Abstract

A general discussion of conformal vector fields in space-times is given. Amongst the topics considered are the maximum dimension of the conformal algebra for space-times which are not conformally flat, the nature of conformal isotropies and a new approach to the theorem of Bilyalov and Defrise-Carter concerning the reduction of the conformal algebra to a Killing or homothetic algebra. Some deficiencies in the original statements of this theorem are discussed (with reference to a general class of counterexamples) and corrected. The proof offered is geometrical in nature and has the advantage of displaying some of the more general features and properties of conformal vector fields and the ways in which they can differ from Killing vector fields.

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On Conformal Vectors

1: Introduction and Notation

This paper investigates, in a geometrical manner, some theoretical aspects of the existence of a Lie algebra of conformal vector fields on a space-time manifold. The topics treated include a discussion of the nature of a conformal isotropy at points of a space-time, a study of the restrictions on the dimension of this Lie algebra and a more precise, extended statement and alternative proof of an important theorem due to Bilyalov [1] and Defrise-Carter [2].

Let $M$ be a space-time with Lorentz metric $g$ of signature $\left(+,+,+,−\right)$. A global vector field $X$ on $M$ is called conformal if in any coordinate domain of $M$ one has

$$X_{;ab} = \phi g_{ab} + F_{ab} \quad (\iff \mathcal{L}_X g = 2\phi g)$$

where $F$ is the conformal bivector ($F_{ab} = -F_{ba}$), $\phi$ is a real valued function on $M$ (the divergence of $X$), $\mathcal{L}$ denotes a Lie derivative, Latin indices take the values $1,2,3,4$ and a semi-colon denotes a covariant derivative with respect to the Levi-Civita connection of $g$. The manifold $M$ and all structures on $M$ are assumed smooth. The vector field $X$ is called proper conformal if $\phi$ is not constant on $M$ and homothetic if $\phi$ is constant on $M$ (proper homothetic if $\phi=\text{constant} \neq 0$ on $M$). If $\phi \equiv 0$ on $M$, $X$ is called Killing. If $\phi_{;ab} = 0$ in every chart of $M$, $X$ is called special conformal. The set of all conformal vector fields on $M$ is a finite-dimensional Lie algebra (the conformal algebra) under the Lie bracket operation and the subsets of special conformal, homothetic and Killing vector fields constitute subalgebras which are named accordingly.

It will be assumed throughout this paper that all space-times $M$ considered admit no local (non-globalisable) conformal vector fields. Thus if $X$ is a conformal vector field on some non-empty open subset $U$ of $M$ then $X$ is the restriction to $U$ of some (global) conformal vector field on $M$. Such an assumption is usually made implicitly in the literature. It will also be
assumed throughout that $M$ is non-flat, in the sense that the curvature tensor will be assumed not to vanish over any non-empty open subset of $M$.

The conformal algebra $\mathcal{A}$ of $M$ gives rise to a local group $G_{\mathcal{A}}$ generated by the local conformal diffeomorphisms of $M$ which arise from the members of $\mathcal{A}$. Thus $G_{\mathcal{A}}$ consists of local maps (where they are defined) of the form

$$p \rightarrow \chi^1_{t_1} (\chi^2_{t_2} (\ldots \chi^k_{t_k}(p) \ldots)) \quad p \in M \tag{6}$$

where $k \in \mathbb{N}$, $t_1, \ldots, t_k \in \mathbb{R}$, $\chi^1_{t}, \ldots, \chi^k_{t}$ are the local conformal diffeomorphisms arising from members $X_1, \ldots, X_k$ of $\mathcal{A}$ and where the usual formal rules for composition of maps and inverses are taken. The orbits in $M$ under $G_{\mathcal{A}}$ are the equivalence classes in $M$ arising from the equivalence relation $p_1 \sim p_2 \iff \exists a \in G_{\mathcal{A}}$ such that $a(p_1) = p_2$ ($p_1, p_2 \in M$). Excluding trivial cases these orbits can be given the structure of connected submanifolds of $M$ and are maximal integral manifolds of the generalised distribution on $M$ which is naturally defined by $\mathcal{A}$ ([5,6] and for an explanation in the present context, see [4]).

If $X$ is a (not identically zero) conformal vector field on $M$ with a zero at $p$, $X(p) = 0$, the associated 1-parameter local group of local diffeomorphisms $\chi_t$ fix the point $p$ ($\chi(p) = p$ for all $t$ where $\chi_t$ is defined). If the function $\phi$ associated with $X$ as in (1) satisfies $\phi(p) = 0$, the fixed point $p$ is called isometric (with respect to $X$), whilst if $\phi(p) \neq 0$, $p$ is called homothetic (with respect to $X$). The algebraic type (timelike, spacelike, null, non-simple) of the conformal bivector $F$ of $X$ at a zero $p$ plays an important rôle in what is to follow. It is noted here that the statement that $X$ is a conformal vector field, the nature (isometric or homothetic) of a zero $p$ of $X$ and the algebraic type of the conformal bivector at a zero $p$ of $X$ are all conformally invariant. Further details on the fixed point structure of conformal vector fields can be found in [7].
2: Preliminary results

In this section some preliminary results concerning space-times admitting conformal vector fields will be collected together.

(i) If the Weyl tensor is nowhere zero on some open subset $U$ of $M$, if $\ell$ is a principal null direction of the Weyl tensor on $U$ and if $X$ is a conformal vector field on $U$ then $\mathcal{L}_X \ell \propto \ell$ on $U$ (see e.g. [8]).

This follows from the conformal nature of the local diffeomorphisms associated with $X$ and the finiteness ($\leq 4$) of the number of principal null directions at each point of $U$.

(ii) Let $\mathcal{A}$ be the Lie algebra of conformal vector fields on $M$ and suppose that the commutator subalgebra of $\mathcal{A}$ consists entirely of Killing vector fields. Then if $X$ is any member of $\mathcal{A}$ with corresponding function $\phi$ as in (1) and $Y$ is any Killing vector field in $\mathcal{A}$, it follows that $\phi_a Y^a = 0$.

