(6.0) Areas by slicing

The area $A$ under a curve of height $f(x)$ between points $a$ and $b$ can be computed by slicing the region into thin strips and approximating these strips by rectangles with height $f(x_i)$, width $\Delta x_i$, and area $f(x_i)\Delta x_i$.

$$A = \lim_{\Delta_i \to 0} \sum_{i} f(x_i)\Delta x_i = \int_{a}^{b} f(x) \, dx$$
(6.1) Volumes by slicing
The slicing technique can be extended to three dimensions to find volumes:

We put an axis through the volume and then slice it perpendicularly to the axis.
• Suppose the volume $V$ is contained between the points $a$ and $b$ on the axis.

• Suppose the slice through $x_i$ has cross-sectional area $A(x_i)$ and thickness $\Delta x_i$, then the volume of the slice is approximately $A(x_i)\Delta x_i$.

• The total volume is approximated by a sum of volumes of such slices as $\Delta x \to 0$

$\text{Volume} = \lim_{\Delta x_i \to 0} \sum_i A(x_i)\Delta x_i = \int_a^b A(x) \, dx$
Example

Find the volume in the first quadrant of the tetrahedron bounded by the three co-ordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$
Example

Find the volume in the first quadrant of the tetrahedron bounded by the three co-ordinate planes and the plane

\[ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \]

\[ A(x) = \frac{1}{2} y(x) z(x) \]
Example

Find the volume in the first quadrant of the tetrahedron bounded by the three co-ordinate planes and the plane

\[ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \]

\[
V = \int_{0}^{a} A(x) \, dx = \int_{0}^{a} \frac{1}{2} (-\frac{b}{a}x + b)(-\frac{c}{a}x + c) \, dx = \frac{abc}{6}
\]
(6.1.1) Solids of Revolution

A solid of revolution is obtained by rotating a curve \( y = f(x) \) about the \( x \)-axis between \( a \) and \( b \).

The cross-sections are disks of radius \( f(x) \), and area \( A(x) = \pi f(x)^2 \).

Volume \( = \int_a^b A(x) \, dx = \pi \int_a^b f(x)^2 \, dx \)
Examples
Find the volume of the solid obtained by rotating the curve \( y(x) = x^2 \) around the \( x \)-axis between 0 and 1.

\[
V = \pi \int_0^1 y(x)^2 \, dx = \pi \int_0^1 x^4 \, dx = \frac{\pi}{5}
\]
Example
Find the volume obtained by rotating the curve \( y(x) = x^2 \) around the \( y \)-axis.
Example
Find the volume obtained by rotating the curve \( y(x) = x^2 \) around the \( y \)-axis.

For rotating about the \( y \)-axis, the formula is \( V = \pi \int_a^b x^2 \, dy \). For this curve \( x(y) = \sqrt{y} \), so
\[
V = \pi \int_0^1 y \, dy = \frac{2}{3}.
\]
Example
Find the volume obtained by rotating the curve \( y(x) = x^2 \) around the \( y \)-axis.

For rotating about the \( y \)-axis, the formula is \( V = \pi \int x(y)^2 \, dy \). For this curve \( x(y) = \sqrt{y} \), so

\[
V = \pi \int_0^1 y \, dy = \frac{\pi}{2}.
\]
Example
Find the volume of the right circular cone with height $h$ and radius of base $r$. 
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Example
Find the volume of the right circular cone with height \( h \) and radius of base \( r \).

In the figure above the cone is obtained by rotating the line \( y = \frac{h}{r}x \) about the \( y \) axis. Thus

\[
V = \pi \int x(y)^2 \, dy = \pi \int_0^h \frac{r}{h}y \, dy = \frac{\pi r^2 h}{3}
\]
Example
Find the volume of the torus obtained by rotating the disc \((x - a)^2 + y^2 = b^2\) around the \(y\)-axis (where \(b < a\)).
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Find the volume of the torus obtained by rotating the disc \((x - a)^2 + y^2 = b^2\) around the \(y\)-axis (where \(b < a\)).

\[
V = \pi \int_{-b}^{b} (x_1(y)^2 - x_2(y)^2) \, dy
\]
Example

Find the volume of the torus obtained by rotating the disc \((x - a)^2 + y^2 = b^2\) around the \(y\)-axis (where \(b < a\)).

