MATH1231 CALCULUS

Session II 2008.

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Integration Techniques

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$$\int g'(x) f'(g(x))\,dx = \int \frac{d}{dx} f(g(x))\,dx = f(g(x)) + C$$

Examples

$$\int 6x(1+x^2)^2\,dx =$$

$$\int \cosh x \cosh(\sinh x) =$$

$$\int 2e^{2x}\cos(e^{2x})\,dx =$$

$$\int \tan(x)\,dx =$$

Examples

$$\int 6x(1+x^2)^2\,dx = (1+x^2)^3 + C$$

$$\int \cosh x \cosh(\sinh x) = \sinh(\sinh x) + C$$

$$\int 2e^{2x}\cos(e^{2x})\,dx = \sin(e^{2x}) + C$$

$$\int \tan(x)\,dx = -\log(\cos x) + C$$

(0.1) Substitution

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$$I = \int f(x)\,dx$$

Put $x = g(u)$ then $f(x) = f(g(u))$ and $dx = g'(u)\,du$.

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Put $x = g(u)$ then $f(x) = f(g(u))$ and $dx = g'(u)\, du$.

$$I = \int f(g(u))g'(u)\, du = \int h(u)\, du$$

This is only useful if $\int h(u)\, du$ is easier than $\int f(x)\, dx$.

Example

$$\int \frac{\cos x}{1+\sin x}\,dx$$

Example

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Put $\boxed{\sin x = u}$ $\quad\Rightarrow\quad$ $\boxed{\cos x\,dx = du}$

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$$\int \frac{\cos x}{1+\sin x}\,dx$$

Put $\boxed{\sin x = u}$ $\Rightarrow$ $\boxed{\cos x\,dx = du}$

$$\int \frac{\cos x}{1+\sin x}\,dx = \int \frac{1}{1+u}\,du = \log(1+u)+C = \log(1+\sin x)+C$$

(0.2) Integration by parts

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$$\int u(x)\frac{dv}{dx}\,dx = u(x)v(x) - \int v(x)\frac{du}{dx}\,dx$$

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Example

$$\int \underbrace{x}_{u(x)}\underbrace{e^x}_{\frac{dv}{dx}}\,dx = \underbrace{x}_{u(x)}\underbrace{e^x}_{v(x)} - \int \underbrace{e^x}_{v(x)} \cdot \underbrace{1}_{\frac{du}{dx}}\,dx = xe^x - e^x + C$$

(0.2) Integration by parts

$$\int u(x)\frac{dv}{dx}\,dx = u(x)v(x) - \int v(x)\frac{du}{dx}\,dx$$

Example

$$\int \underbrace{x}_{u(x)}\underbrace{e^x}_{\frac{dv}{dx}}\,dx = \underbrace{x}_{u(x)}\underbrace{e^x}_{v(x)} - \int \underbrace{e^x}_{v(x)} \cdot \underbrace{1}_{\frac{du}{dx}}\,dx = xe^x - e^x + C$$

Try

$$\int \underbrace{x^3}_{u(x)}\underbrace{e^x}_{\frac{dv}{dx}}\,dx$$

(2.1) **Trigonometric Integrals**

$$\int \sin^3 x \cos^6 x \, dx$$

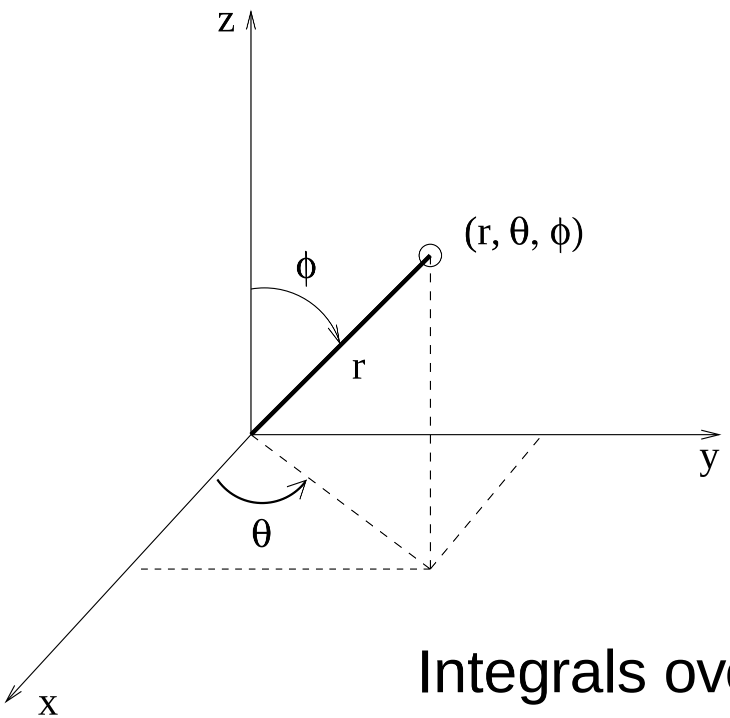
$$\int \cos^8 x \, dx$$

$$\int \tan^3 x \sec^2 x \, dx$$

$$\int \sinh^3 x \cosh^6 x \, dx$$

Who Cares?

Many problems in three dimensional space can be simplified using spherical polar co-ordinates



$$
\begin{aligned}
x &= r \sin\phi \cos\theta \\
y &= r \sin\phi \sin\theta \\
z &= r \cos\phi
\end{aligned}
$$

Integrals over powers of trig functions then arise, e.g., volume calculations, charge distributions, acoustic vibrations, angular momentum of electrons, trajectories of falling space junk

http://sn-callisto.jsc.nasa.gov/index.html

Example

$$
\begin{aligned}
\int \cos^3 x \, dx &= \int \cos x \cos^2 x \, dx \\
&= \int \cos x (1 - \sin^2 x) \, dx \\
&= \int (1 - u^2) \, du \qquad u = \sin x \\
&= u - \frac{1}{3} u^3 \\
&= \cos x - \frac{1}{3} \cos^3 x + C
\end{aligned}
$$

(2.1.1) $\boxed{\int \cos^m x \sin^n x \, dx}$

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- If m is **odd**, say $m = 2k + 1$, then factor

$$\cos^m x \sin^n x = \cos x (\cos^2 x)^k \sin^n x = \cos x (1 - \sin^2 x)^k \sin^n x$$

The substitution $\boxed{u = \sin x, \quad du = \cos x \quad dx}$ gives a polynomial to integrate.

