ACCELERATED COORDINATES

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Abstract: We use the Weyl canonical coordinates for the C-metric to set up the metrics that arise from deforming the C-metric in the same way that the Voorhees-Zipoy metrics arise from the Schwarzschild metric. The charged C-metric is considered and is expressed in terms of the Weyl canonical coordinates also. The twisting C-metric written in the Weir and Kerr form is then treated by a similar technique. This gives rise to some possibly new twisting versions of the C-metric. Finally we present these twisting metrics in the Weyl canonical coordinates.

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1: Introduction

In part of their now classic work \cite{EhlersKundt}, Ehlers and Kundt classified all static type D vacuum spacetimes. They found three classes, which they called A, B and C, differentiated by the isotropy group they admitted. Type A have spacelike isotropy and spacelike 2-surfaces of constant curvature (the positive curvature case being Schwarzschild). Type B are similar but with timelike isotropy and timelike 2-surfaces of constant curvature. The third type contained only one metric, now called the C-metric, which was first discovered by Levi-Civita, and it admits no isotropy.

As Ehlers and Kundt pointed out, all type D static metrics are in fact axisymmetric. That is they admit an Abelian $G_2$ spanned by a hypersurface orthogonal timelike Killing vector and the axially symmetric Killing vector $\partial_\varphi$. Hence they are what are known as Weyl solutions. In the Weyl canonical coordinates (cf. \cite{Weyl}), Weyl solutions can be written as:

$$ds^2 = e^{2(V-U)}(dp^2 + dz^2) + \rho^2 e^{2U}d\varphi^2 - e^{2U}dt^2,$$

where $U$ and $V$ are functions of $\rho$ and $z$ only. The field equations are then

$$\Delta U = 0 \quad \Leftrightarrow \quad U_{,\rho\rho} + \rho^{-1}U_{,\rho} + U_{,zz} = 0,$$

$$V_{,\rho} = \rho(U_{,\rho}^2 - U_{,z}^2), \quad V_{,z} = 2\rho U_{,\rho}U_{,z}.$$  

(1.2a)

(1.2b)

Since the first equation is just Laplace’s equation in cylindrical coordinates for an axially symmetric potential, we can take $U$ to be the Newtonian gravitational potential of some axially symmetric source to get a solution. As is well known, this Newtonian source bears no direct relationship to the actual source in the context of General Relativity. For
example, the Schwarzschild solution is given by the (Newtonian) potential of a finite rod lying along the z-axis. Flat space can also be represented in several different ways (cf. [2]), in particular with $U$ as the (Newtonian) potential of a semi-infinite rod of line density 1/2. In this case we get flat space in accelerated coordinates.

In Kinnersley and Walker [3], the C-metric was interpreted as the field of a uniformly accelerating particle, and this interpretation was extended by Bonnor [4] to look at the (Newtonian) sources of the C-metric by transforming to the Weyl canonical coordinates.

This paper is a sequel to [5]. In that paper we discussed the basic idea of representing static and stationary axisymmetric metrics in terms of rational coordinates, in the same way that the Schwarzschild-Kerr-Tomimatsu-Sato-Yamazaki-Cosgrove [6,7,8] family of solutions are represented in (rational) prolate spheroidal coordinates.

The main thrust of this work is to push the addition theorem for quadratics that is used in Schwarzschild to the addition theorem for cubics; that is from

$$\frac{dx^2}{x^2 - 1} - \frac{dy^2}{y^2 - 1} = \frac{d\rho^2 + dz^2}{R_+ R_-},$$

where $R^2 = \rho^2 + (z + 1)^2$, to

$$\frac{dx^2}{P_3(x)} - \frac{dy^2}{P_3(y)} = \frac{d\rho^2 + dz^2}{R_+ R_- R_l},$$

where the $R_j$ are defined similarly and $P_3$ is a cubic with three real roots, $\pm m$ and $l$.

We note that in the Schwarzschild case, the important polynomial is $x^2 - 1$, which contains no real physical information. It will emerge from our analysis that the same can be made true in the cubic case. In the work of Plebański and Demiański [9] and Weir and Kerr [10], charged and twisting versions of the C-metric were given. Their metrics had a similar rational coordinate expression, but involving a quartic polynomial that contained physically important information (mass, charge, angular momentum). We will recast these metrics in a way that leaves only the acceleration as part of the coordinate system, so that we will have genuine accelerated coordinates analogous to flat space expressed in accelerated coordinates.

In section 2 of this paper, we find a coordinate transformation for the C-metric from cubic coordinates to the Weyl canonical coordinates. The difference between our transformation and Bonnor’s is that our coordinate ranges are more easily dealt with, and we also find the inverse transformation from Weyl to cubic coordinates (cf. [11]). Since it is easy to change the density of the (Newtonian) source in the Weyl canonical coordinates, we use our coordinate transformation to explicitly construct “deformed” C-metric solutions; these bear the same relationship to the C-metric as the Voorhees-Zipoy [12,13] solutions do to the Schwarzschild solution. This is done in section 3. In section 4 we look at the charged C-metric, re-writing it first in cubic coordinates by removing the charge parameter from the original quartic and then transform cubic coordinates to the Weyl canonical coordinates. We repeat this process for the twisting C-metric written in the Weir and Kerr [10] form in section 5, and also present some special cases of this metric that have an interesting form for the angular momentum parameter.
2: Cubic Coordinates

Following Bonnor [4], for the C-metric and its deformations, we take $U$ to be the Newtonian potential of three semi-infinite rods on the $z$-axis: one of density $\varepsilon > 0$ along $[l, \infty)$, one of density $-\delta < 0$ along $(m, \infty)$, where $0 < m < l$, and one of density $\delta > 0$ along $[-m, m]$. The latter two rods give a finite rod of density $\delta$ along $[-m, m]$. We have essentially chosen the origin of $z$ and we could choose a scaling of $z$ and $\rho$ to make $m = 1$, but it will be more convenient not to do so.