\[
V = \pi \int_{-b}^{b} (x_1(y)^2 - x_2(y)^2) \, dy
\]

\[
V = \pi \int_{-b}^{b} (a + (b^2 - y^2)^{1/2})^2 - (a - (b^2 - y^2)^{1/2})^2 \, dy
\]

\[
= 4a\pi \int_{-b}^{b} (b^2 - y^2)^{1/2} \, dy = \frac{4a\pi^2 b^2}{2}
\]
(6.2) Arc length

Slice the length into small pieces, and approximate the length of each piece \( k \) by the length of the chord \( \ell_k \) joining the endpoints: if the slice has width \( \Delta x_k = x_k - x_{k-1} \), then

\[
\ell_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.
\]
The total arc length from $a$ to $b$ is

$$\ell \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \Delta x_k$$

$$\quad \to \quad \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{as} \quad n \to \infty$$

**Definition**

The arc length of the curve $y = f(x)$ between $x = a$ and $x = b$ is

$$\int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx$$

(provided $f$ is differentiable and the integral exists).
Example
Find the arc length of \( y = 2 + 3x^2 \) between \( x = 0 \) and \( x = 1 \).
Example
Find the arc length of $y = 2 + 3x^2$ between $x = 0$ and $x = 1$.

\[
\ell = \int_0^1 \sqrt{1 + y'(x)^2} = \int_0^1 \sqrt{1 + (6x)^2} \, dx
\]
Example
Find the arc length of $y = 2 + 3x^2$ between $x = 0$ and $x = 1$.

\[
\ell = \int_0^1 \sqrt{1 + y'(x)^2} = \int_0^1 \sqrt{1 + (6x)^2} \, dx
\]

put $6x = \sinh u$
Example
Find the arc length of $y = 2 + 3x^2$ between $x = 0$ and $x = 1$.

\[ l = \int_0^1 \sqrt{1 + (6x)^2} \, dx \]

Put $6x = \sinh u$

\[ l = \int_0^{\sinh^{-1} 6} \sqrt{1 + \sinh^2 u} \frac{\cosh u}{6} \, du \]

\[ = \frac{1}{6} \int_0^{\sinh^{-1} 6} \left( \frac{1 + \cosh 2u}{2} \right) \, du \]

\[ = \frac{1}{12} \left( \sinh^{-1} 6 + 6\sqrt{37} \right) \approx 3.24903 \]
Example

Find the arc length of \( y = \ln x \) between \( x = 1 \) and \( x = \sqrt{3} \).
Example

Find the arc length of \( y = \ln x \) between \( x = 1 \) and \( x = \sqrt{3} \).

\[
\ell = \int_{1}^{\sqrt{3}} \sqrt{1 + \left(\frac{1}{x}\right)^2} \, dx = \int_{1}^{\sqrt{3}} \frac{\sqrt{x^2 + 1}}{x} \, dx
\]
Example

Find the arc length of \( y = \ln x \) between \( x = 1 \) and \( x = \sqrt{3} \).

\[
\ell = \int_1^{\sqrt{3}} \sqrt{1 + \left(\frac{1}{x}\right)^2} \, dx = \int_1^{\sqrt{3}} \frac{\sqrt{x^2 + 1}}{x} \, dx
\]

put \( u = \sqrt{x^2 + 1} \) \( \Rightarrow \) \( du = \frac{1}{2} \frac{2x}{\sqrt{x^2 + 1}} \) \( \Rightarrow \) \( dx = \frac{u}{x} \, du \)

\[
\Rightarrow \quad \frac{\sqrt{x^2 + 1}}{x} \, dx = \left(\frac{u}{x}\right)\left(\frac{u}{x}\right) du = \frac{u^2}{x^2} \, du = \frac{u^2}{u^2 - 1} \, du
\]
Example
Find the arc length of $y = \ln x$ between $x = 1$ and $x = \sqrt{3}$.