(2.1.1) $\boxed{\int \cos^m x \sin^n x\, dx}$

- If m is **odd**, say $m = 2k+1$, then factor

$$\cos^m x \sin^n x = \cos x(\cos^2 x)^k \sin^n x = \cos x(1-\sin^2 x)^k \sin^n x$$

The substitution $\boxed{u = \sin x, \quad du = \cos x \quad dx}$ gives a polynomial to integrate.

- If n is **odd**, use the analogous method with the roles of $\sin x$ and $\cos x$ interchanged.

Examples

$$\int \cos^3 x \sin^4 x \, dx =$$

$$\int \sin^3 x \cos^2 x \, dx =$$

$$\int \sin^7 x \, dx =$$

Examples

$$\int \cos^3 x \sin^4 x \, dx = \int (\cos x) \cos^2 x \sin^4 x \, dx$$

$$\int \sin^3 x \cos^2 x \, dx = \int (\sin x) \sin^2 x \cos^2 x \, dx$$

$$\int \sin^7 x \, dx = \int (\sin x) \sin^6 x \, dx$$

Examples

$$\int \cos^3 x \sin^4 x \, dx = \int (\cos x)(1 - \sin^2 x) \sin^4 x \, dx$$

$$\int \sin^3 x \cos^2 x \, dx = \int (\sin x)(1 - \cos^2 x) \cos^2 x \, dx$$

$$\int \sin^7 x \, dx = \int (\sin x)(1 - \cos^2 x)^3 \, dx$$

Examples

$$\int \cos^3 x \sin^4 x \, dx = \int (\cos x) \underbrace{(1 - \sin^2 x) \sin^4 x}_{u=\sin x} \, dx$$

$$\int \sin^3 x \cos^2 x \, dx = \int (\sin x) \underbrace{(1 - \cos^2 x) \cos^2 x}_{u=\cos x} \, dx$$

$$\int \sin^7 x \, dx = \int (\sin x) \underbrace{(1 - \cos^2 x)^3}_{u=\cos x} \, dx$$

$\int \cos^m x \sin^n x \, dx$

• If **both** m and n are **even**, we use the identities

$$\cos^2 x = \frac{1 + \cos 2x}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

to change the integral into a **sum** of integrals of the form

$$\int \cos^j 2x \, dx$$

if j is **odd**, we use the method of the first section

if j is **even**, we use the method of this section again.

Example

$$I = \int \sin^2 x \cos^2 x \, dx$$

Example

$I = \int \sin^2 x \cos^2 x \, dx$

$$I = \int (\frac{1 - \cos 2x}{2})(\frac{1 + \cos 2x}{2}) \, dx$$

Example

$I = \int \sin^2 x \cos^2 x \, dx$

$$I = \int (\frac{1 - \cos 2x}{2})(\frac{1 + \cos 2x}{2}) \, dx$$

$$I = \int (\frac{1 - \cos^2 2x}{4}) \, dx = \frac{1}{4} \int 1 - \cos^2 2x \, dx$$

Example

$I = \int \sin^2 x \cos^2 x \, dx$

$$I = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) dx$$

$$I = \int \left(\frac{1 - \cos^2 2x}{4}\right) dx = \frac{1}{4} \int 1 - \cos^2 2x \, dx$$

$$I = \frac{1}{4} \int 1 - \left(\frac{1 + \cos 4x}{2}\right) dx$$

(2.1.2)

$$\boxed{\int \sin nx \cos mx \, dx}$$

$$\boxed{\int \sin nx \sin mx \, dx}$$

$$\boxed{\int \cos nx \cos mx \, dx}$$

Use the identities

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right) \\
\sin A \sin B &= \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \\
\cos A \cos B &= \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)
\end{aligned}$$

Example

$$\int \sin 3x \sin 2x \, dx = \frac{1}{2} \int \left(\cos(3x - 2x) - \cos(3x + 2x) \right) \, dx$$

(2.1.3) $\boxed{\int \tan^n x \, dx}$ or $\boxed{\int \sec^n x \, dx}$

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Exploit the relationships

$$\sec^2 x = 1 + \tan^2 x, \qquad \frac{d}{dx} \tan x = \sec^2 x, \qquad \frac{d}{dx} \sec x = \sec x \tan x$$

(2.1.3) $\boxed{\int \tan^n x\, dx}$ or $\boxed{\int \sec^n x\, dx}$

Exploit the relationships

$$\sec^2 x = 1+\tan^2 x, \quad \frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\sec x = \sec x \tan x$$

•Reduction Formula (apply repeatedly)

$$\begin{aligned} I_n = \int \tan^n x dx &= \int \tan^{n-2} x(\tan^2 x)dx \\ &= \int \tan^{n-2} x(\sec^2 x - 1)dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \end{aligned}$$

(until) $I_1 = \int \frac{\sin x}{\cos x}\, dx = \ln|\sec x| + C$ (or) $I_0 = \int dx = x + C$

Example

$$I_4 = \int \tan^4 x \, dx$$

(2.1.4) $\boxed{\int \tan^m x \sec^n x \, dx}$

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Examples

(2.1.4) $\boxed{\int \tan^m x \sec^n x \, dx}$

Examples

$$\int \tan^2 x \overbrace{\sec^4 x}^{even} \, dx = \int (\tan^2 x \sec^2 x) \overbrace{\sec^2 x}^{factor} \, dx$$

$$= \int \underbrace{\tan^2 x (1 + \tan^2 x)}_{u = \tan x} \sec^2 x \, dx$$

(2.2) **Reduction formulae**

Suppose we have an integrand that is a function of both an independent variable x and an integral index n

e.g., $\int (\log x)^n \, dx$, $\int x^n e^x \, dx$, $\int x^n \cos x \, dx$

If we now integrate by parts and obtain a similar integrand but with a smaller index n then we have a reduction formula that can be applied repeatedly.