If we let $R^2_+ = \rho^2 + (z - m)^2$ be the square of the distance from an arbitrary point in the flat background $(\rho, z)$ plane to the point $z = m$ on the $z$-axis, and similarly $R^2_- = \rho^2 + (z + m)^2$ and $R^2_l = \rho^2 + (z - l)^2$ be the squares of the distances to the points $z = -m$, $l$ on the $z$-axis respectively. Then we can check that the function $U$ and $V$ of (1.1) are (cf. [5])

$$U = \frac{1}{2} \delta \log \left[ \frac{R_+ - R_- - 2m}{R_+ + R_- + 2m} \right] + \frac{1}{2} \varepsilon \log \left[ \frac{R_l - (z - l)}{2l} \right],$$

$$V = \frac{1}{2} \delta^2 \log \left( \frac{(R_+ + R_-)^2 - 4m^2}{4R_+R_-} \right) + \frac{1}{2} \varepsilon^2 \log \left( \frac{R_l - (z - l)}{2R_l} \right) + \delta \varepsilon \log \left( \frac{R_l - R_- - (m - l)}{R_l - R_+ - (m - l)} \right).$$

Note that we can still add an arbitrary constant to $U$ or $V$ (see also the discussion near the beginning of the next section).

The three important limiting cases for $U$ and $V$ are:

1. Minkowski spacetime in accelerating coordinates can be obtained by either putting $\delta = 0$ and $\varepsilon = 1$ in equations (2.1), or by letting $m \to 0$ and $\varepsilon = \delta = 1$; viz.

$$U = \frac{1}{2} \log \left( \frac{R_l - (z - l)}{2l} \right), \quad V = \frac{1}{2} \log \left( \frac{R_l - (z - l)}{2R_l} \right).$$

Note that when $l$ approaches $\infty$, both $U$ and $V$ approach to 0 and hence we have flat space in cylindrical coordinates.

2. The Schwarzschild solution can be obtained either by putting $\varepsilon = 0$ and $\delta = 1$ in equations (2.1), or by letting $l \to \infty$, $\varepsilon = \delta = 1$; viz.

$$U = \frac{1}{2} \log \left[ \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m} \right], \quad V = \frac{1}{2} \log \left( \frac{(R_+ + R_-)^2 - 4m^2}{4R_+R_-} \right).$$

3. The Curzon-Chazy solution (see for example [2]) can be obtained by taking limits as $m \to 0$ and then either putting $\varepsilon = 0$, or letting $l \to \infty$ in equations (2.1); viz.

$$U = \frac{M}{\sqrt{\rho^2 + z^2}}, \quad V = \frac{M^2 \rho^2}{\sqrt{\rho^2 + z^2}},$$

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where \( M = m \delta \) and we have used the result \( e^x = \lim_{n \to \infty} (1 + \frac{x}{n})^n \).

The key part of the problem is to find a coordinate transformation from the Weyl canonical coordinates so that we can write the solution in rational coordinates involving a cubic polynomial (cf. [5]).

We look for a coordinate transformation from the \((\rho, z)\) plane to the \((x, y)\) plane such that a polynomial \( P_3 \) with three distinct real roots (cf. [5,14,15]) is determined by

\[
\frac{dx^2}{X} - \frac{dy^2}{Y} = \text{const.} \times \frac{(d\rho^2 + dz^2)}{R_+ R_- R_l}, \tag{2.2}
\]

where \( X \equiv P_3(x) > 0, Y \equiv P_3(y) < 0 \). Clearly the left hand side of (2.2) is a flat 2-metric (when it is expressed in terms of elliptic functions) and consequently the right hand side is also flat. If we define a complex variable \( \zeta = z + i\rho \), and let \( \theta_j \), for \( j = +, - \) and \( l \), be the (real) constants \( m, -m \) and \( l \), respectively, then \( R_j = |\zeta - \theta_j| \). This suggests that we take \( \Theta(\zeta) \) to be a cubic of the complex variable \( \zeta \) with real roots at \( \pm m \) and \( l \), i.e. \( \Theta(\zeta) = (\zeta - l)(\zeta^2 - m^2) \), so that the right hand side of (2.2) is (proportional to) \( |d\zeta/\sqrt{\Theta(\zeta)}|^2 \). We will call \( P_3 \) the coordinate cubic and \( \Theta \) the physical cubic when we wish to differentiate between them.

We now demonstrate how to construct (2.2). Consider the differential equation (cf. Cayley [16])

\[
\frac{dx}{\sqrt{X}} + i \frac{dy}{\sqrt{-Y}} = 0. \tag{2.3}
\]

Assume that the general solution is \( u(x, y) = 0 \), where \( u \) is a quadriquadric in \( x \) and \( y \) involving one arbitrary parameter which will be identified as the complex parameter \( \zeta = z + i\rho \) (see also Lagrange [17]):

\[
u = a + 2h(x + y) + g(x^2 + y^2) + 4bxy + 2fxy(x + y) + cx^2y^2 = 0. \tag{2.4a}
\]

The six constants \( a, b, c, f, g, h \) depending on the arbitrary constant are to be determined. This equation can be put into the form

\[
u = \alpha + 2\beta y + \gamma y^2 = \tilde{\alpha} + 2\tilde{\beta} x + \tilde{\gamma} x^2 = 0, \tag{2.4b}
\]

where \( \alpha, \beta \) and \( \gamma \) are the same quadratics in \( x \) that \( \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{\gamma} \) are in \( y \):

\[
\alpha = a + 2hx + gx^2, \quad \beta = h + 2bx + fx^2, \quad \gamma = g + 2fx + cx^2.
\]

