The arithmetic of chaos

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God
“God gave us the integers, the rest is the work of man”

L Kronecker
“God gave us the integers, the rest is the work of man”

L Kronecker
the work of man
the work of man

0 1 2 3 4

\frac{1}{2} \quad \frac{18}{7}
the work of man
the work of man
the work of man

\[ \sqrt{2} \]

\[ \frac{1}{2} \]

\[ \frac{18}{7} \]

\[ \pi \]

59304878120451337437530175389175098028916381028347492817272987

30384347391847329111084745934789430006108457350897213087523809
the continuum of real numbers
the continuum of real numbers
the continuum of real numbers

there is a number for every point of a line
the continuum of real numbers

there is a number for every point of a line

we can do continuum maths: \( \frac{d}{dx} \quad \int dx \)
dynamics

the science of things that move

law of motion
dynamics

the science of things that move

dynamics

the science of things that move

present

law of motion

future
dynamics: the science of things that move

main goal: to predict the future
Dynamics: the science of things that move

Main goal: to predict the future

Present: 0.140523
0.150001
0.212703
0.762610
1.860044

Future
a toy model
a toy model

- multiplication by 10:  \[0.4073 \times 10 = 4.073\]
a toy model

**multiplication by 10:** \[ 0.4073 \times 10 = 4.073 \]

0.4073 \rightarrow 4.073 \rightarrow 40.73 \rightarrow

numbers become large very quickly
a toy model

- multiplication by 10: $0.4073 \times 10 = 4.073$
  
  
  $0.4073 \rightarrow 4.073 \rightarrow 40.73 \rightarrow$

- modular multiplication by 10:
  
  - multiply by 10
  
  - throw away the digit to the left of the decimal point

numbers become large very quickly
a toy model

- multiplication by 10: \(0.4073 \times 10 = 4.073\)

  0.4073 → 4.073 → 40.73

  numbers become large very quickly

- modular multiplication by 10:
  - multiply by 10
  - throw away the digit to the left of the decimal point

  0.12345 → 0.2345 → 0.345

  numbers remain small
chaos

present

0.4076629218?

future
chaos

present

0.4076629218?
0.076629218??
0.76629218???
0.6629218????
0.629218??????
0.29218??????
0.9218???????
0.218????????
0.18??????????
0.8???????????
0.????????????

future

exponential instability
can we beat chaos?
can we beat chaos?

in physics, we measure: 0.4076629218?
can we beat chaos?

in physics, we measure: 0.4076629218??????????????

in mathematics, we compute:

\[ \sqrt{2} = 1.41421356237309504880168872420969807856967187537 \]

6948073176679737990732478462107038850387534327641573...
can we beat chaos?

in physics, we measure: \(0.4076629218\ldots\)

in mathematics, we compute:

\[
\sqrt{2} = 1.41421356237309504880168872420969807856967187537 \\
6948073176679737990732478462107038850387534327641573\ldots
\]

to predict the future, we must be able to compute the present
computer programs to build numbers

print(123123123123123123123123123123123123123123123)

print(123) 15 times
computer programs to build numbers

print(123123123123123123123123123123123123123123123)
dumb

print(123) 15 times
smart
computer programs to build numbers

print(123123123123123123123123123123123123123123123)
dumb

print(123) 15 times
smart

dumb

print(3846264338327950288419716939937510582097494)
dumb
computer programs to build numbers

print(123123123123123123123123123123123)
dumb

print(123) 15 times
smart

print(3846264338327950288419716939937510582097494)
dumb

smart
random numbers

A number is **random** if the **shortest** program that can build its digits is the **dumb** program.

Kolmogorov
A number is **random** if the **shortest** program that can build its digits is the **dumb** program.

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**Kolmogorov**

**random**

**dumb program**

**output**
A number is **random** if the **shortest** program that can build its digits is the **dumb** program.

Kolmogorov

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**random**

**dumb** program | output

**non-random**

smart program | output
do numbers have mass?
do numbers have mass?

Lebesgue
do numbers have mass?

Lebesgue
do numbers have mass?

Lebesgue

0

1

1 Kg

145 g
do numbers have mass?

The total mass of all fractions is zero
DARK MATTER
DARK MATTER

visible  dark

universe

0 25 50 75 100
Theorem:
Theorem: the random numbers account for the total mass of the real number system.
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Random numbers are dark:
Theorem: the random numbers account for the total mass of the real number system.

Random numbers are dark:

- they can’t be defined individually, or talked about
Theorem: the **random numbers** account for the **total mass** of the real number system.

Random numbers are **dark**:

- they can’t be defined individually, or talked about
- they can’t be computed, or stored in computers
Theorem: the random numbers account for the total mass of the real number system.