The proof of this is immediate by applying the commutator to $X$ and $Y$ and using the result that $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$.

(iii) A generalised plane wave space-time is a space-time $M$ with a global coordinate system $u, v, x, y$ and metric

$$ds^2 = dx^2 + dy^2 + 2dudv + [\alpha(u)x^2 + \beta(u)y^2 + \gamma(u)xy]du^2. \quad (7)$$

The global null vector field with components $\ell^a = g^{ab}u_b$ is covariantly constant. The energy-momentum tensor at each $p \in M$ is either zero or of Segre type $\{(211)\}$ with zero eigenvalue (the null fluid type) and the Petrov type at each $p \in M$ is either O or N. Suppose the Petrov type is N everywhere (this will be the relevant case later). Then $M$ admits exactly five independent Killing vector fields and one homothetic vector field all of which have the property of being tangent to the null hypersurfaces of constant $u$, together with at most one other (not necessarily proper) independent conformal vector field not tangent to these hypersurfaces and so the dimension of the conformal algebra is 6 or 7. (A more general result will be given in the next section.) For later use it is noted that if $g$ is a generalised plane wave metric which is of Petrov type N everywhere then for each $p \in M$ there is an open neighbourhood $U$ of $p$ and a real valued function $\sigma(u)$ on $U$ such that the (generalised plane wave) metric $e^{2\sigma}g$ on $U$ has a conformal Lie algebra consisting entirely of homothetic (and Killing) vector fields $[2,7]$. A generalised plane wave space-time can be locally characterised as a non-flat space-time admitting a homothetic vector field with a non-isolated zero $[9,8]$. Also a non-flat space-time admitting a 5-dimensional Lie algebra of Killing vector fields with 3-dimensional null orbits everywhere (see e.g. [19]), or a Lie algebra of Killing vector fields with 3-dimensional null orbits everywhere together with a proper homothetic vector field everywhere tangent to these orbits...
is locally isometric to a generalised plane wave. Plane waves (in vacuum) were first discussed in detail in [10].

(iv) Let $M$ be a space-time which admits an $r$-dimensional Lie algebra $\mathcal{H}$ of homothetic vector fields and suppose that at least one member of $\mathcal{H}$ is proper homothetic. Then the subalgebra $\mathcal{K}$ of $\mathcal{H}$ consisting of Killing vector fields has dimension $r - 1$. Now suppose that in some non-empty open subset $U$ of $M$ the subspace of the tangent space $T_p M$ to $M$ at $p$ spanned by members of $H$ at $p$ has, at each $p \in M$, the same dimension $q$ and, if $q < 4$, the same type (timelike, spacelike or null) and that the same is true for the Killing subalgebra $G$ where the corresponding dimension is $q' \leq q$. Thus the members of $\mathcal{H}$ (respectively $G$) give rise to $q$- (respectively $q'$-) dimensional orbits in $U$. If $r = 11$ then $U$ is flat and clearly $r = 10$ and $r = 9$ are impossible. It also follows that [11] if $r = 7$, $U$ is (part of) a type $N$ homogeneous generalised plane wave or a conformally flat non-homogeneous generalised plane wave or a Robertson-Walker space-time (or its “timelike” equivalent), that if $r = 6$, $U$ is a type $N$ non-homogeneous generalised plane wave and so, necessarily $q = q' = 3$ and that if $r = 5$, necessarily $q = 4$ and $q' = 3$. More details can be found in [11] and are based on results in [8].

3: Conformal Isotropy

Let $X$ be a not identically zero conformal vector field on $M$ with a zero at $p$. If $p$ is an isometric fixed point of $X$ the Weyl tensor at $p$ is either Petrov type D, N or O (cf. [10]). If it is of type N, the conformal bivector at $p$ is a linear combination of the bivectors $\ell_{[axb]}$ and $\ell_{[ayb]}$ where $\ell$, $x$ and $y$ are tangent vectors at $p$ with $\ell$ spanning the (uniquely determined quadruply repeated) principal null direction of the Weyl tensor at $p$ and $x$ and $y$ are orthogonal unit spacelike vectors orthogonal to $\ell$. If it is type D the conformal bivector at $p$ is a linear combination of the bivectors $\ell_{[anb]}$ and $x_{[ayb]}$ where $\{\ell, n, x, y\}$ is a null tetrad at $p$ with $\ell$ and $n$ spanning the (uniquely determined doubly repeated) principal null directions of the Weyl tensor at $p$. These results follow directly from (4) applied at $p$. If, however, $p$ is a homothetic fixed point of $X$ then all the Weyl eigenvalues (Petrov scalars) must vanish at $p$ and the Petrov type at $p$ is thus III, N or O [8]. If the Petrov type is III at $p$ then the conformal bivector at $p$ takes the form $4\phi(p)\ell_{[anb]}$ where $\ell$ spans the (unique triply) repeated and $n$ the (unique) non-repeated principal null direction of the Weyl tensor at $p$ and $\ell^a n_a = 1$. If the Petrov type at $p$ is N the conformal bivector at $p$ is of the form $2\phi(p)\ell_{[anb]}$ where $\ell$ spans the repeated principal null direction of the Weyl tensor at $p$, $n$ is a null vector at $p$ and $\ell^a n_a = 1$.

With these results we can describe all possible conformal isotropies in a straightforward way.

Lemma 1. Let $X_1, \ldots, X_n$ be independent conformal vector fields on $M$ each of which vanishes
at $p$. By taking appropriate linear combinations of these vector fields one can arrange that all but at most one of them have $p$ as an isometric fixed point and of the conformal bivectors at $p$ corresponding to these latter at most two are independent. It follows that, after taking suitable linear combinations of these vector fields, one can assume that the corresponding functions $\phi$ and conformal bivectors $F$ are zero at $p$ for $X_q$ with $q \geq 4$.

The proof of this follows from the results of the first paragraph of this section.

**Lemma 2.** Let $X$ be a not identically zero conformal vector field on $M$ with a zero at $p$ and suppose that the corresponding function $\phi$ and bivector $F$ (see (1)) also vanish at $p$. Then the Weyl tensor vanishes at $p$ [12].