In general if $\int [f(x)]^n g(x) \, dx$ then integrate by parts with $u(x) = [f(x)]^n$ and $\frac{dv}{dx} = g(x)$

Example

$I_n = \int x^n e^x \, dx$

Integrate by parts with $u = x^n$ and $\frac{dv}{dx} = e^x$

$$\begin{aligned} I_n &= \int x^n e^x \, dx \\ &= x^n e^x - n \int x^{n-1} e^x \, dx \\ &= x^n e^x - n I_{n-1} \end{aligned}$$

and $I_0 = e^x$.

Example

$I_n = \int \sin^n x \, dx$.

Integrate by parts with $u = \sin^{n-1} x$ and $\frac{dv}{dx} = \sin x$

$$\begin{aligned} I_n &= -\sin^{n-1} x \cos x + \int ((n-1)\sin^{n-2} x \cos x) \cos x \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n \end{aligned}$$

Now solve for I_n

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \left(\frac{n-1}{n}\right) I_{n-2}.$$

⋆ Wallis's Product

$$\int_0^{\pi/2} \sin^{2m} x\,dx = \left(\frac{2m-1}{2m}\right)\left(\frac{2m-3}{2m-2}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2} \sin^{2m+1} x\,dx = \left(\frac{2m}{2m+1}\right)\left(\frac{2m-2}{2m-1}\right)\cdots\left(\frac{2}{3}\right)$$

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2m}{2m-1}\left(\frac{2m}{2m+1}\frac{\int_0^{\pi/2}\sin^{2m} x\,dx}{\int_0^{\pi/2}\sin^{2m+1} x\,dx}\right)$$

$$\lim_{m\to\infty}\frac{\pi}{2} = \lim_{m\to\infty}\frac{2^2}{3^2}\frac{4^2}{5^2}\frac{6^2}{7^2}\cdots\frac{(2m-2)^2}{(2m-1)^2}2m \qquad \sqrt{\pi} = \lim_{m\to\infty}\frac{(m!)^2 2^{2m}}{(2m)!\sqrt{m}}$$

(2.2.3) Proposition

$$I_{m,n} = \int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \begin{cases} \left(\frac{m-1}{m+n}\right) I_{m-2,n} & \text{provided} \quad m \geq 2 \\ \left(\frac{n-1}{m+n}\right) I_{m,n-2} & \text{provided} \quad n \geq 2 \end{cases}$$

$$I_{1,1} = \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = \frac{1}{2} \qquad I_{1,0} = \int_0^{\pi/2} \cos\theta \, d\theta = 1$$

$$I_{0,1} = \int_0^{\pi/2} \sin\theta \, d\theta = 1 \qquad I_{0,0} = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

Proof: See Notes

Example

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin^4 x \, dx = I_{3,4}$$

Sample Test Question

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n \theta \sec \theta \, d\theta$$

Show that

$$I_n = \frac{1}{n}\left(\sqrt{2} - (n-1)I_{n-2}\right) \qquad n \geq 2$$

Hint

$$\sec \theta \tan \theta = \frac{d}{d\theta}(\sec \theta)$$

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{4}} \underbrace{\tan^{n-1}\theta}_{u} \underbrace{\tan\theta\sec\theta}_{\frac{dv}{d\theta}} \, d\theta \\
&= \tan^{n-1}\theta\sec\theta \,\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta (n-1)\tan^{n-2}\theta\sec^2\theta \, d\theta
\end{aligned}
$$

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{4}} \underbrace{\tan^{n-1}\theta}_{u}\,\underbrace{\tan\theta\sec\theta}_{\frac{dv}{d\theta}}\; d\theta \\
&= \tan^{n-1}\theta\sec\theta\,\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta(n-1)\tan^{n-2}\theta\sec^2\theta\,d\theta
\end{aligned}
$$

$$
\begin{aligned}
I_n &= \sqrt{2} - (n-1)\int_0^{\frac{\pi}{4}} \sec\theta\tan^{n-2}\theta\left(\tan^2\theta + 1\right)\, d\theta \\
&= \sqrt{2} - (n-1)\int_0^{\frac{\pi}{4}} \sec\theta\tan^{n}\theta\,d\theta - (n-1)\int_0^{\frac{\pi}{4}} \sec\theta\tan^{n-2}\theta\,d\theta \\
&= \sqrt{2} - (n-1)I_n - (n-1)I_{n-2}
\end{aligned}
$$

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{4}} \underbrace{\tan^{n-1}\theta}_{u}\,\underbrace{\tan\theta\sec\theta}_{\frac{dv}{d\theta}}\,d\theta \\
&= \tan^{n-1}\theta\sec\theta\,\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta(n-1)\tan^{n-2}\theta\sec^2\theta\,d\theta
\end{aligned}
$$

$$
\begin{aligned}
I_n &= \sqrt{2} - (n-1)\int_0^{\frac{\pi}{4}} \sec\theta\tan^{n-2}\theta\left(\tan^2\theta + 1\right)\,d\theta \\
&= \sqrt{2} - (n-1)\int_0^{\frac{\pi}{4}} \sec\theta\tan^{n}\theta\,d\theta - (n-1)\int_0^{\frac{\pi}{4}} \sec\theta\tan^{n-2}\theta\,d\theta \\
&= \sqrt{2} - (n-1)I_n - (n-1)I_{n-2}
\end{aligned}
$$

solve for I_n

(2.3) **Trigonometric and hyperbolic substitution**

Substitution is inspired guesswork. This table is for inspiration.