$$\ell = \int_{1}^{\sqrt{3}} \sqrt{1 + \left(\frac{1}{x}\right)^2} \, dx = \int_{1}^{\sqrt{3}} \frac{\sqrt{x^2 + 1}}{x} \, dx$$

put $u = \sqrt{x^2 + 1} \Rightarrow du = \frac{1}{2} \frac{2x}{\sqrt{x^2 + 1}} \Rightarrow dx = \frac{u}{x} \, du$

$$\Rightarrow \frac{\sqrt{x^2 + 1}}{x} \, dx = \left(\frac{u}{x}\right)\left(\frac{u}{x}\right) \, du = \frac{u^2}{x^2} \, du = \frac{u^2}{u^2 - 1} \, du$$

$$\ell = \int_{\sqrt{2}}^{2} \frac{u^2}{u^2 - 1} \, du = \int_{\sqrt{2}}^{2} 1 + \frac{1}{u^2 - 1} \, du$$

$$= \int_{\sqrt{2}}^{2} \, du + \frac{1}{2} \int_{\sqrt{2}}^{2} \frac{1}{u - 1} \, du - \int_{\sqrt{2}}^{2} \frac{1}{u + 1} \, du$$

$$= 2 - \sqrt{2} + \ln \left(\frac{\sqrt{2} + 1}{\sqrt{3}}\right)$$
Parametric form

Suppose that a curve is represented in parametric form by $(x(t), y(t))$.

The arc length between $t = a$ and $t = b$ is given by

\[
\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
\]
Example
Find the arc length of the cycloid defined by
\[ x(\theta) = a(1 - \cos \theta) \] and \[ y(\theta) = a(\theta - \sin \theta) \] for \( 0 \leq \theta \leq 2\pi \).
Example ctd.

Note that \( \frac{dx}{d\theta} = a \sin \theta \) and \( \frac{dy}{d\theta} = a(1 - \cos \theta) \) so that the arc length of the cycloid is

\[
\ell = \int_0^{2\pi} \sqrt{a^2 \sin^2 \theta + a^2 (1 - \cos \theta)^2} \, d\theta = a \int_0^{2\pi} \sqrt{2 - 2 \cos \theta} \, d\theta
\]

\[
= a \int_0^{2\pi} \sqrt{2 - 2 \left( \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right)} \, d\theta = a \int_0^{2\pi} 2 \sqrt{\sin^2 \frac{\theta}{2}} \, d\theta
\]

\[
= 4a \int_0^{\pi} \sin \frac{\theta}{2} \, d\theta = 8a \left[ - \cos \frac{\theta}{2} \right]_0^\pi = 8a
\]
Surfer’s Law

The optimal path for a big wave surfer to drop down the face of a big wave, do a bottom turn and return to the top without getting closed out is a cycloid. This manoeuvre is not possible unless $H > 2\frac{v_s^2}{g}$.
Surfing Cycloids?
(6.3) Surface Area – Solids of Revolution

Consider a cone of length \( \ell \) along the side with base radius \( R \). If we cut the cone and flatten it out we see that it is a sector of a circle subtended by an angle \( \theta = \frac{2\pi R}{\ell} \). The area is then \( A = \frac{1}{2} \ell^2 \theta = \pi R \ell \). Similarly the area of the smaller cone of length \( \ell - s \) and base radius \( r \) is \( a = \pi r (\ell - s) \).

The area of the \text{frustrum} (shaded yellow) is the difference \( A - a = \pi (R + r) s \).
Surface Area – Sum of Frustrum Strip Areas

Let \( y = f(x) \) be rotated around the \( x \)-axis between \( x = a \) and \( x = b \).

Slice the surface into strips, and approximate each strip by the **frustrum of a cone** having radii \( y_k \) and \( y_k + \Delta y_k \), and slant height \( \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \).

