asymptotically flat solution, respectively.

When the time-like Killing vector $\partial_t$ in the metric (1.1) is hypersurface-orthogonal, the metric coefficient $A$ can be put equal to zero, and as a consequence the axisymmetric Killing vector $\partial_\phi$ is also hypersurface orthogonal and one has the Weyl metrics. In Weyl’s canonical coordinates, the field equations become

\[
\Delta U = \rho^{-1} (\rho U_{\rho})_{\rho} + U_{zz} = 0 \quad (1.6a)
\]

\[
V_{\rho} = \rho \left( U_{\rho}^2 - U_z^2 \right), \quad V_z = 2 \rho U_{\rho} U_z \quad (1.6b)
\]

Note that the three dimensional Laplace equation (1.6a) is the static form of the Ernst equation (1.2).

The “elementary” axisymmetric static metric has

\[
U = \frac{1}{2} \delta \log \left( R_c - (z - c) \right), \quad V = \frac{1}{2} \delta^2 \log \left( \frac{R_c - (z - c)}{R_c} \right) \quad (1.7)
\]

where $R_c^2 = \rho^2 + (z - c)^2$, with $c$ and $\delta$ real constants. If one interprets $U$ as a Newtonian potential (such an interpretation is purely for convenience in our classification and has no physical meaning in the context of General Relativity) then the “elementary” axisymmetric static metric corresponds to the Newtonian potential of a semi-infinite rod lying along the interval $[c, \infty)$ of the $z$-axis and having density $\delta$. Following the literature, we will call $\delta$ a deformation parameter. Due to the linearity of the Laplace equation (1.6a), we can superpose $n$ oriented semi-infinite rods of arbitrary densities, and still have a solution of (1.6a). Consequently,

\[
U = \sum_{i=1}^{n} \frac{\varepsilon_i \delta_i}{2} \log [R_i - \varepsilon_i (z - c_i)] \quad (1.8a)
\]

where $R_i^2 = \rho^2 + (z - c_i)^2$ and $\varepsilon_i$ gives the orientation of the $i$th rod:

\[
\varepsilon_i = \begin{cases} 
+1 & \text{rod in } +\text{ve } z \text{ direction;} \\
-1 & \text{rod in } -\text{ve } z \text{ direction.} 
\end{cases}
\]
Without any loss of generality, we assume the \( n \) real parameters \( c_i \) satisfy \( c_1 \leq c_2 \leq \ldots \leq c_n \). The \( n \) real constants \( \delta_i \) are the deformation parameters. The metric function \( V \) is given by quadrature and can be written explicitly in the form

\[
V = \sum_{i=1}^{n} \frac{\delta_i^2}{2} \log \left( \frac{R_i - \varepsilon_i(z - c_i)}{R_i} \right) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_i \delta_j \log \left[ \varepsilon_i R_i - \varepsilon_j R_j + (c_i - c_j) \right] \tag{1.8b}
\]

Both \( U \) and \( V \) are determined up to an additive constant.

One can construct asymptotically flat Weyl solutions from an even number \((2n)\) of semi-infinite rods by orientating all of them in the same direction such that the densities of each pair are of the same magnitude but of the opposite signs (i.e. \( \varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{2n} \) and \( \delta_{2i-1} = -\delta_{2i}, i = 1, 2, \ldots, n \)). Thus there are \( n \) finite segments of rods with various densities \((= \delta_{2i-1} \) for the \( i \)th rod). We call such asymptotically flat metrics the \( n \) Voorhees-Zipoy solutions, whereas the \( 2n \) and the \( 2n+1 \) semi-infinite rods solutions with arbitrary densities and orientations will be referred to as the generalised \( n \) Voorhees-Zipoy solutions and the generalised accelerated \( n \) Voorhees-Zipoy solutions respectively. One can show, after some complicated algebra, that in the \( n \) Voorhees-Zipoy solutions equations (1.8) become

\[
U = \sum_{i=1}^{n} \varepsilon_{2i-1} \delta_{2i-1} \frac{1}{2} \log \left( \frac{R_{2i-1} + R_{2i} + (c_{2i-1} - c_{2i})}{R_{2i-1} + R_{2i} - (c_{2i-1} - c_{2i})} \right) \tag{1.9a}
\]

and

\[
V = \sum_{i=1}^{n} \frac{\delta_{2i-1}^2}{2} \log \left( \frac{(R_{2i-1} + R_{2i})^2 - (c_{2i-1} - c_{2i})^2}{4R_{2i-1} R_{2i}} \right) + \Gamma_0 \tag{1.9b}
\]