To prove this one notes that the conditions $X^a(p) = 0$, $\phi(p) = 0$ and $F_{ab}(p) = 0$ imply that $X_{a;b}(p) = 0$ (from (1)), $\phi_{a;b}(p) = 0$ from (3) and lead to a simple expression for $F_{abc;e}(p)$ from (2). These, in turn, lead to the relation

$$X_{a;bc} = g_{ab}\phi_c + g_{ac}\phi_b - g_{bc}\phi_a$$

(8)

at $p$. Then on covariantly differentiating (4) and evaluating at $p$ one finds that at $p$

$$-C_{bed} X_e^{a;cf} + C_{aed} X_c^{e;bf} + C_{bed} X_e^{c;bf} + C_{bec} X_e^{c;df} = 0.$$  (9)

On substituting (8) into (9), contracting over the indices $a$ and $f$ and rearranging using $C_{a[bcde]} = 0$ one arrives at the relation

$$C_{abcd}\phi^d = 0 \quad (\iff C_{ab[cd}\phi_e] = 0)$$

(10)

at $p$ where the bracketed (equivalent) result follows by taking the dual of the first and vice versa. A substitution of the two results of (10) and (8) back into (9) then gives $C_{abcd}\phi_e = 0$ at $p$. Now a conformal vector field $X$ on $M$ is uniquely determined by specifying $X^a$, $F_{ab}$, $\phi$ and $\phi_a$ (equivalently $X^a$, $X^a_{;b}$ and $X^a_{;c}$) at any point of $M$ and so, since $X$ is not identically zero on $M$, it follows that $\phi_a \neq 0$ at $p$ and so the Weyl tensor vanishes at $p$. It is remarked that a vector field with the conditions of lemma 2 is necessarily proper conformal.

**Theorem 1.** The maximum dimension of the Lie algebra of conformal vector fields on a space-time which is not conformally flat is 7.

The proof follows immediately from the two lemmata. It is remarked that this maximum is achieved in a certain of the type N generalised plane wave space-times (cf. section 6(e)). The theorem shows that if the Weyl tensor is non-zero at $p \in M$ then the conformal isotropy at $p$ consists, at most, of “three parameters”, this maximum being 3 for type N, 2 for type D and 1 for type III.
The maximum dimension of the Lie algebra of global vector conformal vector fields on a space-time is 15 but such space-times are necessarily conformally flat. Further considerations of dimension regarding the homothety and special conformal Lie algebras can be found in [11] and [13] respectively whilst a more detailed discussion of the behaviour of conformal vector fields in the vicinity of a zero of such a vector field can be found in [7].

4: General Conformal Algebras

Conformal vector fields are difficult to study because they lack the “linearity” properties that are enjoyed, for example, by affine vector fields (including homothetic and Killing vector fields). Since any vector field $X$ can be “linearised” in some neighbourhood of any $p \in M$ where $X(p) \neq 0$, the difficulties with conformal vector fields appear in neighbourhoods of their zeros. (A more complete discussion of these difficulties and a method of avoiding them in many cases can be found in [7].)

Another method of avoiding the difficulties with conformal vector fields has been suggested [1,2]. In this approach one tries to find a conformal change of the metric on $M$, $g \rightarrow g' = e^{2\sigma} g$ for some function $\sigma$ on $M$, such that the Lie algebra of conformal vector fields on $M$ with respect to $g$ become a Killing (or at least a homothetic) algebra with respect to $g'$. The combined results of [1,2] say that the conformal Lie algebra can be reduced by this method to a homothetic algebra provided the space-time is not conformally flat and that the reduction to a Killing algebra can be achieved when the space-time is neither conformally flat nor a generalised plane wave. As stated, the theorems in [1,2] lack precision and counterexamples can be found [7]. A general class of counterexample is given by those non-conformally flat space-times admitting a proper homothetic vector field with an isolated zero. In fact the zeros of a proper homothetic vector field are either isolated or constitute null geodesics. In the latter case one essentially has generalised plane waves ([9,8] — see section 2(iii)) but in the former case one has counterexamples to the above theorems, provided the space-time is not conformally flat, as is shown by a consideration of equation (11) below in a neighbourhood of the zero.

A specific example of such a space-time was given and discussed in [7]. A number of clauses are required to clarify certain implicit assumptions made in [1,2] as well as a tightening of the conditions and conclusions of the theorem. The aim of the next two sections is to achieve these ends and to give a precise statement (and slightly extension) of the important results in [1,2] and to offer a more geometrical proof which, it is believed, clarifies what is going on.

Suppose that a space-time $M$ admits a conformal algebra $\mathcal{A}$. The method of the previous paragraph, roughly speaking, involves finding a function $\sigma$ such that

$$\sigma_a X^a + \phi = 0$$

(11)
is simultaneously satisfied for all $X \in \mathcal{A}$ where $\phi$ is the divergence of $X$. This would ensure that $\mathcal{A}$ is a Killing algebra for the metric $g' = e^{2\sigma} g$. If (11) can be solved when the right hand side is replaced by a constant for each of the members of $\mathcal{A}$ then $\mathcal{A}$ would be a homothetic algebra for $g'$. However, one must pay careful attention to the nature of the orbits associated with $\mathcal{A}$, the local nature of any such solution $\sigma$ and the Petrov type. Clearly, if $X$ is a conformal vector field which is non-zero at $p \in M$ then there exists an open neighbourhood $U$ of $p$ and a function $\sigma : U \to \mathbb{R}$ satisfying (11). This can be strengthened to the case when $\mathcal{A}'$ is a subalgebra of the conformal algebra $\mathcal{A}$ with $\dim \mathcal{A}' = m \ (1 \leq m \leq 4)$ and where $\mathcal{A}'$ has a basis $X_1, \ldots, X_m$ of conformal vector fields which, when evaluated at $p \in M$ are independent members of $T_p M$; again an open neighbourhood $U$ and function $\sigma$ could be found as above with $\sigma$ satisfying (11) simultaneously for all members of $\mathcal{A}'$ [14]. (The fact that $\mathcal{A}'$ is a subalgebra rather than a subspace is important here.) The problems in solving the system (11) occur when members of $\mathcal{A}$ have zeros.