The area of the \( k \)th frustrum is
\[
\pi (2y_k + \Delta y_k) \sqrt{1 + \left( \frac{\Delta y_k}{\Delta x_k} \right)^2} \Delta x_k.
\]

The **total surface area** is the sum of the areas of the strips in the limit \( \Delta x \to 0 \).
Surface Area Formulae

Cartesian Form

\[ S' = \lim_{\Delta x \to 0} \sum_{k} \pi(2y + \Delta y) \sqrt{1 + \left( \frac{\Delta y}{\Delta x} \right)^2} \Delta x \]
\[ = \int_{a}^{b} 2\pi y(x) \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

Parametric Form

For parametric curves \((x(t), y(t))\)

\[ S = \int_{a}^{b} 2\pi y(t) \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]
Example
Find the surface area of the cone of base radius $r$ and height $h$. 
Example
Find the surface area of the cone of base radius $r$ and height $h$.

rotation about $y$ axis

$$S = \int_0^h 2\pi x(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$
Example ctd.

\[ y = \frac{h}{r} x \quad \Rightarrow \quad x = \frac{r}{h} y \]

\[ \Rightarrow S = \int_0^h 2\pi \frac{ry}{h} \sqrt{1 + \left(\frac{r^2}{h^2}\right)} \, dy \]

\[ = 2\pi \frac{r}{h} \sqrt{1 + \left(\frac{r^2}{h^2}\right)} \int_0^h y \, dy \]

\[ = 2\pi r \frac{h^2}{2} \sqrt{1 + \left(\frac{r^2}{h^2}\right)} \]

\[ = \pi r \sqrt{r^2 + h^2} = \pi r \ell \]

where \( \ell \) denotes the slant height of the cone.
Example – Gabriel’s Horn or Torricelli’s Trumpet
Find the surface area and volume of the hyperboloid formed by rotating $y = \frac{1}{x}$ between $x = 1$ and $x \to \infty$
Example – Gabriel’s Horn or Torricelli’s Trumpet

Find the surface area and volume of the hyperboloid formed by rotating \( y = \frac{1}{x} \) between \( x = 1 \) and \( x \to \infty \).
Example – Gabriel’s Horn or Torricelli’s Trumpet
Find the surface area and volume of the hyperboloid formed by
rotating \( y = \frac{1}{x} \) between \( x = 1 \) and \( x \to \infty \)

\[
S = 2\pi \int_1^\infty y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
= 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \left(\frac{1}{x^2}\right)} \, dx
> 2\pi \int_1^\infty \frac{1}{x} \, dx \to \infty
\]
Example – Gabriel’s Horn or Torricelli’s Trumpet
Find the surface area and volume of the hyperboloid formed by rotating \( y = \frac{1}{x} \) between \( x = 1 \) and \( x \to \infty \)

\[
S = 2\pi \int_1^\infty y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

\[
= 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \left(\frac{1}{x^2}\right)} \, dx
\]

\[
> 2\pi \int_1^\infty \frac{1}{x} \, dx \to \infty
\]

\[
V = \pi \int_1^\infty y(x)^2 \, dx = \pi \int_1^\infty \frac{1}{x^2} \, dx = \pi
\]
(6.4) Averages, moments, centres of mass

Average value of a function

If $f$ is integrable on $[a, b]$ then the average $\bar{f}$ of $f$ on the interval is

$$\bar{f} = \frac{\int_a^b f(x) \, dx}{b - a}$$

Justification

Divide the interval into $n$ subintervals of equal width $\Delta = \frac{b-a}{n}$ and let $x_k$ denote a point in the $k$th subinterval then

$$\frac{\int_a^b f(x) \, dx}{b - a} = \lim_{n \to \infty} \frac{\sum_{k=1}^n f(x_k) \Delta}{b - a} = \lim_{n \to \infty} \frac{\sum_{k=1}^n f(x_k)(b - a)}{(b - a)n}$$

$$= \lim_{n \to \infty} \frac{1}{n} [f(x_1) + f(x_2) + \ldots + f(x_n)] = \bar{f}$$
Example

If \( f(t) \) is a periodic function, ie., \( f(t) = f(t + T) \) then the average value of \( f(t) \) over one period is

\[
\bar{f}(t) = \frac{1}{T} \int_{0}^{T} f(t) \, dt
\]
Example
If \( f(t) \) is a periodic function, i.e., \( f(t) = f(t + T) \) then the average value of \( f(t) \) over one period is

\[
\bar{f}(t) = \frac{1}{T} \int_{0}^{T} f(t) \, dt
\]