integral factor	substitutions		identities
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ $dx = a\cos\theta\,d\theta$	$x = a\cos\theta$ $dx = -a\sin\theta\,d\theta$	$\cos^2\theta + \sin^2\theta = 1$
$\sqrt{a^2 + x^2}$	$x = a\sinh\theta$ $dx = a\cosh\theta\,d\theta$	$x = a\tan\theta$ $dx = a\sec^2\theta\,d\theta$	$\sec^2\theta = \tan^2\theta + 1$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ $dx = a\sec\theta\tan\theta\,d\theta$	$x = a\cosh\theta$ $dx = a\sinh\theta\,d\theta$	$\cosh^2\theta = \sinh^2\theta + 1$

Example

$$I = \int \sqrt{1 - x^2}\, dx$$

Example

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substitution $x = \sin\theta, \qquad \frac{dx}{d\theta} = \cos\theta$

Example

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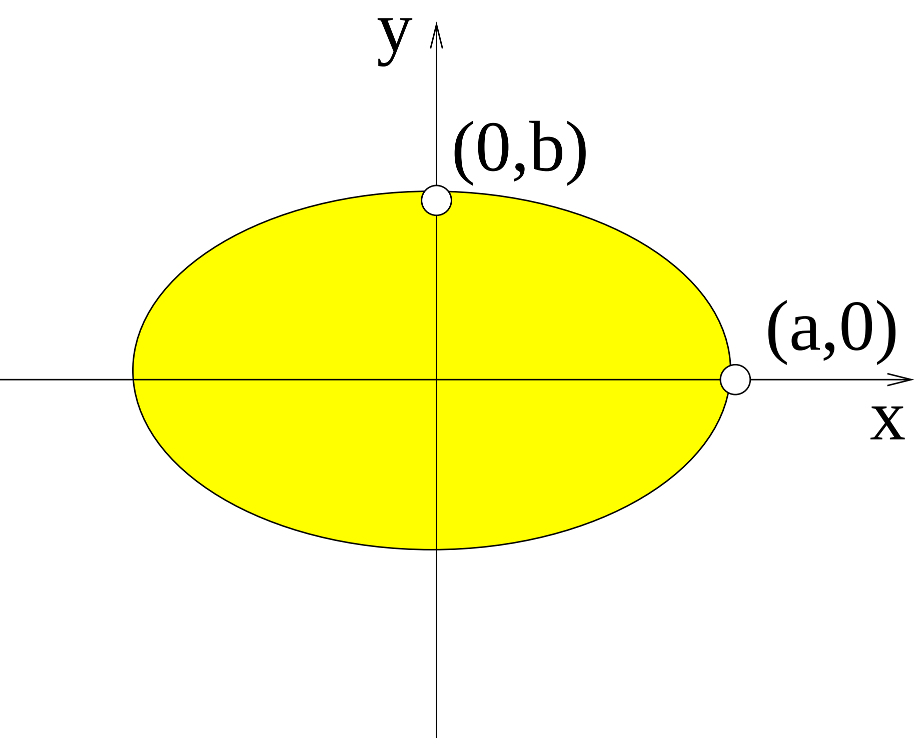
substitution $x = \sin\theta, \qquad \frac{dx}{d\theta} = \cos\theta$

$$
\begin{aligned}
I &= \int \cos\theta \cdot \cos\theta \, d\theta = \frac{1}{2}\int (1 + \cos 2\theta)\, d\theta \\
&= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C = \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + C \\
&= \frac{\sin^{-1} x}{2} + \frac{x\sqrt{1 - x^2}}{2} + C
\end{aligned}
$$

Example – Find the area of the ellipse



$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$A = \pi ab$ exercise

Example

$$I = \int \frac{x^3}{\sqrt{x^2-16}}\, dx$$

substitution

$$x = 4\sec\theta$$

Example

$I = \int \sqrt{x^2 - a^2}\, dx \qquad x \geq a \geq 0$

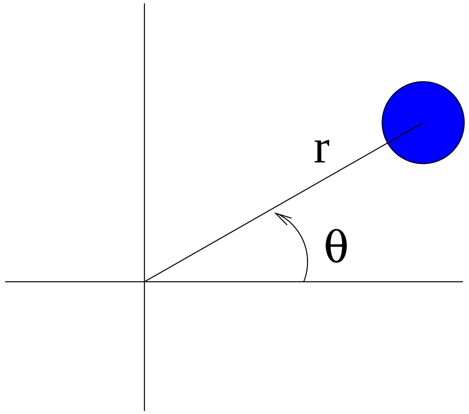
substitution
$x = a \sec\theta$ (see notes)

$$I = \frac{a^2}{2}\left(\frac{x\sqrt{x^2 - a^2}}{a^2} - \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right|\right) + C$$

compare the substitution
$x = a \cosh\theta$

Example – Planetary Motion – Kepler (1609,1619)

The motion of a planet around the sun is described by



$$\theta(r) = \int \frac{\frac{\ell}{r^2}\,dr}{\sqrt{2\mu(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2})}} + C$$

E, ℓ, μ, k, C are constants

What are the possible motions of the planets?

Substitute $u = \frac{1}{r}$ then $du = -\frac{1}{r^2}\,dr$

$$\theta(r) = \int \frac{-\ell\,du}{\sqrt{2\mu(E + ku - \frac{\ell^2}{2\mu}u^2)}} + C$$

Example ctd.