Equation (2.4b) implies that \( (\gamma y + \beta)^2 = \beta^2 - \alpha \gamma \), and similarly \( (\gamma x + \beta)^2 = \tilde{\beta}^2 - \tilde{\alpha} \tilde{\gamma} \). Hence the differential equations \( du = 0 \) and (2.3) are the same if

\[
\beta^2 - \alpha \gamma = \Theta X, \quad \tilde{\beta}^2 - \tilde{\alpha} \tilde{\gamma} = \Theta Y, \tag{2.5}
\]

where \( \Theta \) is a function of the arbitrary parameter \( \zeta \). If we further insist (as we can) that \( u \) is a quadratic when considered as a function of \( \zeta \), namely

\[
u = \lambda + 2\mu \zeta + \nu \zeta^2 = 0, \tag{2.6}
\]

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where $\lambda$, $\mu$ and $\nu$ are quadratic in $x$ and $y$, then we obtain the constraint $\mu^2 - \lambda \nu = XY$. Now consider $du = 0$ as a differential equation with three independent variables $x$, $y$ and $\zeta$ and we find that it is equivalent to the differential equation
\[
\frac{dx}{\sqrt{X}} + i \frac{dy}{\sqrt{-Y}} + \frac{d\zeta}{\sqrt{\Theta(\zeta)}} = 0 ,
\]
which gives equation (2.2) with proportionality factor 1 when we set $\zeta = z + i\rho$ (note that the discriminant of $u$ in (2.6), confirms that $\zeta$ is complex).

It remains to find $a, b, c, f, g, h$ such that $\Theta$ is the above cubic in $\zeta$, and $X$ and $Y$ factorise simply in terms of $x$ and $y$. For reasons that will be apparent later, we wish the coefficient $\nu$ in (2.6) to be a perfect square, and we will insist that $\nu$ is proportional to $(x - y)^2$. We then choose the coordinate cubic so that $X = (mx - l)(x^2 - 1)$, which will enable us to look at the limiting cases mentioned above.

Then writing $a, b, c, f, g, h$ as quadratics in $\zeta$, we expand out and equate coefficients. After a fair amount of elementary algebra, we find that there are only two consistent solutions, one the negative of the other. We take the solution
\[
\begin{align*}
a &= 2l^2 - 2l\zeta + m^2/2 , & b &= \frac{1}{4}(2l\zeta - \zeta^2 + m^2) , & c &= m^2/2 , \\
f &= -\zeta m/2 , & g &= \zeta^2/2 , & h &= -lm + m\zeta/2 .
\end{align*}
\]
The other solution gives exactly the same results in what follows.

The actual coordinate transformation involved can easily be found by substituting (2.7) into (2.4a) and then comparing the resultant expression with that of (2.6). We have $\zeta = \left(-\mu \pm i\sqrt{\nu \lambda - \mu^2}\right)/\nu$ and thus (cf. Bonnor [4])
\[
\begin{align*}
z(x, y) &= -\mu/\nu = (1 - xy)(2l - m(x + y))/(x - y)^2 , \\
\rho(x, y) &= \sqrt{\nu \lambda - \mu^2}/\nu = 2\sqrt{X}\sqrt{-Y}/(x - y)^2 .
\end{align*}
\]

Now consider the $R_j$. A simple calculation shows that $R_j^2 = u(\theta_j)/\nu$, considering $u$ as a quadratic in $\zeta$. Then (again, cf. Bonnor [4])
\[
\begin{align*}
R_+ &= \epsilon_+ (m(xy - x - y - 1) + 2l)/(x - y) , & (2.10a) \\
R_- &= \epsilon_- (m(xy + x + y - 1) - 2l)/(x - y) , & (2.10b) \\
R_l &= \epsilon_l (m(xy + 1) - l(x + y))/(x - y) , & (2.10c)
\end{align*}
\]
where $\epsilon_j = \pm 1$ as required to ensure $R_j > 0$.

We can use these equations to find the inverse transformation from $(\rho, z)$ to $(x, y)$. A re-arrangement of equations (2.10) provides us with three inhomogeneous linear equations for $x$, $y$ and $xy$, with coefficients depending on $m$, $l$ and the $R_j$. Solving these equations gives the inverse transformation (cf. [11]):
\[
\begin{align*}
x &= (2l^2 - m^2) - m(\epsilon_- R_- - \epsilon_+ R_+) + l(\epsilon_- R_- + \epsilon_+ R_+ - 2\epsilon_l R_l)}/\Delta , \\
y &= (2m^2 - l^2) - m(\epsilon_- R_- - \epsilon_+ R_+) + l(\epsilon_- R_- + \epsilon_+ R_+ - 2\epsilon_l R_l)}/\Delta ,
\end{align*}
\]

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where $\Delta = l(\varepsilon_+ R_+ - \varepsilon_- R_-) + m(\varepsilon_- R_+ + \varepsilon_+ R_- - 2\varepsilon_1 R_1)$. Note that $x - y = 4(l^2 - m^2)/\Delta$.