where

\[
\Gamma_0 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\delta_{2i-1} \delta_{2j-1}}{2} \times \log \left[ \frac{(R_{2i-1} + R_{2j})^2 - (c_{2i-1} - c_{2j})^2}{(R_{2i-1} + R_{2j})^2 - (c_{2i-1} - c_{2j})^2} \right] \right]
\]

\[
\cdot \left[ \frac{(R_{2i-1} + R_{2j-1})^2 - (c_{2i-1} - c_{2j-1})^2}{(R_{2i-1} + R_{2j})^2 - (c_{2i-1} - c_{2j})^2} \right] \right]
\]

We have chosen the integration constants such that both \( U \) and \( V \) vanishes when \( \rho \) approaches infinity. Consequently the asymptotic form of the Minkowski metric is given in cylindrical polar coordinates \((t, \rho, z, \varphi)\). We have to emphasise that there are many ways to construct distinct asymptotically flat Weyl solutions from \( n \) semi-infinite rods \((n \geq 2)\).

Many well-known Weyl solutions have the deformation parameter \( \delta_i = 1 \) or \(-1\). For instance, the one semi-infinite rod solutions with \( \delta = 1 \) and \( \varepsilon = \pm 1 \) represent the Minkowski spacetime; the Schwarzschild solution is a Voorhees-Zipoy [8] solution with \( \delta_1 = 1 \); the C-metric, which can be interpreted as a pair of uniformly accelerating Schwarzschild particles, is a three semi-infinite rod solution with \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 \) and \( \delta_1 = \delta_2 = -\delta_3 = 1 \); and the Bach-Weyl [9] two Schwarzschild particles metric has \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 \) and \( \delta_1 = \delta_3 = -\delta_2 = -\delta_4 = 1 \). Hence one can use a Minkowski one semi-infinite rod solution to build up a hierarchy of
Weyl solutions by (1) adding to it another Minkowski one rod solution, or (2) multiplying it by an arbitrary deformation parameter $\delta$. (Compare with figure 1.)

Most of the Weyl solutions have previously been given in the Weyl canonical coordinates, the exception being the C- and the deformed C-metrics. We will establish the connection between the cubic polynomial coordinates in the C-metric and the Weyl canonical coordinates in Theorem 1 (cf. Bonnor-Hoenselaers [10]). The deformed C-metrics are obvious in the Weyl canonical coordinates once this theorem is given, but the corresponding form in cubic polynomial coordinates are more obscure and are given in section 2. It is straightforward to verify that our results coincide with those of Plebański and Kerr quoted in [11].

The generalisation of the Weyl solutions to their stationary axisymmetric vacuum counterparts is a formidable task, due to the non-linearity of the Ernst equation (1.2) in these cases. Figure 2 is the “stationary counterpart” to figure 1, most of these solutions are in principle known, from application of HKX [6] transformations, for example; few seem to have been explicitly investigated.

The outstanding feature in figure 2 is the two rod case, i.e. the Kerr and the Tomomatsu-Sato-Cosgrove class, which has only one (arbitrary) deformation parameter. This case has been so thoroughly investigated because these solutions can be expressed easily in prolate spheroidal coordinates. The prolate spheroidal coordinates in our picture are the quadratic counterparts to the cubic coordinates in the C-metric. Our aim is to look for a similar ansatz which can extend the twisting-C metric of Kinnersley to its deformed generalisations, and
similarly for the double-Kerr. Partial results are given below.

2: Basic Results

We begin by looking at the relationship between the Weyl canonical coordinates and the prolate spheroidal coordinates in the two rod cases. The key part of the transformation is the (flat) 2-metric

$$\frac{d\rho^2 + dz^2}{\sigma^2 R_+ R_-} = \frac{dx^2}{x^2 - 1} - \frac{dy^2}{y^2 - 1}$$

where $x$ and $y$ are the prolate spheroidal coordinates. The transformation is given by

$$\rho^2 = \sigma^2(x^2 - 1)(1 - y^2)$$
$$z = \sigma xy$$
$$2\sigma x = R_+ + R_-$$
$$2\sigma y = R_+ - R_-$$
$$R_\pm = \rho^2 + (z \pm \sigma)^2$$