> From the definition with \( a = t \) and \( b = t + T \)

\[
I(t) = \bar{f}(t) = \frac{\int_{t}^{t+T} f(s) \, ds}{t + T - t} = \frac{1}{T} \int_{t}^{t+T} f(s) \, ds
\]

\[
\Rightarrow I'(t) = \frac{1}{T} \frac{d}{dt} \int_{t}^{t+T} f(s) \, ds = \frac{1}{T} (f(t + T) - f(t)) = 0
\]
Example

If \( f(t) \) is a periodic function, i.e., \( f(t) = f(t + T) \) then the average value of \( f(t) \) over one period is

\[
\bar{f}(t) = \frac{1}{T} \int_{0}^{T} f(t) \, dt
\]

>From the definition with \( a = t \) and \( b = t + T \)

\[
I(t) = \bar{f}(t) = \frac{\int_{t}^{t+T} f(s) \, ds}{t + T - t} = \frac{1}{T} \int_{t}^{t+T} f(s) \, ds
\]

\[
\Rightarrow I'(t) = \frac{1}{T} \frac{d}{dt} \int_{t}^{t+T} f(s) \, ds = \frac{1}{T} (f(t + T) - f(t)) = 0
\]

Hence \( I(t) \) is a constant (independent of \( t \)) so choose \( t = 0 \) and \( \bar{f}(t) = I(t) = I(0) = \frac{1}{T} \int_{0}^{T} f(s) \, ds \)
Centre of mass of a rod
Consider a rigid rod (weightless) with two masses $m_1, m_2$ at $x_1, x_2$ supported at $x_0$

Each mass $m_k$ exerts a turning force at $x_0$ called its moment about $x_0$ and given by $m_k(x_k - x_0)$.

The system is in balance if the total moment about $x_0$ is zero.

The balance point is called the centre of mass.
The condition for balance with two masses is

\[ m_1(x_1 - x_0) + m_2(x_2 - x_0) = 0 \]

\[ \Rightarrow m_1x_1 + m_2x_2 = (m_1 + m_2)x_0 \]

\[ \Rightarrow \bar{x} = x_0 = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \]

The centre of mass with \( n \) masses is

\[ \bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\text{total moment about } x_0}{\text{total mass}} \]
Continuous Rod

What is the centre of mass for a continuous rod?

For a continuous rod define the mass per unit length $\rho(x)$ at each point $x$

$$\rho(x) = \lim_{\Delta l(x) \to 0} \frac{\Delta m(x)}{\Delta l(x)}$$

$\Delta l(x)$ is the length of a small segment of the rod around $x$
$\Delta m(x)$ is the mass of this segment.

The centre of mass is defined by

$$\overline{x} = \frac{\text{total moment about the origin}}{\text{total mass}}.$$
Total Mass of Rod
Consider the rod located on an interval $0 \leq x \leq L$ on the $x$-axis with mass density function $\rho(x)$.

The total mass is the sum of masses for each segment in the limit $\Delta x \to 0$ hence

$$M = \int_0^L \rho(x) \, dx.$$
Total Moment of Rod
Consider the segment from $x$ to $x + \Delta x$. If $\Delta x$ is small then all points in the interval $[x, x + \Delta x]$ are approximately the same distance $x$ from the origin, so the moment of this segment about the origin is approximately

$$x \Delta M \approx x \rho(x) \Delta x.$$

The total moment is the sum of moments for all segments in the limit $\Delta x \to 0$ hence

$$\text{total moment} = \int_0^L x \rho(x) \, dx.$$
Centre of Mass of Rod

\[
\bar{x} = \frac{\text{total moment about the origin}}{\text{total mass}}
\]

\[
= \frac{\int_0^L x \rho(x) \, dx}{M}
\]

\[
= \frac{\int_0^L x \rho(x) \, dx}{\int_0^L \rho(x) \, dx}
\]
Example – Centroid

If the density is a constant, say $\rho_0$, then the centre of mass is called the centroid

$$
\bar{x} = \frac{\rho_0 \int_0^L x \, dx}{\rho_0 \int_0^L dx} = \frac{L}{2}.
$$

In this case the centre of mass is at the midpoint of the rod. In other words, the centroid of an interval is at its midpoint.
Example

The density of a 4 metre non-uniform metal rod, measured in kg/m, is given by \( \rho(x) = 2\sqrt{x} \) where \( x \) is the distance from the left hand end of the rod.