We have four regions to consider in the $(x, y)$ plane, as dictated by the signature requirements (i.e. by $X > 0$ and $Y < 0$):

(I) $1 < y < l/m$, $|x| < 1$
(II) $1 < y < l/m$, $x > l/m$
(III) $y < -1$, $|x| < 1$
(IV) $y < -1$, $x > l/m$

Now the zeros of the numerators of the $R_i$ all fall on the corners of or outside these feasible regions, and the $R_i$ are infinite only when $x = y$, i.e. at the $x = y = 1$ corner of Region(I), the $x = y = l/m$ corner of Region(II) and the $x = y = -1$ corner of Region(III). To ensure the $R_j$ are positive the $\varepsilon_j$ must have the following signs:

- (I) $\varepsilon_+ = -$, $\varepsilon_- = +$, $\varepsilon_1 = -$
- (II) $\varepsilon_+ = -$, $\varepsilon_- = +$, $\varepsilon_1 = -$
- (III) $\varepsilon_+ = +$, $\varepsilon_- = -$, $\varepsilon_1 = +$
- (IV) $\varepsilon_+ = +$, $\varepsilon_- = -$, $\varepsilon_1 = -$

The picture of these four regions on the $(x, y)$ plane is given in figure 1.
Which region do we choose? Region(I) is the one we settle on for two reasons. Firstly, when we calculate the metric component \( g_{00} = e^{2U} \) for the case of equal density, \( \varepsilon = \delta \) in equation (2.1a), we find that the it equals to \((-Y/l)\delta/(y-x)^2\) in this region, comparing with the result for the C-metric where \( e^{2U} = -(Y/l)/(y-x)^2 \) (so here \( \delta = 1 \)). The second reason we choose to look at Region(I) is that the important points in the \((\rho,z)\) plane are given by finite values in the \((x;y)\) plane: \((x;y) = (0,0), (0,m), (l,m)\), respectively, and the point \((x,y) = (1,1)\) corresponds to points at infinity in the \((\rho,z)\) plane.

We can show that the spacetimes that we obtain from the four regions are in fact isometric. We proceed as follows. Suppose that \( P_3(x) = (x-\alpha)(x-\beta)(x-\gamma) \) is a cubic. Using a linear fractional transformation we can swap two of the roots of \( P_3 \), say \( \beta \) and \( \gamma \), and map \( \beta \) to \( 0 \) and conversely. We let

\[
f(x) = \frac{\gamma x + \alpha \beta - \alpha \gamma - \beta \gamma}{x - \gamma}.
\]

We find that \( P_3(f(x)) = (\beta - \gamma)^2(\alpha - \gamma)^2P_3(x)/(x-\gamma)^4 \) and that \( dx^2/P_3(x) \) is invariant under this transformation.

Consider Region(II). If we wish to map it to Region(I), we must swap roots (we consider \( \infty \) as a root) of the coordinate cubic using the above transformation with \( \alpha = 1, \beta = l/m \) and \( \gamma = -1 \). We apply this transformation separately to both \( x \) and \( y \) and we find not only that the differentials \( dx^2/X \) and \( dy^2/Y \) are invariant, but so are \( \rho \) and \( z \). Furthermore, the functions \( R_i \) with signs as appropriate for Region(II) are mapped in such a way that their overall signs change to be those appropriate for Region(I). Since the metric is written solely in terms of these functions, it is clear that the spacetimes are isometric.

Similar results hold for the other two regions. For Region(III) we set \( \alpha = 1, \beta = -1 \) and \( \gamma = l/m \). For Region(IV) we set \( \alpha = -1, \beta = l/m \) and \( \gamma = 1 \). It follows that as all the metrics from the three regions are isometric to that from Region(I), they are all isometric.

Note that in the above analysis if we choose \( m = 1 \), then the physical and coordinate cubics are the same. However with the two cubics separated as we have them, the limiting cases are easier to deal with. If we set \( m = 0 \), then the finite rod in the “Weyl picture” vanishes and we have prolate spheroidal coordinates with \( X = l(1-x^2) > 0 \) and \( -Y = l(y^2-1) > 0 \). We can obtain the transformation from the \((\rho,z)\) plane to the \((x,y)\) plane together with the inverse transformation by simply setting \( m = 0 \) in equations (2.8) to (2.12). In particular for \( \delta = 1 \), we get the flat space in prolate spheroidal coordinates with \( |y| > 1 \) and \( |x| < 1 \). If we let \( l \to \infty \), then the semi-infinite rod will vanish and we will be left with prolate spheroidal coordinates again. However the actual coordinate transformation which will lead us back to the usual form of the Schwarzschild solution in prolate spheroidal coordinates is more involved.
3: The deformed C-Metrics

We can now progress to calculate the metric functions from equations (2.1). We will write all the factors that turn up as positive; for example, in Region(I) \( x < y \), so we prefer \( y - x \) to \( x - y \). We find the following.

\[
e^{2U} = \frac{1}{l^{\varepsilon}(y - x)^{2\varepsilon}}(l - my)^{\delta}(y^2 - 1)^{\varepsilon}(l - mx)^{(\varepsilon - \delta)} ,
\]

\[
e^{2V} = \frac{K^2(m + l)^{2\delta \varepsilon}(2m)^{-2\delta^2}}{(R_- R_+)^{\delta l^2} R_i^{2\varepsilon}(y - x)^{2(\delta^2 + \varepsilon^2)}(l - my)^{\delta\varepsilon}(y^2 - 1)^{\varepsilon^2}} ,
\]

\[
\rho^2 e^{-2U} = \frac{4l^{\varepsilon}(-Y)X}{(y - x)^{1 - 2\varepsilon}}(l - mx)^{\delta - \varepsilon}(l - my)^{-\varepsilon}(y^2 - 1)^{-\varepsilon} ,
\]

\[
e^{2(V - U)} = \frac{l^{\varepsilon} K^2(m + l)^{2\delta \varepsilon}(l - mx)^{(\delta - \varepsilon)(\delta - \varepsilon + 1)}}{2^{2\delta^2} (R_- R_+)^{\delta l^2} R_i^{2\varepsilon} m^{2\delta^2} (y - x)^{2(\delta^2 + \varepsilon^2 - \varepsilon)}(l - my)^{\delta^2 - \delta}(y^2 - 1)^{\varepsilon^2 - \varepsilon}} ,
\]

where \( K \) is a constant to be determined by a condition which ensures regularity on the \( z \)-axis (see equations (3.3) and (3.4) below). From which

\[
e^{2(V - U)}(d\rho^2 + dz^2) = \frac{2^{2 - 2\delta^2} l^{\varepsilon} K^2(m + l)^{2\delta \varepsilon}}{(R_- R_+)^{\delta l^2} R_i^{2\varepsilon} m^{2\delta^2} (y - x)^{2(\delta^2 + \varepsilon^2 - \varepsilon)}} (l - mx)^{(\delta - \varepsilon)(\delta - \varepsilon + 1)}
\]

\[
(l - my)^{\delta^2 - \delta}(y^2 - 1)^{\varepsilon^2 - \varepsilon} \left( \frac{dx^2}{X} - \frac{dy^2}{Y} \right) .
\]