Complete the square in the denominator

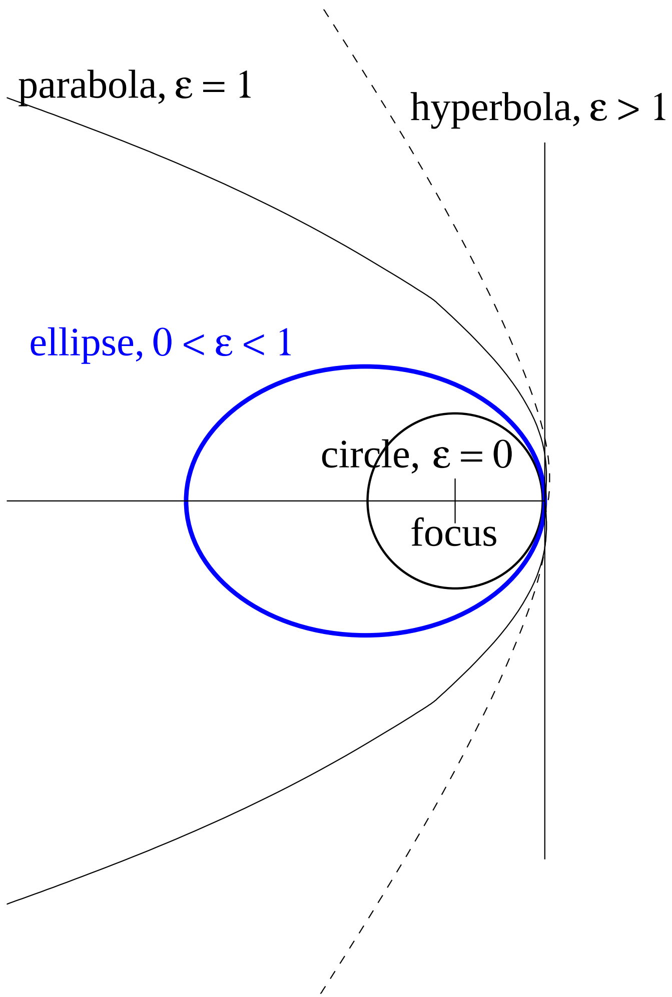
$$\theta(r) = \int \frac{-du}{\sqrt{(\frac{k^2\mu^2}{\ell^4} + \frac{2\mu E}{\ell^2}) - (u - \frac{k\mu}{\ell^2})^2}}$$

Substitution

$$(u - \frac{k\mu}{\ell^2}) = \sqrt{(\frac{k^2\mu^2}{\ell^4} + \frac{2\mu E}{\ell^2})}\cos\phi, \quad du = -\sqrt{(\frac{k^2\mu^2}{\ell^4} + \frac{2\mu E}{\ell^2})}\sin\phi\, d\phi$$

$$\theta(r) = \int \frac{\sqrt{(\frac{k^2\mu^2}{\ell^4} + \frac{2\mu E}{\ell^2})}\sin\phi\, d\phi}{\sqrt{(\frac{k^2\mu^2}{\ell^4} + \frac{2\mu E}{\ell^2})}\sin\phi} = \phi$$

Example ctd.



$$\frac{\alpha}{r} = 1 + \epsilon\cos\theta$$

$$\alpha = \frac{\ell^2}{\mu k}$$

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

(2.4) **Rational functions and partial fractions**

A rational function has the form

$$\frac{p(x)}{q(x)} \qquad \text{where} \quad p(x), q(x) \quad \text{are polynomial functions}$$

Example

$$\int \frac{x-3}{(x-2)(x^2-4x+8)}\,dx$$

Any rational function can be integrated by **systematic** reduction with **partial fractions**

- Step 1 – **SIMPLIFY** using long division

$$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)} \qquad \text{deg} \quad r(x) \quad < \quad \text{deg} \quad q(x)$$

- Step 1 – **SIMPLIFY**
  using long division

$$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)} \qquad \text{deg} \quad r(x) \quad < \quad \text{deg} \quad q(x)$$

- Step 2 – **FACTORIZE**
  the denominator $q(x)$ as product of **linear** $(x + a)$ and **irreducible quadratic** $(x^2 + bx + c), b^2 - 4c < 0$ factors.

- Step 1 – **SIMPLIFY**
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- Step 2 – **FACTORIZE**
  the denominator $q(x)$ as product of **linear** $(x + a)$ and **irreducible quadratic** $(x^2 + bx + c), b^2 - 4c < 0$ factors.

- Step 3 – **DECOMPOSE**
  $\frac{r(x)}{q(x)}$ as a **sum** of rational functions whose denominators contain only one of the factors of $q(x)$ from step 2.

(2.4.1) $\boxed{q(x) = \alpha(x+a)}$ – linear

deg $r(x)$ $<$ deg $q(x)$ $\Rightarrow r(x) = r =$ const

$$\int \frac{r(x)}{q(x)}\,dx = r\int \frac{dx}{\alpha(x+a)}$$

$$= \frac{r}{\alpha}\ln|x+a| + C \qquad (x \neq -a)$$

Example $\quad I = \int \frac{2x^3+7x^2+3}{2x+3}\,dx$

$$
\begin{array}{r@{}l}
 & \quad\;\; x^2 + 2x - 3 \\
\cline{2-2}
2x+3 \big) & \; 2x^3 + 7x^2 \qquad\quad + 3 \\
 & -2x^3 - 3x^2 \\
\cline{2-2}
 & \qquad\qquad 4x^2 \\
 & \qquad\quad -4x^2 - 6x \\
\cline{2-2}
 & \qquad\qquad\quad -6x + 3 \\
 & \qquad\qquad\qquad 6x + 9 \\
\cline{2-2}
 & \qquad\qquad\qquad\quad 12
\end{array}
$$

$$
\begin{aligned}
I &= \int (x^2 + 2x - 3) + \frac{12}{2x+3}\,dx \\
&= \frac{x^3}{3} + x^2 - 3x + 6\ln|2x+3| + C
\end{aligned}
$$

(2.4.2) $\boxed{q(x) = \alpha(x+a_1)(x+a_2)\ldots(x+a_n)}$

– distinct linear factors

We can write

$$\frac{r(x)}{q(x)} = \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \cdots + \frac{A_n}{x+a_n}$$

Then

$$\int \frac{r(x)}{q(x)}\,dx = \sum_j \int \frac{A_j}{x+a_j}\,dx$$

How to find A_j?

To find the constants A_j
For example, to find A_2, multiply both sides of ($\star$) by $x + a_2$ then

$$\frac{r(x)}{\alpha(x+a_1)(x+a_3)\cdots(x+a_n)} = A_1\frac{x+a_2}{x+a_1} + A_2 + \cdots + A_n\frac{x+a_2}{x+a_n}$$

Now substitute $x = -a_2$.