Throughout the next two sections it will be assumed (i) that the conformal algebra $\mathcal{A}$ of $M$ is such that the subspace $\mathcal{A}_p$ of $T_p M$ spanned by the members of $\mathcal{A}$ evaluated at $p \in M$ is the same dimension and (if this dimension is less than 4) the same type (timelike, spacelike or null) at each $p \in M$, (ii) that the Petrov type (I, II, III, D, N, O) is the same at each $p \in M$ (so that one can speak of the Petrov type of $M$) and (iii) that $M$ admits no local (non-globalisable) conformal vector fields as described in section 1. The assumption (i) guarantees, by Frobenius’ Theorem, that the orbits are everywhere of dimension $k = \dim \mathcal{A}_p$ and of the same type. Suppose that $\dim \mathcal{A} = r \geq k$ and let $X_\alpha$ for $1 \leq \alpha \leq k$ be independent members of $\mathcal{A}$ which span $\mathcal{A}_p$ for some $p \in M$. Then the $X_\alpha$ span $\mathcal{A}_q$ for each $q$ in some open neighbourhood $U$ of $p$. Suppose also that $X_\mu$ for $k + 1 \leq \mu \leq r$ extend the $X_\alpha$ to a basis for $\mathcal{A}$. Then one may write $X_\mu = \psi^{1}_\mu X_1 + \ldots + \psi^{k}_\mu X_k$ in $U$ where the $\psi^\alpha_{\mu}$ are smooth functions on $U$. The second equation in (1) then gives, for the $X_\alpha$ $(1 \leq \alpha \leq k)$ and the $X_\mu$ $(k + 1 \leq \mu \leq r)$ respectively, in a local coordinate neighbourhood $N$ with $p \in N \subseteq U$ [1]

\begin{align}
g_{ab,c}X_\alpha^c + g_{cb}X_\alpha^c, a + g_{ac}X_\alpha^c, b = 2\phi_\alpha g_{ab} \\
g_{cb}\psi^\alpha_{\mu,a}X_\alpha^c + g_{ac}\psi^\alpha_{\mu,b}X_\alpha^c = \Delta_\mu g_{ab}
\end{align}

(12) (13)

where $\alpha$ and $\mu$ range as above, $X_\alpha^c$ are the components of $X_\alpha$ in $N$, $\phi_\alpha$ is the divergence of $X_\alpha$, a summation over $\alpha$ is implied in (13) and where (smooth) functions $\Delta_\mu$ on $N$ are defined by

$$\Delta_\mu = \phi_\mu - \psi^\alpha_{\mu}\phi_\alpha \quad (1 \leq \alpha \leq k, \ k + 1 \leq \mu \leq r).$$

(14)

It can then be shown [1,12] that if $\Delta_\mu \equiv 0$ on $N$ there exists an open set $N'$ such that $p \in N' \subset N$ and a function $\sigma : N' \to \mathbb{R}$ such that, in $N'$, $\sigma$ satisfies (11) for all $X \in \mathcal{A}$ (restricted to $N'$),
that is, the members of $\mathcal{A}$ when restricted to $N'$ constitute the Killing algebra for the metric $e^{2\sigma}g$ on $N'$. A consequence of this result is ([1] cf. [11]) that if the orbits associated with $\mathcal{A}$ are neither 4-dimensional nor 3-dimensional and null, a local solution $\sigma$ of (11) for all $X \in \mathcal{A}$ about any point $p \in M$ always exists. This follows because at any $p \in M$ one may always choose a member $v \in T_pM$ which is orthogonal and non-null. Transvection of (13) by $v^av^b$ then gives $\Delta_\mu(v^av^c) = 0$. It follows that $\Delta_\mu \equiv 0$ on $M$ and the result is proven.

The Jacobi identities satisfied by the members of $\mathcal{A}$ imply certain differential identities for the functions $\Delta_\mu$ ([1] cf. [12]). Of special importance here is the case when $\dim \mathcal{A}_p = 4 \ \forall \ p \in M$. In this case there is a single orbit under $G_{\mathcal{A}}$, namely $M$ itself and the $\Delta_\mu$ satisfy

$$\Delta_{\mu,a} = B^\nu_{\mu a} \Delta_\nu$$

where the $B^\nu_{\mu a}$ are (smooth) functions on $N$ and $k + 1 \leq \mu, \nu \leq r$.

**Lemma 3.** Suppose the conformal algebra $\mathcal{A}$ of $M$ satisfies $\dim \mathcal{A}_p = 4 \ \forall \ p \in M$ and suppose $X_1, \ldots, X_4$ are members of $\mathcal{A}$ which span $T_{p'}M$ at some $p' \in M$. Then on extending $X_1, \ldots, X_4$ to a basis for $\mathcal{A}$ and using the above notation, if $\Delta_\mu = 0$ at $p'$ for $5 \leq \mu \leq r = \dim \mathcal{A}$ then there exists an open neighbourhood $U$ of $p'$ on which $\Delta_\mu \equiv 0$ for $5 \leq \mu \leq r$.

To prove this let $N$ be any connected (and hence path connected) coordinate domain containing $p'$. Then any point $q$ of $N$ can be joined to $p'$ by a smooth curve lying in $N$. Then for any such curve $x: I \to N$, $x: t \to x^a(t)$, where $I$ is an open interval in $\mathbb{R}$, $(0, 1) \subset I$, $x(0) = p'$, $x(1) = q$ and the $x^a$ are the coordinates functions of $N$, define a (smooth) function $X: I \to \mathbb{R}^{r-4}$ by $X(t) = (\Delta_5(x(t)), \ldots, \Delta_r(x(t)))$. Then $\dot{X} = (x^aP_a)X$ where a dot denotes differentiation with respect to $t$ and

$$P_a(t) = \begin{pmatrix}
B^5_{5a}(x(t)) & \cdots & B^r_{5a}(x(t)) \\
\vdots & \ddots & \vdots \\
B^5_{ra}(x(t)) & \cdots & B^r_{ra}(x(t))
\end{pmatrix}$$

(16)

It then follows (see e.g. [15] chapter 15, theorem 1.2) since $X(a) = 0$ that $X(t) = 0$ on $I$ and so $X(q) = 0$. Hence $\Delta_\mu \equiv 0$ for $5 \leq \mu \leq r$ on $N$. 
5: Reduction of the Conformal Algebra

In this section a more precise statement and proof of the theorem of Bilyalov and Defrise-Carter [1,2] will be given.