We now consider the problem of regularity on the \( z \)-axis. We want the axisymmetric Killing vector \( \eta^a = \partial_{\phi} \) to have period \( 2\pi \), which is true only if (cf. [2])

\[
C = N_{a\dot{a}} N^{a\dot{a}} / 4N \rightarrow 1 ,
\]

where \( N = \eta_{a\dot{a}} \eta^{a\dot{a}} = \rho^2 e^{2U} \), as the symmetry axis is approached. For Weyl metrics \( C = ((1 - \rho U_{\rho})^2 + (\rho U_z)^2)e^{-2V} \), so this becomes (cf. [2])

\[
\lim_{\rho \rightarrow 0} e^{-2V} = 1 .
\]

On the segments \([-m, m] \) and \([l, \infty) \) of the \( z \)-axis where the rods are situated, we do not expect (3.3) to hold there (in fact \( e^{2V} \rightarrow 0 \) there). The other parts of the \( z \)-axis are \((-\infty, -m) \), corresponding to \( x = 1 \), and \([m, l) \) corresponding to \( x = -1 \). Substituting these values into (3.1b) gives

\[
\lim_{x \rightarrow -1} e^{2V} = 2^{-4\delta^2} K^2 m^{-2\delta^2} ,
\]

\[
\lim_{x \rightarrow 1} e^{2V} = 2^{-4\delta^2} K^2 \left( \frac{l + m}{l - m} \right)^{2\delta \varepsilon} m^{-2\delta^2} .
\]
So if we set \( K = 2^{4\delta^2} \frac{(l - m)/(l + m)}{2\delta^2\varepsilon} m^{2\delta^2} \) the metric is regular on the semi-infinite gap on the \( z \)-axis, but there is a singularity on the finite gap. In fact there is no way to have regularity on both gaps unless either \( \delta \) or \( \varepsilon \) is zero. Note that with this value of \( K \) we have \( \lim_{x \to -1} e^{2\varepsilon} = \left[ \frac{(l + m)/(l - m)}{2\delta^2\varepsilon} \right]^{2\delta^2\varepsilon} \) and so the orbits of the axisymmetric Killing vector have periodicity less than \( 2\pi \) if \( \delta \varepsilon > 0 \). This conical stress singularity is the same as the "nodal crease" in Kinnersley and Walker’s analysis [3].

With this value of \( K \), equation (3.1) becomes

\[
e^{2(V-U)}(d\rho^2 + dz^2) = \frac{2^{2\delta^2} l^{\varepsilon}(l - m)^{2\delta^2 \varepsilon}}{(R_--R_+)^{\delta^2-1}R_-^{\varepsilon-1} (y - x)^{2(\delta^2 + \varepsilon^2 - \varepsilon)}} \times (l - my)^{\delta^2-\delta}(y^2 - 1)^{\varepsilon^2-\varepsilon} \left( \frac{dx^2}{X} - \frac{dy^2}{Y} \right) \] (3.2')

Then the metric is

\[
ds^2 = 2^{2\delta^2} l^{\varepsilon}(l - m)^{2\delta^2 \varepsilon} d\sigma^2 + 4l^{\varepsilon} X(l - mx)^{\delta-\varepsilon} (y - x)^{1-2\varepsilon} (l - my)^{1-\delta}(y^2 - 1)^{1-\varepsilon} \] \[ + \left( \frac{(l - mx)^{\varepsilon-\delta}}{l^{\varepsilon}(y - x)^{2\varepsilon}} (l - my)^{\delta}(y^2 - 1)^{\varepsilon} \right) dt^2 . (3.5)
\]

The C-metric is then given by \( \varepsilon = \delta = 1 \), i.e.

\[
ds^2 = \frac{1}{(y - x)^2} \left\{ 4l(l - m)^2 \left( \frac{dx^2}{X} - \frac{dy^2}{Y} \right) + 4lXd\varphi^2 + Y \frac{dt^2}{l^2} \right\} . (3.6)
\]

Note that the simple coordinate transformation

\[
\varphi = 2^{\delta^2-1}(l - m)^{\delta^2\varepsilon} \tilde{\varphi} , \hspace{1cm} \hspace{1cm} (3.7a) \\
t = 2^{\delta^2} l^{\varepsilon}(l - m)^{\delta^2 \varepsilon} \tilde{t} , \hspace{1cm} (3.7b)
\]

will give a common factor to the metrics. In the case of the C-metric we get

\[
ds^2 = \frac{4l(l - m)^2}{(y - x)^2} \left\{ \frac{dx^2}{X} - \frac{dy^2}{Y} + X d\varphi^2 + Y dt^2 \right\} . (3.8)
\]

Under this rescaling of \( \varphi \) the scalar \( C \) in (3.3) will be scaled by \( 2^{2\delta^2-2}(l - m)^{2\delta^2} \), and so the periodicity of the new axisymmetric Killing vector \( \partial_{\varphi} \) will not be \( 2\pi \).
4: Charged C metric

Metrics (3.6) and (3.8) are also solutions of the vacuum field equations for any general coordinate cubic $P_3$ (i.e. one with arbitrary coefficients), and as is well known they are solutions of the non-null Einstein-Maxwell equations (with aligned electromagnetic field) when $P_4$ is a general quartic. If we define

$$X = (1 - fx)(mx - l)(x^2 - 1) ,$$

(4.1)

and $Y \equiv X(y)$, then (3.6) is an Einstein-Maxwell spacetime and the uncharged C-metric is given by $f = 0$.

