Duality for discrete integrable systems

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Abstract
A new class of discrete dynamical systems is introduced via a duality relation for discrete dynamical systems with a number of explicitly known integrals. The dual equation can be defined via the difference of an arbitrary linear combination of integrals and its upshifted version. We give an example of an integrable mapping with two parameters and four integrals leading to a (four-dimensional) dual mapping with four parameters and two integrals. We also consider a more general class of higher-dimensional mappings arising via a travelling-wave reduction from the (integrable) MKdV partial-difference equation. By differencing the trace of the monodromy matrix we obtain a class of novel dual mappings which is shown to be integrable as level-set-dependent versions of the original ones.

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1. Introduction
Discrete integrable systems have received a lot of attention in the last two decades. Areas of physics in which discrete integrable systems prominently feature include statistical mechanics and discrete analogues of integrable systems in classical mechanics or solid state physics [5–8, 22–25].

Some of the early papers dealt with the problem of discretizing integrable partial differential equations, such as the (modified) Korteweg–de Vries (MKdV) equation and the sine-Gordon equation, while retaining their integrability. This led to integrable partial difference equations (PΔEs) [12–15]. Later papers studied integrable ordinary difference equations (OΔEs), both autonomous (i.e. integrable maps) [5–8, 10, 21] and non-autonomous (e.g. discrete Painlevé equations, for a review see [11]).
In this paper we start from a (primary) OΔE with one or more first integrals, and construct a dual OΔE which also has one or more integrals. While neither the primary OΔE nor the dual need be integrable in order for this construction to work, here we will be mainly interested in the case where the primary OΔE is integrable. The question we subsequently seek to answer is whether or not the ensuing dual equation is also integrable.

Accordingly, we focus on a new class of discrete dynamical systems which can be obtained by means of a duality relation from a known discrete dynamical system which possesses a number of integrals. The general idea is as follows. We consider a discrete dynamical system given by the $d$th-order OΔE:

$$u_{n+d} = f(u_n, u_{n+1}, \ldots, u_{n+d-1}, \{p_i\}),$$

(1)

where $\{p_i\}$ is a set of $l$ parameters occurring in the system and $f : \mathbb{R}^{d+l} \to \mathbb{R}$. By the standard method, we can alternatively view (1) as defining a map

$$V_{n+1} = F(V_n, \{p_i\}),$$

(2)

in which

$$V_n := (u_n, u_{n+1}, \ldots, u_{n+d-1})$$

(3)

is a $d$-dimensional vector and $F : \mathbb{R}^{d+l} \to \mathbb{R}^d$.

Suppose the dynamical system is known to have the integrals

$$I_j := I_j(V_n, \{p_i\}) = I_j(V_{n+1}, \{p_i\}),$$

(4)

with $j = 1, \ldots, m$. Then we can form a linear combination

$$\sum_{i=1}^{m} \alpha_j I_j =: I(V_n, \{p_i\}, \{\alpha_j\}),$$

(5)

which, for arbitrary $\alpha_j$, is an integral of system (1).

Taking the difference between the integral $I$ and its upshifted version, we derive a relation of the type

$$I(V_{n+1}, \{p_i\}, \{\alpha_j\}) - I(V_n, \{p_i\}, \{\alpha_j\}) = L(V_n, V_{n+1}, \{p_i\}) L^*(V_n, V_{n+1}, \{p_i\}, \{\alpha_j\}),$$

(6)

in which

$$L(V_n, V_{n+1}, \{p_i\}) = u_{n+d} - f(u_n, u_{n+1}, \ldots, u_{n+d-1}, \{p_i\}) = 0$$

(7)

is equivalent to the original dynamical equation (1).

In fact, for system (1), the left-hand side of (6) must vanish and one would expect the right-hand side to contain a factor such that the vanishing of that factor is equivalent to the dynamical equation (1). Apart from this, the right-hand side may contain another factor such that the vanishing of this second factor also ensures that the left-hand side of (6) is zero.

Starting from a specific example of a dynamical mapping (1) possessing integrals, it is not clear to which extent the second factor would contain an interesting dependence on the field $V_n$. However, if this dependence is interesting, the vanishing of the second factor could be equivalent to another dynamical equation which by construction of (6) may be called the dual equation of the original (1). The dual equation automatically has one integral which is given by $I(V_n, \{p_i\}, \{\alpha_j\})$ but, depending on the presence or absence of the original parameters $\{p_i\}$ in $L^*$, there may be more integrals.

At this stage the description of how to obtain the dual equation and the nature of the resulting dual equation is rather general\(^5\). However, in this paper, we will show on the basis of

\(^5\) For example, the dual equation may be almost trivial.
some more sophisticated specific examples that indeed new dynamical systems with interesting properties can be derived from the differencing of integrals described by (6).

In section 2, we will first consider what we call a motivating example. Our starting point is a low-dimensional mapping arising from the integrable partial difference MKdV equation of [6] by a travelling-wave reduction. It has four integrals and the dual mapping has four parameters and two integrals but may not be integrable.

The following sections 3–6 are devoted to a more general class of higher-dimensional mappings arising from the MKdV partial-difference equations treated in [6]. We consider the integrals arising from different powers of the spectral parameter occurring in the trace of the monodromy matrix $T$. Defining the dual equation by (6) with $I = \text{Trace } T$, we find that these provide some new dynamical systems which can be considered as generalizations of known integrable mappings and which have a number of interesting integrals. In fact, we can establish the integrability of the ensuing dual equations. We do this by deriving a Lax representation for it which can be obtained from the Lax representation of the original system by some simple substitutions. Furthermore, it is interesting to note that in this case the dual equation is a so-called level-set-dependent (LSD) version of the original equation. This terminology refers to a higher-dimensional generalization of the work of [3, 4] on QRT [7, 8] and other mappings, where it was shown that a large class of such mappings amounts to a LSD version of the McMillan mapping. The work of [3, 4] and of [9] represents another way of associating two dynamical systems with integrals to one another.