In effect, we find A_2 by covering up the factor $(x + a_2)$ in the denominator of $\frac{r(x)}{q(x)}$ and then substituting $x = -a_2$:

$$A_2 = \frac{r(-a_2)}{\alpha(-a_2+a_1)\boxed{\boxed{(x+a_2)}}(-a_2+a_3)\cdots(-a_2+a_n)}$$

Example $I = \int \frac{dx}{(x-2)(x-3)}$

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

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Hence

$$\begin{aligned} I &= \int \frac{-dx}{x-2} + \int \frac{dx}{x-3} \\ &= \ln|x-3| - \ln|x-2| + C = \ln\left|\frac{x-3}{x-2}\right| + C. \end{aligned}$$

(2.4.3) $\boxed{q(x) = \alpha(x+a)^k}$ – a power of a linear factor

We can write

$$\frac{r(x)}{q(x)} = \frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_k}{(x+a)^k} \qquad \star\star$$

Multiply both sides of ($\star\star$) by $(x+a)^k$ then substitute $x = -a$ to find A_k

Multiply both sides of ($\star\star$) by $(x+a)^k$ then **differentiate** and substitute $x = -a$ to find A_{k-1}

Multiply both sides of ($\star\star$) by $(x+a)^k$ then **differentiate twice** and substitute $x = -a$ to find A_{k-2} . . .

Example

$$\int \frac{x^2-3}{(x-1)^3}\,dx$$

(2.4.4) $\boxed{q(x) = \alpha(x+a_1)^{k_1}(x+a_2)^{k_2}\dots(x+a_n)^{k_n}}$

– powers of distinct linear factors

$$\begin{aligned}\frac{r(x)}{q(x)} &= \frac{A_{11}}{x+a_1} + \frac{A_{12}}{(x+a_1)^2} + \cdots + \frac{A_{1k_1}}{(x+a_1)^{k_1}} + \\ &\quad + \frac{A_{21}}{x+a_2} + \cdots + \frac{A_{2k_2}}{(x+a_2)^{k_2}} + \cdots \\ &\quad \cdots + \frac{A_{n1}}{x+a_n} + \cdots + \frac{A_{nk_n}}{(x+a_n)^{k_n}}\end{aligned}$$

Finding the constants A_{jk} involves solving simultaneous equations.

Example

$I = \int \frac{dx}{x(x+1)^3}$

$$\frac{1}{x(x+1)^3} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

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multiply by $x(x+1)^3$

$$1 = A(x+1)^3 + Bx(x+1)^2 + Cx(x+1) + Dx$$

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multiply by $x(x+1)^3$

$$1 = A(x+1)^3 + Bx(x+1)^2 + Cx(x+1) + Dx$$

substitute $x = 0 \longrightarrow 1 = A$

substitute $x = -1 \longrightarrow 1 = -D$

$$1 = (x+1)^3 + Bx(x+1)^2 + Cx(x+1) - x$$

Example ctd.
We need two more equations for B and C

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substitute $x = 1 \longrightarrow 1 = 8 + 4B + 2C - 1 \Rightarrow -3 = C + 2B$
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solve simultaneously then $B = -1, \qquad C = -1$

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We need two more equations for B and C

substitute $x = 1 \longrightarrow 1 = 8 + 4B + 2C - 1 \Rightarrow -3 = C + 2B$
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solve simultaneously then $B = -1, \qquad C = -1$

Finally

$$\int \frac{1}{x(x+1)^3} = \int \frac{1}{x} - \int \frac{1}{x+1} - \int \frac{1}{(x+1)^2} - \int \frac{1}{(x+1)^3}$$

$$= \ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + \frac{1}{2(x+1)^2} + C$$

Sample Class Test Problem

$$I = \int \frac{x+3}{(x+1)(x+2)^2}\,dx$$

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$$\frac{x+3}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\frac{(x+3)(x+1)}{(x+1)(x+2)^2} = \frac{A(x+1)}{x+1} + \frac{B(x+1)}{x+2} + \frac{C(x+1)}{(x+2)^2}$$

Sample Class Test Problem

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$$\frac{x+3}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

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- $x \to -1 \Rightarrow A = 2$

$$\frac{(x+3)(x+2)^2}{(x+1)(x+2)^2} = \frac{A(x+2)^2}{x+1} + \frac{B(x+2)^2}{x+2} + \frac{C(x+2)^2}{(x+2)^2}$$

$$\frac{(x+3)(x+2)^2}{(x+1)(x+2)^2} = \frac{A(x+2)^2}{x+1} + \frac{B(x+2)^2}{x+2} + \frac{C(x+2)^2}{(x+2)^2}$$

- $x \to -2 \Rightarrow C = -1$

$$\frac{(x+3)(x+2)^2}{(x+1)(x+2)^2} = \frac{A(x+2)^2}{x+1} + \frac{B(x+2)^2}{x+2} + \frac{C(x+2)^2}{(x+2)^2}$$

- $x \to -2 \Rightarrow C = -1$

$$\frac{x+3}{(x+1)(x+2)^2} = \frac{2}{x+1} + \frac{B}{x+2} + \frac{-1}{(x+2)^2}$$

$$\frac{(x+3)(x+2)^2}{(x+1)(x+2)^2} = \frac{A(x+2)^2}{x+1} + \frac{B(x+2)^2}{x+2} + \frac{C(x+2)^2}{(x+2)^2}$$

- $x \to -2 \Rightarrow C = -1$

$$\frac{x+3}{(x+1)(x+2)^2} = \frac{2}{x+1} + \frac{B}{x+2} + \frac{-1}{(x+2)^2}$$

- $x = 0 \Rightarrow B = -2$

$$
\begin{aligned}
I &= \int \frac{2}{x+1}\,dx - \int \frac{2}{x+2}\,dx - \int \frac{1}{(x+2)^2}\,dx \\
&= 2\log|x+1| - 2\log|x+2| + \frac{1}{x+2} + C
\end{aligned}
$$