Using (3.8) we now have the charged C-metric in the same form as that in Kinnersley and Walker [3], apart from the form of the quartic, which just arises from affine transformations of $x$ and $y$. We identify the mass and charge in terms of our parameters $m, l$ and $f$ by going to retarded half-null coordinates

$$Au = t - \int^y Y^{-1}(\tilde{y})d\tilde{y} ,$$

(4.2a)

$$Ar = 1/(y - x) ,$$

(4.2b)

where $A^{-1} = 2\sqrt{l}(l - m)$. The metric (3.8) becomes

$$ds^2 = r^2 \left( \frac{dx^2}{X} + Xd\varphi^2 \right) + H(r, x)du^2 + 2dudr - 2Ar^2dudx ,$$

(4.3)

where $H(r, x) = A^2r^2X(x + 1/Ar)$. Define the following tetrad

$$\ell_a = du ,$$

(4.4a)

$$n_a = \frac{H}{2}du + dr - Ar^2dx ,$$

(4.4b)

$$m_a = \frac{r}{\sqrt{2}} \left( \frac{dx}{\sqrt{X}} + i\sqrt{X}d\varphi \right) .$$

(4.4c)

Calculating the Weyl and Ricci tensors in this tetrad (using EXCALC in REDUCE) gives only two non-zero Newman-Penrose scalars:

$$\Psi_2 = -\frac{1}{r^3} \left( \frac{(m + fl)}{2A} - \frac{2mf}{A} \right) + \frac{1}{r^4} \frac{mf}{2A^2} ,$$

(4.5a)

$$\Phi_{11} = \frac{1}{r^4} \frac{mf}{2A^2} .$$

(4.5b)

Here the tetrad is aligned to give the canonical forms of the Weyl and Ricci tensors. By comparison with Kinnersley and Walker [3], the mass is $(m + fl)/2A$ and the square of the electric charge is $mf/2A^2$. Clearly the flat space limit is given by $m = f = 0$ (we cannot have $m = l = 0$). The mass term is zero if $m + fl = 0$. From (4.3) it is straightforward to analyse the horizons and other geometric properties, with the restriction on $m$ and $l$.
removed, as in Kinnersley and Walker [3], and Ashtekar and Dray [17]. The advantage of this formulation is that the coordinate ranges are easily specified. For example, the flat space limit shows that the constant $l$ in here is the acceleration as defined in [3].

However, in a similar manner to the way we transformed the regions of the C-metric in section 1, we can use linear fractional transformations to change the quartic of the charged C-metric to a cubic (cf. Lun [14]). In order to use the $(\rho, z)$ coordinate transformation we will aim to make the cubic that appears after the transformation $P_3(w) = (w^2 - 1)(mw - l)$, and to do this we redefine the coefficients so that we begin with the quartic

$$P_4(w) = \frac{(w^2 - 1)(1 - fw)(w(m + lf) - (l + mf))}{1 - f^2}.$$  

Then setting \[ x = \frac{\dot{x} + f}{1 + f\dot{x}} \quad \text{and} \quad y = \frac{\dot{y} + f}{1 + f\dot{y}}, \]  
we get the metric in the form

$$ds^2 = \frac{2}{(\dot{x} - \dot{y})^2} \left\{ L(\dot{x})L(\dot{y})(l - m)^2 \left( \frac{d\dot{x}^2}{P_3(\dot{x})} - \frac{d\dot{y}^2}{P_3(\dot{y})} \right) + \frac{1}{L(\dot{x})L(\dot{y})} \left[ P_3(\dot{x})L(\dot{y})^2d\varphi^2 + P_3(\dot{y})L(\dot{x})^2dt^2 \right] \right\},$$

where $L(w) = (1 + fw)^2/(1 - f^2)$. Clearly, $f = 0$ gives the C-metric (3.6).

Since this is a static axisymmetric solution, we can look for the Weyl canonical coordinates. By definition $-\rho^2$ is the scalar $2\xi^a\eta_b\xi^a\eta^b$ where $\xi^a$ is the timelike and $\eta^a$ the axisymmetric Killing vectors [2]. A simple calculation shows that in the coordinates of equation (4.7), this scalar is $4P_3(\dot{x})P_3(\dot{y})/(\dot{x} - \dot{y})^4$ as it is in the C-metric (see equation (2.9)). The coordinate $z$ is then the harmonic conjugate, exactly as in equation (2.8).

We can therefore use equations (2.11) to transform this metric into the Weyl canonical coordinates. The functions $L$ and $P$ will be very messy, but clearly we can write

$$L(\dot{x}) = \alpha(\rho, z)/\Delta^2,$$
$$P_3(\dot{x}) = Q_1(\rho, z)/\Delta^3,$$
$$L(\dot{y}) = \beta(\rho, z)/\Delta^2,$$
$$P_3(\dot{y}) = Q_2(\rho, z)/\Delta^3,$$

where $\alpha, \beta, Q_1$ and $Q_2$ are polynomials in the $R_j$ and $\Delta$ is as in equations (2.11) and (2.12). In fact with $k = 2(l^2 - m^2)$,

$$Q_1 = k^2(R_+ + R_ - + 2m)(R_ - R_ + (m + l))(R_ + R_ - (m - l)),$$
$$Q_2 = k^2(R_+ + R_ - - 2m)(R_ - R_ + (m + l))(R_ + R_ - (m - l)).$$

The metric is then

$$ds^2 = \frac{1}{2k^2} \left\{ \frac{\alpha\beta(l - m)^2}{\Delta^2 R_+ R_- R_l} (d\rho^2 + dz^2) + \frac{1}{\Delta} \left( Q_1 \frac{\alpha}{\beta} d\varphi^2 + Q_2 \frac{\beta}{\alpha} dt^2 \right) \right\}.$$
Note that by comparison with the form (1.1) of the metric that \( \rho^2 = -Q_1 Q_2 / (4 \Delta^2 k^4) \) and the Ernst potential \( e^{2\mathcal{U}} \) is \( Q_2 \alpha / (2 \beta \Delta k^2) \).