**Theorem 2.** Let $(M, g)$ be a space-time which admits an $r$-dimensional conformal algebra $\mathcal{A}$ and suppose that the Petrov type and the dimension and nature of the orbits associated with $\mathcal{A}$ are the same for each $p \in M$ and that $M$ admits no local (non-globalisable) conformal vector fields. Then for each $p \in M$ there exists an open neighbourhood $U$ of $p$ and a function $\sigma: U \to \mathbb{R}$ such that $\mathcal{A}$ (restricted to $U$) is a Lie algebra of special conformal vector fields on $U$ with respect to the metric $g' = e^{2\sigma} g$ on $U$. If the Petrov type is not O, the above local scaling function $\sigma$ can always be chosen so that $\mathcal{A}$ restricts to a Lie algebra of homothetic vector fields with respect to $g'$ on $U$ and if $(M, g)$ is not locally conformally related to a generalised plane wave space-time about any $p \in M$ the above local scaling can always be chosen so that $\mathcal{A}$ restricts to a Lie algebra of Killing vector fields with respect to $g'$ on $U$.

To prove this suppose firstly that $M$ is of Petrov type I, II or D. Then for any $p \in M$ there exists an open neighbourhood $U$ of $p$ and a nowhere zero real or complex (necessarily smooth [16]) Petrov scalar (Weyl eigenvalue) on $U$. This can be used to construct a solution $\sigma$ to (11) on $U$ for all $X \in \mathcal{A}$ by the method given in [2] and so one need only consider the cases when the Petrov type is III, N or O. Clearly if $M$ is type O there can be no solution for $\sigma$ of (11) for all $X \in \mathcal{A}$ by an elementary consideration of dim $\mathcal{A}$. The stated result when $M$ is of type O follows since then $g$ is locally conformally related to a flat metric and in a flat space-time all conformal vector fields are, from (3), necessarily special conformal.

Suppose that $M$ is of Petrov type III. It follows from the results of section 3 that if $p \in M$ is a zero of any non trivial member of the conformal algebra $\mathcal{A}$ of $M$, it is necessarily a homothetic fixed point and any non trivial isotropy is “1-parameter”. Also, if dim $\mathcal{A}$ equals the dimension of the orbits of $\mathcal{A}$ then, as pointed out above, a local solution $\sigma$ of (11) for all $X \in \mathcal{A}$ can be found about any $p \in M$. It follows that the only cases to consider are when dim $\mathcal{A} = 5$ (with a single orbit $M$) and dim $\mathcal{A} = 4$ (and the orbits 3-dimensional and null). Now this latter case is easily eliminated. To see this one notes that the smooth null 3-dimensional distribution on $M$ determined by $\mathcal{A}$ gives rise to a smooth null vector field $k$ defined in some open neighbourhood $U$ of any $p \in M$ and which is tangent to the orbits everywhere in $U$ [16]. Also, there exists a non-trivial member $X$ of $\mathcal{A}$ which vanishes at $p$ and is orthogonal to $k$ everywhere in $U$. Hence, in $U$, one covariantly differentiates the equation $k^a X_a = 0$, substitutes for $X_{a;b}$ from (1) and obtains $F_{ab} k^b = \phi(p) k_a$ at the zero $p$ of $X$. The first paragraph of section 3 then shows that this contradicts the Petrov type III assumption.

So consider the case when dim $\mathcal{A} = 5$ and choose four independent members $X_1, \ldots, X_4$ of
A such that, at \( p \)

\[
\begin{align*}
X_1(p) &= \ell \\
X_2(p) &= n \\
X_3(p) &= x \\
X_4(p) &= y \\
\end{align*}
\]  

(17)

where \((\ell, n, x, y)\) is a null tetrad at \( p \) with \( \ell \) (respectively \( n \)) the repeated (respectively the non-repeated) principal null direction of the Weyl tensor at \( p \). These can be extended to a basis for \( A \) by the addition of a conformal vector field \( X_5 \) which can be chosen to vanish at \( p \) and so satisfies, at \( p \) (see the first paragraph of section 3),

\[
X_{5a;b} = \phi(p)g_{ab} + 4\phi(p)\ell_{[a}n_{b]} \quad (\phi(p) \neq 0),
\]

(18)

One may now use the Schmidt method (see e.g. [19]) to write down the following Lie brackets in some neighbourhood of \( p \).

\[
\begin{align*}
[X_1, X_5] &= 3\phi(p)X_1 + AX_5 \\
[X_2, X_5] &= -\phi(p)X_2 + BX_5 \\
[X_3, X_5] &= \phi(p)X_3 + CX_5 \\
[X_4, X_5] &= \phi(p)X_4 + DX_5 \\
\end{align*}
\]

(19)

for constants \( A, B, C, D \in \mathbb{R} \). The remainder of the Lie brackets can then be written down and the Jacobi identities imposed (after first noting that, by replacing \( X_1, \ldots, X_4 \) with \( X_1 + (A/3\phi(p))X_5, \ldots, X_4 + (D/\phi(p))X_5 \) one may assume that \( A = B = C = D = 0 \) in (19)). The resulting restrictions on the structure constants can then be shown to lead to a 4-dimensional Abelian ideal \( \mathcal{A} \) of \( A \) spanned by \( X_1, \ldots, X_4 \) and satisfying \( \dim \mathcal{A}_p = 4 \) [17]. Hence there exists a neighbourhood \( V \) of \( p \) and a function \( \sigma \) on \( V \) such that the space-time \( (V, e^{2\sigma}g) \) admits a 4-dimensional Abelian Killing algebra \( \mathcal{A} \) (restricted to \( V \)) satisfying \( \dim \mathcal{A}_q = 4 \) for all \( q \in V \). It follows that this last space-time is flat and hence that the Weyl tensor of the original space-time vanishes in \( V \), contradicting the type III assumption. The conclusion is that if \( M \) is of Petrov type III no conformal isotropy is possible and a local solution \( \sigma \) of (11) for all \( X \in \mathcal{A} \) can be found about any \( p \in M \).

