MATH1231 CALCULUS
Session II 2007.

Dr John Roberts (notes written by A./Prof. Bruce Henry)

Red Center Room 3065
Jag.Roberts@unsw.edu.au
(5.2) Definition
A power series (around $x = 0$) is a function of the form

\[ P(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n \]

$a_0, a_1, \ldots$ are fixed real constants

$x$ is a real variable.

All the standard functions of calculus can be written \textbf{exactly}
in the form of \textbf{power series} over a finite domain.
Example

\[ \tan x \]

\[ x \]

\[ x + \frac{1}{3} x^3 \]

\[ x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^{17} \]
How do we determine the convergence of a power series?

How do we find the power series that represents a given function?

How do we find the error in approximating the function with a finite number of terms in the power series?
Example

\[ P(x) = \sum_{n=0}^{\infty} x^n \quad \text{geometric series} \]

If \( |x| \geq 1 \) then \( P(x) \) diverges

If \( |x| < 1 \) then \( P(x) = \frac{1}{1-x} \)

\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad \text{for} \quad |x| < 1 \]
Example

\[ P(x) = \sum_{n=0}^{\infty} \frac{x^n}{n} \]

Ratio test (for absolute convergence)

\[
\lim_{n \to \infty} \left| \frac{x^{n+1}}{n+1} \right| = \lim_{n \to \infty} \frac{n|x|}{n + 1} \to |x|
\]

* If \(|x| > 1\) then \(P(x)\) diverges
* If \(|x| < 1\) then \(P(x)\) converges

What about the end points?
* If \(x = 1\) then \(P(x) = \sum \frac{1}{n}\) diverges (harmonic)
* If \(x = -1\) then \(P(x) = \sum \frac{(-1)^n}{n}\) converges (Leibniz)
Example

\[ P(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

Ratio test (for absolute convergence)

\[
\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \right| = \lim_{n \to \infty} \frac{|x|}{n + 1} \to 0 \quad \forall x
\]

\[ P(x) \text{ converges for all } x \]
(5.3) Radius of Convergence

All power series (around $x = 0$) converge on a symmetric domain $x \in (-R, R)$ where $R$ is called the radius of convergence.

**Theorem**

i) $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely whenever $|x| < R$

ii) $\sum_{n=0}^{\infty} a_n x^n$ diverges whenever $|x| > R$

If the series converges for all $x$ then $R = \infty$
If the series converges only for $x = 0$ then $R = 0$
Interval of Convergence

The power series

\[ P(x) = \sum_{n=0}^{\infty} a_n x^n \]

may or may not converge at the end points \( x = R \) and \( x = -R \) these points need to be tested separately

The interval \((-R, R)\) together with those endpoints where the power series converges is called the interval of convergence

The interval of convergence is the domain of the function

\[ P(x) = \sum_{n=0}^{\infty} a_n x^n \]
Theorem

\[ R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{if this limit exists} \]

Proof

Consider the ratio test for \( \sum a_n x^n \)

\[
\lim_{n \to \infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|
\]

The series diverges if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x| > 1 \)

The series converges if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x| < 1 \)

\[ \Rightarrow |x| < \lim_{n \to \infty} \frac{1}{\left| \frac{a_{n+1}}{a_n} \right|} = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = R \]
(5.4) Power series in powers of \((x - x_0)\)

**Definition**

A power series around \(x_0\) is a function of the form

\[
P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots = \sum_{n=0}^{\infty} a_n(x - x_0)^n
\]

\(a_0, a_1, \ldots\) are fixed real constants

\(x\) is a real variable.

**Note**

The results for power series around \(x_0\) can be obtained from results for power series around \(0\) after a change of variables \(u = x - x_0\).
Define the radius of convergence

\[ R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \]

then

\[ \sum_{n=0}^{\infty} a_n(x - x_0)^n \] converges absolutely for \( |x - x_0| < R \)

\[ \sum_{n=0}^{\infty} a_n(x - x_0)^n \] diverges for \( |x - x_0| > R \)

Also check convergence at the end points \( x = x_0 \pm R \)
Example
Find the interval of convergence for

\[ P(x) = \sum_{n=0}^{\infty} \left( \frac{2^n}{3^n + 1} \right) (x - 2)^n \]
Example
Find the interval of convergence for

\[ P(x) = \sum_{n=0}^{\infty} \left( \frac{2^n}{(3^n + 1)n} \right) (x - 2)^n \]
(5.5) Differentiation and Integration of Power Series

If \( P(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \) for \(|x - x_0| < R\) then

1. \( P \) is continuous and differentiable for \(|x - x_0| < R\)
   \[
   \frac{dP}{dx} = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}
   \]

2. \( P \) is integrable on \(|x - x_0| < R\)
   \[
   \int P(x) \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C
   \]

convergent power series can be differentiated and integrated term by term.
Example

\[ f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ \Rightarrow f'(x) = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \]

\[ = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x) \]