The Schwarzschild solution has Ernst potential

$$e^{2U} = \frac{(R_+ + R_- - 2\sigma)}{(R_+ + R_- + 2\sigma)} = \frac{(x - 1)}{(x + 1)}$$
In prolate spheroidal coordinates the potential $\xi$ has the very simple form $\xi = 1/x$. The Voorhees-Zipoy solutions have

$$e^{2U} = \left(\frac{R_+ + R_- - 2\sigma}{R_+ + R_- + 2\sigma}\right)^\delta = \left(\frac{x-1}{x+1}\right)^\delta.$$ 

Similarly, the $\xi$ potential is

$$\xi = \frac{(x+1)^\delta - (x-1)^\delta}{(x+1)^\delta + (x-1)^\delta}.$$

The extension of these metrics to include the rotation parameter leads to the Kerr (with $\delta = 1$), Tomimatsu-Sato (integral $\delta$) and Cosgrove (arbitrary $\delta$) metrics. For example the Kerr metric in prolate spheroidal coordinates has the simple $\xi$ potential $$(px - iqy)^{-1}$$ where $p^2 + q^2 = 1$. Here $q$ represents the rotation parameter: if $q = 0$ the metric is Schwarzschild (see [14]). In the Weyl canonical coordinates, the Ernst potential of the Kerr metric takes the form

$$\xi = \frac{e^{-i\theta}(R_+ - m) + e^{i\theta}(R_- - m)}{e^{-i\theta}(R_+ - m) - e^{i\theta}(R_- - m)}$$

where $e^{i\theta} = p + iq$ and $\sigma = mp$.

Now we consider the case of a 2-metric where the quadratic in equation (2.1) is replaced by a cubic, i.e.

$$\frac{d\rho^2 + dz^2}{R_1 R_2 R_3} = \frac{dx^2}{P_3(x)} - \frac{dy^2}{P_3(y)}$$

(2.2)

where $P_3$ is some cubic and $R_i^2 = \rho^2 + (z - c_i)^2$.

The case where the cubic is replaced by a quartic is closely linked to this case because of the following result [15]:

**Theorem 1**

There are coordinate systems $(x, y)$, $(X, Y)$, $(\rho, z)$ and $(\sigma, Z)$ such that

$$\frac{dx^2}{P_3(x)} - \frac{dy^2}{P_3(y)} = \frac{dX^2}{P_4(X)} - \frac{dY^2}{P_4(Y)} = \frac{dp^2 + dz^2}{R_1 R_2 R_3} = \frac{d\sigma^2 + dZ^2}{S_1 S_2 S_3 S_4}$$

where $P_4$ is a quartic and $S_i^2 = \Sigma^2 + (Z - C_i)^2$ for some constants $C_i$.

The result can be proved elegantly by a straightforward adaptation of a result due to Lagrange given in detail in Cayley [16]; essentially one uses a fractional linear transformation that maps one of the roots of the quartic to infinity.

In these coordinates the C-metric has the Ernst potential

$$e^{2U} = P_3(y)/(x - y)^2 = \frac{(R_3 - (z - c_3))(R_1 - (z - c_1))}{(R_2 - (z - c_2))}$$

where $c_1 > c_2 > c_3$ are the roots of the cubic $P_3$ and correspond to the ends of the rods, as above. The function $V$ can be found by applying the transformation equations to (1.8b).
The deformed generalisations of the C-metric in Weyl coordinates can be found from (1.8). In the cubic coordinates the Ernst potential takes the form

\[ e^{2U} = \frac{2^{2\delta_3}}{(y-x)^{4\delta_3}}(c_1 - c_2y)^{2\delta_1}(y^2 - c_2^2)^{2\delta_3}(c_1 - c_2x)^{2(\delta_3 - \delta_1)} \]

where we have arranged that \( c_3 = -c_2 \), and have assumed that the density \( \delta_2 = -\delta_3 \). These deformation parameters are, of course, continuous.

The extension of this metric to include rotation parameters using, e.g. HKX transformations, give rise to the Kinnersley metric and its generalisations with various deformation parameters. Note that the Kinnersley metric is written with a quartic, but from Theorem 1 it can be written in terms of a cubic — although the expression is not as pleasant.

The importance of the prolate spheroidal coordinates is that they allow the “deformed Kerr” (Tomimatso-Sato-Cosgrove) metrics to be expressed in terms of rational functions of these coordinates, and hence in a way far better suited to computer (and human) manipulation. The application of the cubic and quartic coordinates to the generalised Kinnersley metrics is expected similarly to lead to more tractable expressions in these cases.

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**References**