Random numbers are dark:

- they can’t be defined individually, or talked about
- they can’t be computed, or stored in computers
- they can’t be proved to be random
Theorem: the random numbers account for the total mass of the real number system.

Random numbers are dark:

- they can’t be defined individually, or talked about
- they can’t be computed, or stored in computers
- they can’t be proved to be random

Random numbers are not meant for humans.
Does chaos exist?
Does chaos exist?

in mathematics:

- chaos exists, but it can’t be observed
Does chaos exist?

in mathematics:
- chaos exists, but it can’t be observed

in the physical world:
- there is no continuum
- the above constructs do not apply
whose fault is it?
whose fault is it?
whose fault is it?

Euler → Gauss → Gates
whose fault is it?

Gates -> Euler -> Gauss
God gave us the integers: part II
the work of man: arithmetic

0 1

(2,1)

0 2 1

2 1 0
the work of man: arithmetic
the work of man: arithmetic
the work of man: arithmetic
the work of man: arithmetic
is there chaos in arithmetic?
is there chaos in arithmetic?

\[
\frac{1}{97} = 0.010309278350515463917525773195876288659793814432989690721649484536082474226804123711340206185567\ldots
\]
is there chaos in arithmetic?

\[ \frac{1}{97} = 0.010309278350515463917525773195876288659793814432989690721649484536082474226804123711340206185567 \ldots \]

a pseudo-random sequence
is there chaos in arithmetic?

\[ \frac{1}{97} = 0.0103092783505154639175257731958762886597938144 \]
\[ \quad 32989690721649484536082474226804123711340206185567 \ldots \]

a pseudo-random sequence

program:

\[
\begin{array}{c}
1 \\
97
\end{array}
\]

input unpacking
is there chaos in arithmetic?

\[
\frac{1}{97} = 0.010309278350515463917525773195876288659793814432989690721649484536082474226804123711340206185567 \ldots
\]

a pseudo-random sequence
is there chaos in arithmetic?

\[ \frac{1}{97} = 0.0103092783505154639175257731958762886597938144 \]
\[ 32989690721649484536082474226804123711340206185567 \ldots \]

a pseudo-random sequence

program:

1
97

input  unpacking
is there chaos in arithmetic?

\[
\frac{1}{97} = 0.010309278350515463917525773195876288659793814432989690721649484536082474226804123711340206185567\ldots
\]

a pseudo-random sequence

program:

input unpacking

Gauss

smart, but slow
is there chaos in arithmetic?

\[
\frac{1}{97} = 0.010309278350515463917525773195876288659793814432989690721649484536082474226804123711340206185567 \ldots
\]

a pseudo-random sequence

program:

\[
\begin{array}{c|c}
1 & \text{input unpacking} \\
97 &
\end{array}
\]

smart, but slow
predicting the future: obstructions
predicting the future: obstructions

I can’t predict the future because my computer program...
predicting the future: obstructions

I can’t predict the future because my computer program...

is too big
predicting the future: obstructions

I can’t predict the future because my computer program...

- is too big

- takes too long
predicting the future: obstructions

I can’t predict the future because my computer program...

is too big

digits of random numbers

\[ \sqrt{2} - 1 = 0.4076629218018315039\ldots \]

takes too long

Kolmogorov
I can’t predict the future because my computer program...

- is too big

\[ \sqrt{2} = 1.4142135623730950488 \ldots \]

- takes too long

\[ \frac{1}{97} = 0.0103092783505154639 \ldots \]

Gauss

Kolmogorov
I can’t predict the future because my computer program...

- is too big
- takes too long

Digits of random numbers

\[ \sqrt{2} = 0.407662921884315039 \ldots \]

Digits of fractions

\[ \frac{1}{97} = 0.0103092783505154639 \ldots \]
I can’t predict the future because my computer program...

- **is too big**
  - $\sqrt{2} = 1.41421356237309504880168872420969...$
  - $\sqrt{2} - 1 = 0.6948073176679737990732478462107038850387534327641573...$

- **takes too long**
  - Fractions: $\frac{1}{97} = 0.0103092783505154639...$

How long is too long?
predictions vs. observations

From A, will I ever reach B? If so, how long will it take?
predictions vs. observations

From A, will I ever reach B? yes!
If so, how long will it take? 7 time steps
predictions vs. observations

From A, will I ever reach B? yes!
If so, how long will it take? 7 time steps

"chaotic motion cannot be predicted: it can only be observed"

Joseph Ford
predictions vs. observations

From A, will I ever reach B?  yes!
If so, how long will it take?  7 time steps

“chaotic motion cannot be predicted: it can only be observed”

Joseph Ford

quantum computers?
God gave us the integers: the end