(2.4.5) $\boxed{q(x) = \alpha(x^2 + bx + c), \quad b^2 - 4c < 0}$

– an irreducible quadratic

If $b^2 - 4c \geq 0$ then the quadratic factorizes so an earlier case applies

w.l.o.g. write $\boxed{r(x) = L(2x + b) + M}$

$$\int \frac{r(x)}{q(x)}\,dx = \frac{L}{\alpha}\int \frac{2x+b}{x^2+bx+c}\,dx + \frac{M}{\alpha}\int \frac{dx}{x^2+bx+c}$$

$$= \frac{L}{\alpha}\ln|x^2+bx+c| + \frac{M}{\alpha\sqrt{c-\frac{b^2}{4}}}\tan^{-1}\left(\frac{x+\frac{b}{2}}{\sqrt{c-\frac{b^2}{4}}}\right) + C$$

$I = \int \frac{dx}{x^2+bx+c}$

$$x^2 + bx + c = (x + \frac{b}{2})^2 + (c - \frac{b^2}{4}) \quad \text{complete the square}$$

$$= (c - \frac{b^2}{4}) \left[\left(\frac{(x + \frac{b}{2})}{\sqrt{c - \frac{b^2}{4}}} \right)^2 + 1 \right]$$

$I = \frac{1}{(c-\frac{b^2}{4})} \int \frac{dx}{\left[\left(\frac{(x+\frac{b}{2})}{\sqrt{c-\frac{b^2}{4}}} \right)^2 + 1 \right]}$

substitute $\tan\theta = \frac{(x+\frac{b}{2})}{\sqrt{c-\frac{b^2}{4}}}$ then $\sec^2\theta \, d\theta = \frac{1}{\sqrt{c-\frac{b^2}{4}}} dx$

$I = \frac{1}{(c-\frac{b^2}{4})} \int \frac{\sqrt{c-\frac{b^2}{4}} \sec^2\theta \, d\theta}{\tan^2\theta+1} = \frac{\theta}{\sqrt{c-\frac{b^2}{4}}} = \frac{1}{\sqrt{c-\frac{b^2}{4}}} \tan^{-1} \frac{(x+\frac{b}{2})}{\sqrt{c-\frac{b^2}{4}}}$

Sample Class Test Problem

$$I = \int \frac{x}{x^2 - 4x + 13}\, dx$$

You are given that

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C, \qquad a \neq 0$$

Sample Class Test Problem

$$I = \int \frac{x}{x^2 - 4x + 13}\, dx$$

You are given that

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C, \qquad a \neq 0$$

First step, write as

$$\int \frac{L(2x-4)}{x^2 - 4x + 13}\, dx + \int \frac{M}{x^2 - 4x + 13}\, dx$$

$\boxed{L = \frac{1}{2}}$ and $\boxed{M = 2}$

$$I = \frac{1}{2}\log|x^2 - 4x + 13| + \int \frac{2}{x^2 - 4x + 13}\,dx$$

complete the square

$$\int \frac{2}{x^2 - 4x + 13}\,dx = \int \frac{2}{(x-2)^2 - 4 + 13}\,dx = \int \frac{2}{(x-2)^2 + 3^2}\,dx$$

put $(x-2) = 3\tan\theta$ or put $(x-2) = u$ and use table result

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C, \qquad a \neq 0$$

(2.4.6) $\boxed{q(x) = \alpha(x^2 + bx + c)^k, \quad b^2 - 4c < 0}$

– irreducible quadratic powers

$$\frac{r(x)}{q(x)} = \frac{B_1 x + C_1}{x^2 + bx + c} + \frac{B_2 x + C_2}{(x^2 + bx + c)^2} + \cdots + \frac{B_k x + C_k}{(x^2 + bx + c)^k}$$

(2.4.6) $\boxed{q(x) = \alpha(x^2+bx+c)^k, \quad b^2-4c<0}$

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to find constants B_j, C_j multiply by $(x^2+bx+c)^k$ and equate coefs of equal powers of x

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to find constants B_j, C_j multiply by $(x^2+bx+c)^k$ and equate coefs of equal powers of x

This yields a sum of integrals $\quad B_jx+C_j = L(2x+b)+M$

$$\int \frac{B_jx+C_j}{(x^2+bx+c)^j}\,dx = L\int \frac{2x+b}{(x^2+bx+c)^j}\,dx + M\int \frac{dx}{(x^2+bx+c)^j}$$

the first integral is easy with the substitution $u = x^2+bx+c$.

$$I = \int \frac{dx}{(x^2+bx+c)^j} = \frac{1}{(c-\frac{b^2}{4})^j} \int \frac{dx}{\left[\left(\frac{(x+\frac{b}{2})}{\sqrt{c-\frac{b^2}{4}}}\right)^2 + 1\right]^j}$$

substitute $\tan\theta = \frac{(x+\frac{b}{2})}{\sqrt{c-\frac{b^2}{4}}}$ then $\sec^2\theta\, d\theta = \frac{1}{\sqrt{c-\frac{b^2}{4}}}\, dx$

$$I = \frac{\sqrt{c-\frac{b^2}{4}}}{(c-\frac{b^2}{4})^j} \int \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^j} = \frac{\sqrt{c-\frac{b^2}{4}}}{(c-\frac{b^2}{4})^j} \int \cos^{2j-2}\theta\, d\theta$$

(2.4.7) $q(x) = \alpha(x+a_1)^{k_1}\cdots(x+a_m)^{k_m}(x^2+b_1x+c_1)^{\ell_1}\cdots(x^2+b_nx+c_n)^{\ell_n}$