But \( f'(x) = f(x) \) with \( f(0) = 1 \) has solution \( f(x) = e^x \)

\[ \Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]
Theorem

If \( P(x) = \sum a_n(x - x_0)^n \quad |x - x_0| < R_1 \)
and \( Q(x) = \sum b_n(x - x_0)^n \quad |x - x_0| < R_2 \)

then for \( |x - x_0| < \min(R_1, R_2) \)

(i) \( (P + Q)(x) = \sum (a_n + b_n)(x - x_0)^n \)
(ii) \( (P \cdot Q)(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)(x - x_0) + \\
+ (a_0 b_2 + a_1 b_1 + a_2 b_0)(x - x_0)^2 + \\
+ (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)(x - x_0)^3 + \ldots \)
Example – February 1998

\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \quad \forall \quad |x| < R \in \mathbb{R}^+ \]

a) By integrating find a power series for \( \log(1-x) \quad \forall \quad |x| < R. \)

b) Write down the power series for \( \log(1+x) \quad \forall \quad |x| < R \)

c) Deduce that

\[ \log \left( \frac{1+x}{1-x} \right) = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \right) \quad \forall \quad |x| < R \]

d) Find \( R \)

e) Use the first two terms of the series in part c) to find a rational number to approximate \( \log 2 \)
(5.7) Taylor Series and Maclaurin Series

Many well known functions $f(x)$ can be written as power series, $f(x) = \sum_{n=0} a_n x^n$.

How do we determine the coefficients $a_n$?

**Definition**
Suppose that $f(x)$ is infinitely differentiable at $x_0$ then

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

is the **Taylor series of $f$ about $x_0$**
Definition
A Taylor series of $f$ about $x_0 = 0$ is called a Maclaurin series.

Theorem
If $f$ has any power series about $x_0$ then it is the Taylor series

Note
All standard functions $f(x)$ are equal to their Taylor series $P(x)$. 
Proof of Theorem

Suppose that $f(x)$ is infinitely differentiable at $x_0$ and suppose that $f(x)$ can be written as a power series. Then

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$\Rightarrow f(x_0) = \sum_{n=0}^{\infty} a_n 0^n = a_0 0^0 = a_0$$

$$f'(x) = \sum_{n=0}^{\infty} na_n (x - x_0)^{(n-1)} = \sum_{n=1}^{\infty} na_n (x - x_0)^{(n-1)}$$

$$\Rightarrow f'(x_0) = \sum_{n=1}^{\infty} na_n 0^{(n-1)} = 1 \times a_1 0^0 = 1 \times a_1$$
Proof Ctd

\[ f''(x) = \sum_{n=2}^{\infty} (n - 1)na_n(x - x_0)^{(n-1)} \]

\[ \Rightarrow f''(x_0) = \sum_{n=2}^{\infty} (n - 1)na_n0^{(n-2)} = 1 \times 2 \times a_2 \]

\[ f'''(x) = \sum_{n=3}^{\infty} (n - 2)(n - 1)na_n(x - x_0)^{(n-3)} \]

\[ \Rightarrow f'''(x_0) = \sum_{n=3}^{\infty} (n - 2)(n - 1)na_n0^{(n-3)} = 1 \times 2 \times 3 \times a_3 \]

\[ \vdots \]

\[ f^{(j)}(x_0) = j!a_j \]

\[ \Rightarrow a_n = \frac{f^{(n)}(x_0)}{n!} \]
Example Find the Maclaurin Series for \( \sin x \).

\[
\begin{align*}
    f(x) &= \sin x & f(0) &= 0 \\
    f'(x) &= \cos x & f'(0) &= 1 \\
    f''(x) &= -\sin x & f''(0) &= 0 \\
    f'''(x) &= -\cos x & f'''(0) &= -1 \\
    f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0
\end{align*}
\]

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1}
\]

\[
= 0 + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}
\]
(5.8) Common Taylor Series

\[
\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k \quad |x| < 1
\]

\[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad |x| < \infty
\]

\[
\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad |x| < \infty
\]

\[
\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad |x| < \infty
\]
Taylor Series to Remember

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \ldots \quad |x| < 1
\]

\[
\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \quad |x| < 1
\]

\[
e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots
\]
(5.9) Convergence of Taylor Series

Definition
The \textit{\textit{n}}\textsuperscript{th} \textit{order Taylor polynomial} for \textit{f} around \textit{x}₀ is

\[ P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \]

Definition
The \textbf{remainder} is the difference

\[ R_n(x) = f(x) - P_n(x) \]

Note
The \textbf{remainder} is also called the \textbf{error}. It is the error in approximating a function with its \textit{n}\textsuperscript{th} order Taylor polynomial
Taylor’s Theorem

If \( f(x) \) is (at least) \( n + 1 \) times differentiable on \( I = (x_0 - a, x_0 + a) \) then for each \( x \in I \) there exists \( c \) between \( x \) and \( x_0 \) so that

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}
\]

and

\[
f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}
\]

Note

If \( n = 0 \) then Taylor’s Theorem is the same as the Mean Value Theorem.
Corollary

If $f$ is infinitely differentiable and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $x$ when $|x - x_0| < R$ then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \text{ when } |x - x_0| < R$$

where $R$ is the radius of convergence.
Example
Find the Taylor series for $e^x$ about $x = 1$ and show that it converges to $e^x$ for all $x$.