(a) What total mass lies to the left of the point \( x \) m from the left hand end?

(b) What is the total mass of the rod?

(c) What is its average density?

(d) Where is the centre of mass of the rod?
Solution

(a) The mass to the left of $x$ is

$$R \int_0^t \, dt = \frac{4}{3} \times 3 = 2 \text{ kg}.$$ 

(b) The total mass is

$$R \int_0^4 \, dt = \frac{32}{3} \text{ kg}.$$ 

(c) The average density is

$$\frac{1}{4} \int_0^4 \, dt = \frac{8}{3} \text{ kg/m}.$$ 

(d) The $x$-coordinate of the centre of mass is

$$x = \frac{R \int_0^4 \, dt}{R \int_0^4 \, dt} = \frac{2}{4};$$

so the centre of mass is located $\frac{2}{4} \text{ m}$ from the lefthand end of the rod.
Solution

(a) The mass to the left of \( x \) is \( \int_0^x \rho(t) \, dt = \frac{4}{3} x^{3/2} \) kg.
Solution

(a) The mass to the left of $x$ is $\int_0^x \rho(t) \, dt = \frac{4}{3} x^{3/2}$ kg.

(b) The total mass is $\int_0^4 \rho(t) \, dt = \frac{32}{3}$ kg.
Solution

(a) The mass to the left of $x$ is $\int_0^x \rho(t) \, dt = \frac{4}{3} x^{3/2}$ kg.

(b) The total mass is $\int_0^4 \rho(t) \, dt = \frac{32}{3}$ kg.

(c) The average density is $\frac{1}{4} \int_0^4 \rho(t) \, dt = \frac{8}{3}$ kg/m.
Solution

(a) The mass to the left of $x$ is $\int_0^x \rho(t) \, dt = \frac{4}{3} x^{3/2}$ kg.

(b) The total mass is $\int_0^4 \rho(t) \, dt = \frac{32}{3}$ kg.

(c) The average density is $\frac{1}{4} \int_0^4 \rho(t) \, dt = \frac{8}{3}$ kg/m.

(d) The $x$-coordinate of the centre of mass is

$$\bar{x} = \frac{\int_0^4 t \rho(t) \, dt}{\int_0^4 \rho(t) \, dt} = 2.4,$$

so the centre of mass is located 2.4 m from the lefthand end of the rod.
Problem

Consider a rod of length 6 starting at 0 and made by welding together three metal rods $A, B, C$ of lengths 2, 1, 3 respectively.

Suppose that the three rods have densities

$\rho_A(x) = x^2 \text{ kg/m}$, $\rho_B(x) = 3 \text{ kg/m}$, $\rho_C(x) = \frac{4}{x^2} \text{ kg/m}$.

Find the centre of mass.
Solution
First write the density $\rho$ of the welded rod as

$$\rho(x) = \begin{cases} 
  x^2 & \text{if } x \in [0, 2] \\
  3 & \text{if } x \in (2, 3] \\
  \frac{4}{x^2} & \text{if } x \in (3, 6]
\end{cases}$$
Solution
First write the density $\rho$ of the welded rod as

$$\rho(x) = \begin{cases} 
x^2 & \text{if } x \in [0, 2] \\
3 & \text{if } x \in (2, 3) \\
\frac{4}{x^2} & \text{if } x \in (3, 6) 
\end{cases}$$

Total mass $\quad = \int_0^6 \rho(x) \, dx$
Solution
First write the density $\rho$ of the welded rod as

$$\rho(x) = \begin{cases} 
x^2 & \text{if } x \in [0, 2] \\
3 & \text{if } x \in (2, 3] \\
\frac{4}{x^2} & \text{if } x \in (3, 6] 
\end{cases}$$

Total mass $= \int_0^6 \rho(x) \, dx$

Total moment about the origin $= \int_0^6 x \rho(x) \, dx$
Solution
First write the density $\rho$ of the welded rod as

$$\rho(x) = \begin{cases} 
  x^2 & \text{if } x \in [0, 2] \\
  3 & \text{if } x \in (2, 3) \\
  \frac{4}{x^2} & \text{if } x \in (3, 6)
\end{cases}$$

Total mass
$$= \int_0^6 \rho(x) \, dx$$

Total moment about the origin
$$= \int_0^6 x \rho(x) \, dx$$

Centre of mass
$$= \frac{\text{Total moment}}{\text{Total mass}}$$
Example – Cricket Bat Sweet Spot

How far from the end of the bat should a batter hit a ball to minimize the impact felt by the hands on the bat?
The motion of the bat can be represented as translational motion of the centre of mass of the bat plus rotational motion about the centre of mass

• If the ball strikes the centre of mass position then the bat will just have translational motion pushing back on the batters hand.