Now suppose that \( M \) is of Petrov type N. The difficulty in this case arises from the extra possibilities (up to “3 parameters”) for the conformal isotropy. However, if \( p \in M \) is a homothetic fixed point for some \( X \in \mathcal{A} \) then one can always choose a basis for \( \mathcal{A} \) such that \( p \) is homothetic with respect to only one member of this basis and isometric to any other basis member (and there will be at most two of them) which vanishes at \( p \). Where appropriate, in what is to follow, this choice will be assumed made.

If there is just the single orbit \( M \) associated with \( \mathcal{A} \) then from what has been said above only the cases \( 5 \leq \dim \mathcal{A} \leq 7 \) need be considered since, otherwise (i.e. \( \dim \mathcal{A} = 4 \)), one has a local solution \( \sigma \) of (11) in some neighbourhood of any \( p \in M \) and for all \( X \in \mathcal{A} \).

If \( \dim \mathcal{A} = 5 \) one can choose a basis \( X_1, \ldots, X_5 \) for \( \mathcal{A} \) such that, if \( p \in M \), an equation like (17) holds at \( p \) when \( \ell \) spans the repeated principal null direction of the Weyl tensor at \( p \),
where $X_5(p) = 0$ and where, if $p$ is a homothetic fixed point with respect to $X_5$, one has at $p$ (section 3)

$$X_{5,a;b} = \phi(p)g_{ab} + 2\phi(p)\ell_{[a}n_{b]} \quad (\phi(p) \neq 0) \tag{20}$$

By applying the Schmidt method and the Jacobi identities as in the Petrov type III case one finds by taking appropriate linear combinations that $\mathcal{A}$ admits a 4-dimensional subalgebra $\mathcal{A}'$ such that $\dim \mathcal{A}_p' = 4$ and such that $\mathcal{A}'$ contains the commutator subalgebra of $\mathcal{A}$, i.e. $\mathcal{A}'$ is an ideal of $\mathcal{A}$. Hence there exists an open neighbourhood $U$ of $p$ and a function $\sigma'$ on $U$ which satisfies (11) for all $X \in \mathcal{A}'$. It follows that the space-time $(U, e^{2\sigma'}g)$ admits the (restrictions of the) members of $\mathcal{A}'$ as Killing vectors fields on $U$ and (result (ii) of section 2) the (restrictions of the) members of $\mathcal{A}$ as homothetic vector fields. That this is impossible follows from the last result in (iv) of section (2). If, on the other hand, $p$ is an isometric fixed point of $X_5$ one can use equation (14), the discussion following it and lemma 3 to see that $\psi^\alpha_5(p) = 0$, $\Delta_5(p) = 0$ (and hence $\Delta_5 \equiv 0$ in some neighbourhood of $p$) and to conclude the existence of a solution $\sigma$ of (11) in some open neighbourhood of $p$ and for all $X \in \mathcal{A}$.

Now suppose that $\dim \mathcal{A} = 6$ and let $p \in M$. One can choose a basis $X_1, \ldots, X_6$ for $\mathcal{A}$ such that $X_1(p), \ldots, X_4(p)$ span $T_pM$ and $X_5(p) = X_6(p) = 0$. If the fixed point $p$ is isometric with respect to both $X_5$ and $X_6$ the argument is similar to that given for an isometric fixed point above and a local solution $\sigma$ of (11) for all $X \in \mathcal{A}$ results. If, however $X_5$ and $X_6$ are chosen such that $p$ is homothetic with respect to $X_5$ and isometric with respect to $X_6$ then one can arrange that an equation like (17) holds at $p$ with $\ell$ spanning the repeated principal null direction of the Weyl tensor at $p$ and that, at $p$, equation (20) holds and also $X_{6,a;b} = 2\ell_{[a}x_{b]}$. Again one applies that Jacobi identities and finds a 5-dimensional subalgebra $\mathcal{A}'$ of $\mathcal{A}$ satisfying $\dim \mathcal{A}_p' = 4$. The argument used earlier then reveals the existence of an open neighbourhood $U$ of $p$ and a function $\sigma$ on $U$ such that the space-time $(U, e^{2\sigma'}g)$ admits the members of $\mathcal{A}'$ as Killing vector fields and the members of $\mathcal{A}$ as homothetic vector fields. This is forbidden by the results of section 2.

Finally suppose that $\dim \mathcal{A} = 7$ and let $p \in M$. Then one may choose a basis $X_1, \ldots, X_7$ for $\mathcal{A}$ such that $X_1(p), \ldots, X_4(p)$ span $T_pM$ and $X_5(p) = X_6(p) = X_7(p) = 0$ with $p$ a homothetic fixed point with respect to $X_5$ and an isometric fixed point with respect to $X_6$ and $X_7$. One may also arrange that (17) and (20) hold as before and that, at $p$, $X_{6,a;b} = 2\ell_{[a}x_{b]}$, $X_{7,a;b} = 2\ell_{[a}y_{b]}$. Finally, proceeding as before one finds an open neighbourhood $U$ of $p$ and a function $\sigma$ on $U$ such that the space-time $(U, e^{2\sigma'}g)$ admits $\mathcal{A}$ as a 7-dimensional homothetic algebra with a 6-dimensional ideal $\mathcal{A}'$ of Killing vector fields such that $\dim \mathcal{A}_p' = 4$. It follows from result (iv) of section 2 that the latter space-time is a homogeneous generalised plane wave.