To find the Taylor series we compute the coefficients defined by

$$a_n = \frac{f^{(n)}(x_0)}{n!} = \frac{e^{x_0}}{n!} = \frac{e}{n!}$$

Then the Taylor series is

$$e^x = \sum_{n=0}^{\infty} \frac{e}{n!} (x - 1)^n = \sum_{n=0}^{N} \frac{e}{n!} (x - 1)^n + R_N(x)$$

where

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N + 1)!} (x - x_0)^{(N+1)} = \frac{e^c}{(N + 1)!} (x - 1)^{(N+1)}$$
Example ctd.

Now $c$ lies between $x$ and 1 hence $e^c$ is finite and

$$\lim_{N \to \infty} \frac{(x - 1)^{(N+1)}}{(N + 1)!} \to \lim_{N \to \infty} \frac{(x - 1)}{(N + 1)} \frac{(x - 1)}{N} \ldots \frac{(x - 1)}{1} \to 0$$

so that $\lim_{N \to \infty} R_N(x) \to 0$

Thus from Taylor’s theorem we have

$$e^x = \sum_{n=0}^{\infty} \frac{e}{n!} (x - 1)^n \quad \forall \quad |x - 1| < R$$

where the radius of convergence is

$$R = \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \frac{e}{n!} \frac{(n + 1)!}{e} = \lim_{n \to \infty} \frac{n + 1}{1} \to \infty$$
(5.10) Estimations using the remainder term

If we approximate a function by a Taylor Polynomial then the remainder term provides the error in this approximation

\[ |f(x) - P_n(x)| = |R_n(x)| \equiv |f^{(n+1)}(c)| \left| \frac{(x - x_0)^{n+1}}{(n + 1)!} \right| \]

where \( c \) is between \( x \) and \( x_0 \)

It is usually not possible to find \( c \) but it is often possible to find an upper bound on the error term.
Example

The Taylor series for \( \cos x \) about \( x_0 = 0 \) is

\[
\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}
\]

Find the lowest degree Taylor polynomial about \( x_0 = 0 \) to provide an approximation for \( \cos x \) that differs by less than 0.001 for all \( x \) in the interval \( -\frac{\pi}{6} < x < \frac{\pi}{6} \).
(5.11) Applications to maxima, minima, indeterminate forms

**Theorem**

Suppose that $f$ is $n$ times differentiable at $x_0$ and $f'(x_0) = 0$. Then if $k \leq n$ and

$$f''(x_0) = f'''(x_0) = \ldots = f^{(k-1)}(x_0) = 0 \quad \text{but} \quad f^{(k)}(x_0) \neq 0$$

we have

a local minimum at $x_0$ if $k$ is even and $f^{(k)}(x_0) > 0$

a local maximum at $x_0$ if $k$ is even and $f^{(k)}(x_0) < 0$

an inflexion minimum at $x_0$ if $k$ is odd

**Proof** based on Taylor’s Theorem – see notes
Example

The Taylor series for $\sin x$ about $x_0 = 0$ is

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k + 1)!}$$

Show that $f(x) = \sin x^6$ has a stationary point at $x = 0$ and determine whether it is a maximum, minimum or point of inflexion?
Application to L’Hopital’s Rule

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \ldots}{g(x_0) + g'(x_0)(x-x_0) + \frac{1}{2!}g''(x_0)(x-x_0)^2 + \ldots}
\]

If \( f(x_0) = g(x_0) = 0 \) but \( f'(x_0) \neq 0, g'(x_0) \neq 0 \) then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x_0)}{g'(x_0)}
\]

If \( f(x_0) = g(x_0) = f'(x_0) = g'(x_0) = 0 \) but \( f''(x_0) \neq 0, g''(x_0) \neq 0 \) then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f''(x_0)}{g''(x_0)}
\]
Example

The Taylor series for $\ln(1 + x)$ about $x_0 = 0$ is

$$\ln(1 + x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

Find

$$\lim_{x \to 0} \frac{\ln(1 + x) - x}{x^2}$$

$$\lim_{x \to 0} \frac{\ln(1 + x) - x}{x^2} = \lim_{x \to 0} \frac{-\frac{x^2}{2} + \frac{x^3}{3} - \ldots}{x^2} = -\frac{1}{2}$$
(5.12) MAPLE Notes

taylor(expr, x=a, k); computes Taylor series of expr about x=a up to the term of order k

convert(taylor(expr, x=a, k), polynom); computes the Taylor polynomial of order \( k-1 \) for expr about x=a

readlib(coeftayl);

coefftayl(expr, x=a, k); computes the \( k \)th coefficient in the Taylor series expansion expr about x=a.