• If the ball strikes near the tip then there will be strong rotational motion causing the fingers to open up.
There is a point of balance (the centre of percussion) between the backwards translational motion of the hand and the forwards rotational motion where no impulsive force is felt.

Swing the bat from a point of pivot where you would grip the bat and time the period $\tau$. 
Example ctd.

The distance from the point of pivot to the minimum impact distance (centre of percussion) is

\[ \ell = \frac{\pi^2 g}{4\pi} \]

Note if \( \bar{x} \) is the distance from the point of pivot to the centre of mass, \( I \) is the rotational moment of inertia around an axis through the point of pivot and \( M \) is the total mass of the bat then

\[ \ell = \frac{I}{M \bar{x}} \]
Example ctd.

- maximum width 4.25 inches
- maximum length 38 inches
- cricket bat willow (extremely light) has a density 417 kg per cubic metre
- profile designed to have greatest thickness at the centre of percussion usually about 180mm from the bottom
- the centre of mass is usually about 350mm from the bottom
Example – Racing Yachts
Racing yachts have large sails that provide large moments that need to be balanced by a large mass well below the water line – hence the **keel bulb**
(6.5) Length and area in polar coordinates

**Area**

The area of the wedge is

\[
\frac{\Delta \theta}{2\pi} \times (\text{area of circle of radius } r) = \frac{1}{2} r^2 \Delta \theta.
\]

Hence the area “enclosed” by the curve \( r = f(\theta) \) for \( \theta_1 \leq \theta \leq \theta_2 \) is

\[
\frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta.
\]
Arc Length

>From the diagram the arc length of the segment is

\[ (\Delta \ell)^2 \approx (\Delta r)^2 + (r \Delta \theta)^2 \]

\[ = \left( \left( \frac{\Delta r}{\Delta \theta} \right)^2 + r^2 \right)(\Delta \theta)^2 \]

The total arc length is the sum of arc lengths of all segments

\[ \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta \]
All the arc length formulæ we have seen can be obtained by manipulating the formal expression

\[
\text{arc length} = \int \sqrt{(dx)^2 + (dy)^2}. \]

The connection between cartesian and polar coordinates is given by \( x = r \cos \theta, \ y = r \sin \theta \), so that

\[
\begin{align*}
 dx &= (-r \sin \theta + \frac{dr}{d\theta} \cos \theta) \, d\theta \\
 dy &= (r \cos \theta + \frac{dr}{d\theta} \sin \theta) \, d\theta
\end{align*}
\]

\[
\text{arc length} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta
\]
Example

For the cardioid $r = 1 + \sin \theta$

(a) Calculate the area enclosed.
(b) Find the length of the curve.
Example
Find the length of the parabolic spiral \( r = a\theta^2 \) between \( \theta = 0 \) and \( \theta = \pi \).

\[
\ell = \int_0^{\pi} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta
\]

\[
= \int_0^{\pi} \sqrt{a^4\theta^4 + (2a\theta)^2} \, d\theta = a \int_0^{\pi} \sqrt{\theta^4 + 4\theta^2} \, d\theta
\]

\[
= a \int_0^{\pi} \theta \sqrt{\theta^2 + 4} \, d\theta = a \left[ \frac{(\theta^2 + 4)^{3/2}}{3} \right]_0^\pi
\]

\[
= a \left( \frac{\pi^2 + 4)^{3/2}}{3} - 8 \right) \approx .52278a
\]
Example – The Logarithmic Spiral

\[ r = ae^{b\theta} \]
Example – Goat Feeding

A goat is tethered by a rope of length $L$ to a post on the outside of a circular fence of radius $R$. What is the total area of the grazing region available to the goat?