Suppose then that $M$ is of Petrov type N and the orbits associated with the conformal
algebra $\mathcal{A}$ are 3-dimensional. Earlier results in this section show that one need only consider the cases when the orbits are null and $4 \leq \dim \mathcal{A} \leq 6$. So as in the Petrov type III case let $p \in M$ and let $\ell$ be a nowhere zero smooth vector field in some open neighbourhood $U$ of $p$ which is tangent to the orbits everywhere in $U$. Then if a non-trivial member $X \in \mathcal{A}$ vanishes at $p$ (as one must) then the equation $X^a \ell_a = 0$ together with (1) shows that $F_{ab}^{\ell b} = \phi(p) \ell_a$ at $p$ and so, whether $\phi(p)$ is or is not zero, $\ell$ spans the repeated principal null direction of the Weyl tensor everywhere in $U$ (section 3).

So consider the case when $\dim A = 4$ and let $X_1, \ldots, X_4$ be a basis for $\mathcal{A}$. Suppose that $X_1$, $X_2$ and $X_3$ span $\mathcal{A}_p$ and that $p$ is a homothetic fixed point with respect to $X_4$. If $\{\ell, n, x, y\}$ is a null tetrad at $p$ such that $\ell$, $x$, $y$ span $\mathcal{A}_p$ and one can arrange that

$$X_1(p) = \ell \quad X_2(p) = x \quad X_3(p) = y \quad X_4(p) = 0 \quad (21)$$

$$X_{4a;b} = \phi(p) g_{ab} + 2 \phi(p) \ell_{[a} n_{b]} \quad (\phi(p) \neq 0) \quad (22)$$

Using the Schmidt method one finds that

$$[X_1, X_4] = 2 \phi(p) X_1 + a X_4 \quad [X_2, X_4] = \phi(p) X_2 + b X_4$$

$$[X_3, X_4] = \phi(p) X_3 + c X_4 \quad (a, b, c \in \mathbb{R}) \quad (23)$$

As before, one can arrange that $a = b = c = 0$ by an obvious redefinition of $X_1$, $X_2$ and $X_3$. The remaining commutators and the Jacobi identities can then be calculated and simplified using the Jacobi identities and it turns out that $X_1$, $X_2$ and $X_3$ generate an ideal $\mathfrak{A}$ of $A$ and $\dim \mathfrak{A}_p = 3$. One may now find a solution $\sigma$ of (11) for all $X \in \mathfrak{A}$ in some neighbourhood of $p$ and then the members of $\mathfrak{A}$ become Killing vectors with respect to the metric $\mathfrak{g} = e^{2\sigma} g$. Choosing local coordinates such that the orbits of $A$ are labelled by $u =$constant, where $u$ is one of the coordinates, one then finds using result (ii) of section 2 that $X_4$ satisfies (1) (with respect to $\mathfrak{g}$) with $\phi = \phi(u)$. Thus one has the situation (with respect to $\mathfrak{g}$) of having Killing vector fields say $Y_1, Y_2, Y_3$ with $Y_1(p)$ null and a conformal vector field $Y_4$, the last satisfying $Y_4(p) = 0$ and equation (1) with $\phi$ nowhere zero in some connected coordinate neighbourhood $U$ of $p$ such that there exists smooth functions $\alpha$, $\beta$, $\gamma$ on $U$ with $Y_4 = \alpha Y_1 + \beta Y_2 + \gamma Y_4$ and $\alpha(p) = \beta(p) = \gamma(p) = 0$. On substituting this expression for $Y_4$ into (1) and transvecting by $v^a v^b$ where $v$ is any member of $T_q M$ ($q \in U$) orthogonal to the gradients $\alpha_a$, $\beta_a$, $\gamma_a$ at $q$ one obtains $\phi(q) v_a v^a = 0$. Thus the set of all such $v$ is a non-trivial subspace $N_q$ of $T_q M$ consisting entirely of null vectors (and is hence 1-dimensional) and so $\alpha_a$, $\beta_a$, $\gamma_a$ are independent at each $q \in U$. Further, a transvecting of the same equation with $\ell_q^b$, where $\ell_q$ spans the unique null direction tangent to the orbits at $q$ shows that $\ell_q \notin N_q$ for all $q \in U$. Thus, by shrinking $U$ if necessary, there exists a smooth null vector field $Z$ on $U$ such that $Z(q)$ spans $N_q$ at each $q \in U$ and such that $\alpha$, $\beta$, $\gamma$
are zero along any integral curve of \(Z\) in \(U\) through \(p\). Thus \(Y_4\) also vanishes along this integral curve. If \(p' \in U\) lies on this integral curve a consideration of equation (1) for \(Y_4\) at \(p\) and \(p'\) and use of the bracket relation for the \(Y\)s shows that \(\phi\) essentially plays the rôle of a structure constant along this curve and so \(\phi(p) = \phi(p')\). Hence, with the above result that \(\phi = \phi(u)\), one sees that \(\phi\) is constant in \(U\). It follows from [8,9] (see section 2 result (iii)) that since \(Y_4\) is a proper homothetic vector field whose zeros are not isolated, some open neighbourhood of the zero \(p\) is isometric to a generalised plane wave space-time and, recalling the assumption about local conformal vector fields, this contradicts the fact that \(\dim A = 4\).

The only way to avoid this contradiction is to assume that no homothetic fixed points (as described above) exist. In this case (21) still holds but with \(\phi(p) = 0\) and \(X_1, X_2\) and \(X_3\) span the orbits of \(A\) in some open neighbourhood \(W\) of \(p\). If \(q \in W\) define \(Y_q = X_4 + \sum_{i=1}^{3} \lambda_i X_i\) where the \(\lambda_i \in \mathbb{R}\) are chosen so that \(Y_q(q) = 0\) (and hence \(\phi(q) = 0\), by the above assumption, where \(\phi(q)\) is the divergence of \(Y_q\)). But then (1) and (14) imply

\[
\phi_q(q) = \phi + \sum_{i=1}^{3} \lambda_i \phi_i \quad \Delta Y_q = \Delta 4
\]  \hspace{1cm} (24)

where \(\phi_i\) is the divergence of \(X_i\) and the notation \(\Delta Y_q\) is obvious. But, \(\Delta Y_q(q) = 0\) and so \(\Delta 4 \equiv 0\) on \(W\) since the choice of \(q\) was arbitrary. It now follows from the result immediately following (14) that there exists an open neighbourhood \(W'\) of \(p\) and a function \(\sigma: W' \to \mathbb{R}\) such that \(A\) is a Killing algebra for the metric \(e^{2\sigma} g\).