– general case

$$
\begin{aligned}
\frac{r(x)}{q(x)} \quad = \quad & \frac{A_{11}}{x+a_1} + \frac{A_{12}}{(x+a_1)^2} + \cdots + \frac{A_{1k_1}}{(x+a_1)^{k_1}} + \cdots \\
& + \frac{A_{m1}}{x+a_m} + \frac{A_{m2}}{(x+a_m)^2} + \cdots + \frac{A_{mk_m}}{(x+a_m)^{k_m}} + \\
& + \frac{B_{11}x+C_{11}}{x^2+b_1x+c_1} + \frac{B_{12}x+C_{12}}{(x^2+b_1x+c_1)^2} + \cdots + \frac{B_{1\ell_1}x+C_{1\ell_1}}{(x^2+b_1x+c_1)^{\ell_1}} + \cdots \\
& + \frac{B_{n1}x+C_{n1}}{x^2+b_nx+c_n} + \frac{B_{n2}x+C_{n2}}{(x^2+b_nx+c_n)^2} + \cdots + \frac{B_{n\ell_n}x+C_{n\ell_n}}{(x^2+b_nx+c_n)^{\ell_n}}
\end{aligned}
$$

Example

$$\int \frac{3x^2-5x+3}{(x-1)(x^2-2x+2)}\,dx$$

(2.5) **Further Substitution Techniques**

Look for a substitution that will convert the integrand to a rational function and then use the method of partial fractions.

Example

$$I = \int \frac{dt}{1 + t^{1/4}}$$

Substitute $x = t^{1/4}$ (or $x^4 = t$) then $dt = 4x^3\,dx$

$$I = \int \frac{4x^3}{1 + x}\,dx$$

Example ctd. $I = \int \frac{4x^3}{1+x}\,dx$

$$
\begin{array}{r l}
 & 4x^2 - 4x + 4 \\
\hline
x+1) & 4x^3 \\
 & -4x^3 - 4x^2 \\
\hline
 & -4x^2 \\
 & 4x^2 + 4x \\
\hline
 & 4x \\
 & -4x - 4 \\
\hline
 & -4
\end{array}
$$

$$
\begin{aligned}
I &= \int 4x^2 - 4x + 4\,dx - \int \frac{4}{1+x}\,dx \\
&= \frac{4}{3}x^3 - 2x^2 + 4x - 4\log|1+x| \\
&= \frac{4}{3}t^{3/4} - 2t^{1/2} + 4t^{1/4} - 4\log|1+t^{1/4}|
\end{aligned}
$$

(2.5.1) $\boxed{I = \int \frac{d\theta}{a+b\sin\theta+c\cos\theta}}$ Half angle substitution $\boxed{t = \tan\frac{\theta}{2}}$

(2.5.1) $\boxed{I = \int \frac{d\theta}{a+b\sin\theta+c\cos\theta}}$ Half angle substitution $\boxed{t = \tan\frac{\theta}{2}}$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = 2\frac{\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{\tan^2\frac{\theta}{2}+1} = \frac{2t}{1+t^2}$$

(2.5.1) $\boxed{I = \int \frac{d\theta}{a + b\sin\theta + c\cos\theta}}$ Half angle substitution $\boxed{t = \tan\frac{\theta}{2}}$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = 2\frac{\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{\tan^2\frac{\theta}{2} + 1} = \frac{2t}{1 + t^2}$$

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{4t^2}{(1 + t^2)^2}} = \frac{\sqrt{1 + 2t^2 + t^4 - 4t^2}}{1 + t^2} = \frac{1 - t^2}{1 + t^2}$$

(2.5.1) $\boxed{I = \int \frac{d\theta}{a+b\sin\theta+c\cos\theta}}$ Half angle substitution $\boxed{t = \tan\frac{\theta}{2}}$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = 2\frac{\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{\tan^2\frac{\theta}{2}+1} = \frac{2t}{1+t^2}$$

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$$dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = \frac{1+\tan^2\frac{\theta}{2}}{2}d\theta \quad \Rightarrow d\theta = \frac{2dt}{1+t^2}$$

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$$dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = \frac{1+\tan^2\frac{\theta}{2}}{2}d\theta \qquad \Rightarrow d\theta = \frac{2dt}{1+t^2}$$

$$I = \int \frac{\frac{2dt}{1+t^2}}{a+b\frac{2t}{1+t^2}+c\frac{1-t^2}{1+t^2}} = \boxed{\int \frac{2}{a(1+t^2)+2bt+c(1-t^2)}\,dt}$$

Example

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$$

(2.6) **MAPLE** excells at integration and partial fractions but it isn't omniscient

and it doesn't always simplify well

```
> # Example 1
> g:=x->x^3/((x+1)^2*(x^2+1));
```

$$x \mapsto \frac{x^3}{(x+1)^2(x^2+1)} := x \mapsto \frac{x^3}{(x+1)^2(x^2+1)}$$

```
> convert(g(x),parfrac,x);
```

$$(x+1)^{-1} - 1/2\ (x+1)^{-2} - 1/2\ \left(x^2+1\right)^{-1}$$

```
> int(g(x),x);
```

$$\ln(x+1) - 1/2\ \arctan(x) + 1/2\ (x+1)^{-1}$$

```
> # Example 2
> f:=x->x*(x^2+5)^8;
```

$$x \mapsto x\left(x^2+5\right)^8 := x \mapsto x\left(x^2+5\right)^8$$

```
> int(f(x),x);
> <br>
```

$$1/18\,x^{18} + 5/2\,x^{16} + 50\,x^{14} + \frac{1750}{3}\,x^{12} + 4375\,x^{10} + 21875\,x^8 + \frac{218750}{3}\,x^6 + 156250\,x^4 + \frac{390625}{2}\,x^2$$

```
> simplify(%);
```

$$1/18\,x^{18} + 5/2\,x^{16} + 50\,x^{14} + \frac{1750}{3}\,x^{12} + 4375\,x^{10} + 21875\,x^8 + \frac{218750}{3}\,x^6 + 156250\,x^4 + \frac{390625}{2}\,x^2$$

```
> h:=x->1/18*(x^2+5)^9;
```

$$x \mapsto 1/18\ \left(x^2+5\right)^9 := x \mapsto 1/18\ \left(x^2+5\right)^9$$

```
> diff(h(x),x);
```

$$x\left(x^2+5\right)^8$$