The C-metric will be given by \( f = 0 \). In this case we have \( \alpha = \beta = \Delta^2 \), and the metric takes the form

\[
d s^2 = \frac{1}{2k^2} \left\{ \frac{\Delta^2 (l - m)^2}{R_- R_+ R_l} (d\rho^2 + dz^2) + \frac{Q_1}{\Delta} d\varphi^2 + \frac{Q_2}{\Delta} dt^2 \right\},
\]

which may be compared to equations (3.1). In particular, from (3.1d) with \( \delta = \varepsilon = 1 \) and the value of \( K \) from (3.4b) we have

\[
e^{2(\mathcal{V} - \mathcal{U})} = \frac{4l(l - m)^2}{R_- R_+ R_l (y - x)^2} = \frac{1}{R_- R_+ R_l} \frac{\Delta^2}{8(l + m)^2}.
\]

5: The twisting C-metric

We can now proceed to look at a twisting version of the C-metric due to Weir and Kerr [7]; viz.

\[
d s^2 = \frac{1}{(x - y)^2} \left\{ g \left[ \frac{d x^2}{\mathcal{P}_4(x)} - \frac{d y^2}{\mathcal{P}_4(y)} \right] + \frac{1}{g} \left[ \mathcal{P}_4(x) (d\varphi + S(y) dt)^2 + \mathcal{P}_4(y) (dt - S(x) d\varphi)^2 \right] \right\}.
\]

Here \( \mathcal{P}_4(w) = f w^4 + dw^3 + cw^2 + bw + a \), \( g = 1 + S(x) S(y) \), \( S(w) = q^2(w - r) \) and \( a, b, c, d, f, q, r \) are real constants. We assume that \( f > 0 \) and to preserve signature we stick to a patch in the \( (x, y) \) plane where \( \mathcal{P}_4(x) > 0 \) and \( \mathcal{P}_4(y) < 0 \).

The vacuum field equations reduce to the single algebraic requirement that

\[
\Phi_{11} = 0 \iff q^4 \mathcal{P}_4(r) + f = 0.
\]

Note that in Weir and Kerr [7], \( r = 0 \) (which we can always assume to be so by a shift of origin of \( x \) and \( y \)) and so \( f = -q^4 a \); we prefer to write this as \( q = (-f/a)^{1/4} \) (whence the twist parameter \( q \) is part of the coordinates) so that \( f = 0 \) gives the C-metric as static limit. However, we could assume that \( f = 0 \) so that the fundamental polynomial \( X \) was a cubic. Again \( q = 0 \) is the C-metric, but we could also leave \( q \) arbitrary and make \( r \) a root of the polynomial in order to satisfy the field equations. This metric is then also a twisting form of the C-metric, which it becomes if \( q = 0 \).

We can calculate the twist potential in these cases. Let \( \xi^a = \partial_t \) be the timelike Killing vector. We define \( \omega_a = \star (\xi_a \wedge d\xi_a) \), where \( \star \) is the Hodge dual operator. It follows from the field equations that \( d\omega_a = 0 \), and so \( \omega_a = d\omega \) for the twist potential \( \omega \). In the case of the general quartic with \( r = 0 \), we find that this scalar \( \omega \) is given by

\[
\omega = \frac{2y q^2 a}{(x - y)} - \frac{y^2 q^2}{(x - y) g} (xyd - b).
\]

If we have a cubic instead, then we can assume that it is written as \( P_3(x) = (mx - l)(x^2 - 1) \), where \( l > m \), by the usual affine transformation of \( x \) and \( y \) which will not alter
the form of the metric. Hence in this case we have three choices for the constant $r$: 1, $-1$ or $l/m$. We find that
\[ \omega = -q^2(y - r)^2 R(r)/g \] with $g$ as in (5.1), and the $R(r)$ are
\[ R(1) = (m(xy - x - y - 1) + 2l)/(x - y) = R_+ , \]
\[ R(-1) = (m(xy + x + y - 1) - 2l)/(x - y) = R_- , \]
\[ R(l/m) = (m(xy + 1) - l(x + y))/(x - y) = R_l . \]

These latter functions we recognise as (to within a sign) the distances $R_j$ considered in section 2. In the patch of the $(x, y)$ plane called Region(I) in section 2, for the C-metric, the finite rod is given by $y = l/m$ and the semi-infinite one by $y = 1$. Thus for $r = 1$ the twist potential is zero on the part of the $z$-axis corresponding to the semi-infinite rod, and for $r = l/m$ it is zero on the part corresponding to the finite rod (cf. Hoenselaers [11])

In order to make some headway in the interpretation of the various coordinate patches of the twisting C-metric, we would like to introduce the Weyl canonical coordinates into metric (5.1). A convenient way of doing this is to realise that the appearance of a quartic in the metric is accidental. From the work of Lun [14] used in the charged C-metric, the twisting C-metric can be transformed so that the fundamental polynomial is a cubic. From there, we find that the Weyl canonical coordinates are related to new “cubic” coordinates in the same way that the coordinates of the static C-metric are to that metric’s Weyl canonical coordinates: this will also take the twist out of the coordinates.