The case when \(\dim A = 5\) is similar and the argument need only be sketched. One selects members \(X_1, \ldots, X_5\) of \(A\) with \(X_1, X_2, X_3\) as above and with \(p\) homothetic with respect to \(X_4\) and isometric with respect to \(X_5\). The Schmidt method then reveals that \(X_1, X_2, X_3\) and \(X_5\) can be chosen to generate a 4-dimensional ideal \(A'\) of \(A\) such that \(A'_p = A_p\), and one applies the previous argument to \(A'\) to see that \(A'\) is a Killing algebra of \(g' = e^{2\sigma} g\) for an appropriate function and neighbourhood of \(p\). As before, \(X_4\) is then homothetic with respect to \(g'\) This implies that \(g'\) is (locally) a generalised plane wave metric, contradicting the dimension of \(A\). The only way to avoid this contradiction is to proceed as in the previous paragraph, working on \(X_4\) and \(X_5\). Again a contradiction ensues.

The case when \(\dim A = 6\) is mostly similar and the argument will only be sketched. If \(p \in M\) there exists a basis \(X_1, \ldots, X_6\) of \(A\) such that \(X_1, X_2, X_3\) vanish and \(X_4, X_5, X_6\) do not vanish, at \(p\) and such that the fixed point \(p\) is homothetic with respect to \(X_1\) and isometric with respect to \(X_2\) and \(X_3\). Further this basis may also be chosen so that \(X_2, \ldots, X_6\) form a 5-dimensional subalgebra of \(A\). Hence \(A\) admits a 3-dimensional subalgebra \(A'\). One then shows the existence of an open dense subset \(M'\) of \(M\) such that each \(p' \in M'\) is contained in an open neighbourhood \(U\) in \(M\) with the property that the orbits of \(A'\) are either 3-dimensional...
(and hence null) at each point of $U$ or 2-dimensional and null at each point of $U$. The former case leads to $g$ being locally conformally related to a plane wave in some open neighbourhood of $p'$ (as the argument in the previously considered case shows) whilst the latter leads to $g$ being locally conformally related to a metric of the type studied in [18] (see also [19]) in some open neighbourhood of $p'$. If $V$ is an open subset of $M$ in which the metric is of the latter type then $V$ may be chosen to be a coordinate neighbourhood, one coordinate of which is $u$ where $u$ labels the null orbits of $A$. For such a metric there is necessarily a nowhere zero null conformal vector field $k$ on $V$ [18, see also 19] and a smooth function $\sigma: V \to \mathbb{R}$ such that $u_a = e^{2\sigma} k_a$. Then if $X \in A$, $\mathcal{L}_X u_a = 0$ and the bracketed equation in (1) gives

$$\mathcal{L}_X k^a = \mathcal{L}_X \left(e^{-2\sigma} g^{ab} u_b \right) = -2(\sigma_b X^b + \phi) k^a.$$  \hspace{1cm} (25)

But since $k$ and $X$ are conformal, so is their Lie bracket (i.e. the left hand side of (25)) and it follows that $\sigma_a X^a + \phi$ is constant in $V$ for each $X \in A$. Thus each member of $A$ is homothetic in $V$ with respect to the metric $g' = e^{2\sigma} g$. It follows that $g'$ is locally isometric to a generalised plane wave metric (section 2(iii)) and so, from the remarks at the bottom of p. 280 in [2], every $p \in M'$, and hence every $p \in M$ is contained in an open neighbourhood of $M$ which is conformally related to a generalised plane wave. This completes the proof.

6: Summary and Discussion

The following remarks arise in connection with this paper. Some have been noted before [4,7] but they are collected together here for convenience.

(a) In [1] no mention was made about the reduction to homotheties and generalised plane wave space-times. This occurred in [2]. The only essential addition in theorem 2 is the remark regarding the reduction to special conformal vector fields. Also certain analyticity assumptions were made in [1] but neither in [2] nor here (although such an assumption would simplify certain proofs!).

(b) It is clear that there can be no solution $\sigma$ of (11) in any neighbourhood of a point $p \in M$ for any $X \in A$ which has $p$ as a homothetic fixed point. More generally one of the main problems in trying to strengthen theorem 2 arises when one tries to remove the conditions regarding the constancy of the dimension and type of the orbits associated with the conformal algebra. The “singular points” at which the dimension or the type (or both) of the orbits change can cause problems and a detailed study of the fixed points of conformal vector fields is hindered by the fact that one may not be guaranteed the “linearity” that one always has for affine (including homothetic and Killing) vector fields. More details about these points can be found in [7].
(c) Theorem 2 only guarantees local functions $\sigma$ with the appropriate properties. Examples where only local (i.e. non-global) functions $\sigma$ are possible and where global functions $\sigma$ always exist can be found in [7].

(d) In spite of the conditions of theorem 2 it can be shown that any space-time possesses an open dense subset $U$ such that any $p \in U$ lies in an open neighbourhood of $M$ in which all the conditions of theorem 2 are satisfied [4]. In this sense theorem 2 is “valid almost everywhere” in $M$.

(e) One can add a little to theorem 1 by noting that if a space-time satisfies the conditions of theorem 2 and is such that its conformal algebra is 7-dimensional then it is locally conformally related to a Petrov type N homogeneous generalised plane wave space-time. This follows from the results given in this paper.

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References

[17] It should be remarked here that, quite generally, every 5-dimensional Lie algebra admits a 4-dimensional subalgebra (and every 4-dimensional Lie algebra admits a 3-dimensional subalgebra). These results can be deduced from J. Patera, R.T. Sharp, P. Winternitz and H. Zassenhaus, *J. Math. Phys.* 17 (1976) p. 986 and A.A. Sagle and R.E. Walde "Introduction to Lie Groups and Lie Algebras"  *Academic Press*  (1973). However the 4-dimensional subalgebra (in fact ideal) \( \mathfrak{A} \) is directly constructed here and similar constructions are made elsewhere in the paper.