In order to use that coordinate transformation we will aim to make the cubic that appears after the transformation $P_3(w) = (w^2 - 1)(mw - l)$, and to do this we begin, as in the charged C-metric case, with the quartic
\[ P_4(w) = \frac{(w^2 - 1)(1 - fw)(w(m + lf) - (l + mf))}{1 - f^2} . \]

Then set
\[ x = \frac{\hat{x} + f}{1 + f\hat{x}} \quad \text{and} \quad y = \frac{\hat{y} + f}{1 + f\hat{y}} . \] (5.5)

As before, we get the metric in the form
\[ ds^2 = \frac{2}{(\hat{x} - \hat{y})^2} \left\{ G(\hat{x}, \hat{y}) \left( \frac{d\hat{x}}{P_3(\hat{x})} - \frac{d\hat{y}}{P_3(\hat{y})} \right)^2 + \frac{1}{G(\hat{x}, \hat{y})} \left[ P_3(\hat{x}) \left[ L(\hat{y})d\varphi + T(\hat{y})dt \right]^2 + P_3(\hat{y}) \left[ L(\hat{x})dt - T(\hat{x})d\varphi \right]^2 \right] \right\} , \] (5.6)

where the constant conformal factor 2 has been added for convenience and here
\[ L(w) = (1 + fw)^2/(1 - f^2) , \]
\[ G(\hat{x}, \hat{y}) = g(x(\hat{x}), y(\hat{y})) L(\hat{x})L(\hat{y}) = L(\hat{x})L(\hat{y}) + T(\hat{x})T(\hat{y}) , \]
\[ T(w) = S(w)L(w) = \frac{q^2}{1 - f^2} (w(1 - rf) + f - r)^2 . \]
Clearly, \( q = f = 0 \) gives the usual C-metric, \( q = 0 \) gives the charged C-metric of equation (4.8) and \( f = 0 \) with \( q \neq 0 \) and \( r \) a root of \( P_4 \) gives the new metrics mentioned above.

It was claimed above that the Weyl canonical coordinates for this metric are the same as for the C-metric. This follows from the definition of \( -\rho^2 \) as \( 2\xi(\eta^b \xi^a) \). A simple calculation shows that in the coordinates of equation (5.5), this scalar is \( 4P_3(\hat{x})P_3(\hat{y})/(\hat{x} - \hat{y})^4 \) as it is in the C-metric (see equation (1.9)). The coordinate \( z \) is then the harmonic conjugate, exactly as in equation (2.9). We can therefore use equations (2.11) and (2.12) to transform this metric into the Weyl canonical coordinates: the result is unedifyingly messy in the general case.

However, with \( f = 0 \) we have \( L(w) = 1 \), so we can write

\[
T(\hat{x}) = q^2 \alpha(\rho, z)/\Delta^2, \quad T(\hat{y}) = q^2 \beta(\rho, z)/\Delta^2, \quad G = 1 + q^2 \alpha \beta/\Delta^4 = \gamma(\rho, z)/\Delta^4, \\
P_3(\hat{x}) = Q_1(\rho, z)/\Delta^3, \quad P_3(\hat{y}) = Q_2(\rho, z)/\Delta^3, \quad k = 2(l^2 - m^2),
\]

where \( \alpha, \beta, \gamma, Q_1 \) and \( Q_2 \) are polynomials in the \( R_j \) (the \( Q_i \) are as in section 4, but the \( \alpha \) and \( \beta \) are not) and \( \Delta \) is as in equation (2.12). Then

\[
ds^2 = \frac{2}{k^2} \left\{ \frac{\gamma}{\Delta^2 R_- R_+ R_i} (d\rho^2 + dz^2) + \frac{1}{\gamma \Delta} \left[ Q_1 \left[ \Delta^2 d\varphi + q^2 \beta dt \right]^2 + Q_2 \left[ \Delta^2 dt - q^2 \alpha d\varphi \right]^2 \right] \right\}.
\]

As can easily be seen, if \( q = 0 \) this reduces to the C-metric as in equations (5.9), modulo some constant conformal factors.

6: Conclusion

We have demonstrated how the “deformed” C-metrics, the charged C-metric and the twisting C-metrics can be rewritten in accelerated coordinates, which involve a cubic with only one essential parameter \( m/l \); viz. the ratio of mass to acceleration. The Ernst equation can now be written in accelerated coordinates. Hence metrics (3.5), (4.7) and (5.7) would give rise to the corresponding Ernst potentials. It is also well known that prolate spheroidal coordinates play a major role to allow the “deformed” Kerr metrics [19] to be expressed in terms of rational functions of these coordinates. Hence in these coordinates, the “deformed” Kerr metrics are in a way better suited to computerized algebraic manipulations. The application of accelerated coordinates to find the “deformed” twisting C-metrics is expected to lead to more tractable expressions in these cases.

In section two, we have given a detailed analysis on how to derive the transformation from the Weyl canonical coordinates to accelerated coordinates together with the inverse transformation. In there we have formulated the problem by relating it to the addition theorem considered by Lagrange, Abel and Riemann. This has the advantage that one can make use of existing results on Riemann surfaces, elliptic and hyperelliptic functions. One can also infer from [14] that the analysis there can be extended to quartics with the double Schwarzschild metric and the double Kerr metric (cf. [11]) taking the place of the C-metric.
and the twisting C-metric, respectively, in identifying the essential parameters to be used in the coordinate quartic. Preliminary calculations suggest that extension of the ansatz is possible but it is not known whether all the metric coefficients are now rational functions of the quartic coordinates. Finally, the intriguing question is in what form the generalization of the results in section 2 will take when quintics and higher degree polynomials are used instead.

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References