The treatment given in this paper can also be applied to a large variety of other dynamical systems and various possible extensions and specific comments are given in a final discussion.

2. A motivating example: creating the dual of a 4D map

Consider the following fourth-order difference equation:

$$V_4 = V_0 \frac{q V_1 - p V_3}{q V_5 - p V_1},$$

(8)

In (8) and throughout this section, $V_j, j = 0, 1, 2, 3, 4$ is shorthand for $V_{n+j}, n \in \mathbb{Z}$, and $p$ and $q$ are parameters. This equation can be obtained as a reduction of the so-called MKdV PΔE [6]. It is equivalent to the following map $(V_0, V_1, V_2, V_3) \mapsto (V'_0, V'_1, V'_2, V'_3)$ in four dimensions:

$$L:\ V'_0 = V_1 \quad V'_1 = V_2$$
$$V'_2 = V_3 \quad V'_3 = V_0 \frac{q V_1 - p V_3}{q V_5 - p V_1}$$

(9)

One checks that the map $L$ has the following four integrals of motion (i.e. $I_\alpha(V_1, V_2, V_3, V_4) = I_\alpha(V_0, V_1, V_2, V_3)$ etc). They each depend linearly on the parameters $p$ and $q$, which we highlight by writing them:

$$I_a = I_{a,q} q - I_{a,p} p \quad I_b = I_{b,q} q - I_{b,p} p \quad I_c = I_{c,q} q - I_{c,p} p$$

(10) (11) (12)

$$I_d = I_{d,q} q - I_{d,p} p$$

(13)
with

\[ I_{\alpha,p} = \frac{V_1}{V_0} + \frac{V_2}{V_1} + \frac{V_3}{V_2} + \frac{V_4}{V_3} \]  

(14)

\[ I_{\beta,p} = \frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_4}{V_2} + \frac{V_5}{V_3} \]  

(15)

\[ I_{\gamma,p} = V_3 V_0 \]  

(16)

\[ I_{\delta,p} = V_3^{-1} V_0^{-1} \]  

(17)

\[ I_{\alpha,q} = \frac{V_2}{V_0} + \frac{V_3}{V_1} \]  

(18)

\[ I_{\beta,q} = \frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_4}{V_2} + \frac{V_5}{V_3} + \frac{V_6 V_1}{V_4 V_3} \]  

(19)

\[ I_{\gamma,q} = V_0 V_1 + V_1 V_2 + V_2 V_3 \]  

(20)

\[ I_{\delta,q} = V_0^{-1} V_1^{-1} + V_1^{-1} V_2^{-1} + V_2^{-1} V_1^{-1}. \]  

(21)

These integrals may be inferred from the work of Hydon [2], but they can also be checked directly using two obvious symmetries of (8): \( S_1 : V_i \mapsto \lambda V_i, \lambda \in \mathbb{R} \) and \( S_2 : V_i \mapsto V_i^{-1} \). This is equivalent to saying that \( L \) of (9) commutes with \( S_1 \) and \( S_2 \), whence if \( L \) has an integral \( I \) it also has an integral \( I \circ S_i, i = 1, 2 \). Using this, the integrals follow by constructing homogeneous expressions in the \( V_i \) of degree 0 (i.e. \( I_{\alpha}, I_{\beta} \)), 2 (i.e. \( I_{\gamma} \)) and \(-2 \) (i.e. \( I_{\delta} \)). We now take the linear combination of these integrals\(^6\):

\[ I(V_0, V_1, V_2, V_3; p, q; \alpha, \beta, \gamma, \delta) = \alpha I_{\alpha} + \beta I_{\beta} + \gamma I_{\gamma} + \delta I_{\delta}, \]  

(22)

and difference it, meaning we consider the difference between \( I \) and its upshifted version \( I' \) with \( V_j \mapsto V_{j+1} \). Since the separate integrals satisfy

\[ I_{\alpha} - I'_{\alpha} = \left[ \left( q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_3} \left( q - p \frac{V_3}{V_1} \right) \right] \left( \frac{V_1}{V_0} - \frac{V_3}{V_1} \right), \]  

(23)

\[ I_{\beta} - I'_{\beta} = \left[ \left( q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_3} \left( q - p \frac{V_3}{V_1} \right) \right] \left( \frac{V_2}{V_0} \left( 1 + \frac{V_1}{V_5} \right) - \frac{V_4}{V_3} \right) \left( 1 + \frac{V_1}{V_5} \right), \]  

(24)

\[ I_{\gamma} - I'_{\gamma} = \left[ \left( q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_3} \left( q - p \frac{V_3}{V_1} \right) \right] \left( -V_1 V_4 \right), \]  

(25)

\[ I_{\delta} - I'_{\delta} = \left[ \left( q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_3} \left( q - p \frac{V_3}{V_1} \right) \right] \left( V_0^{-1} V_5^{-1} \right), \]  

(26)

and so vanish if (8) is satisfied, we find

\[ I(V_1, V_2, V_3, V_4; p, q; \alpha, \beta, \gamma, \delta) - I(V_0, V_1, V_2, V_3; p, q; \alpha, \beta, \gamma, \delta) \]

\[ = L(V_0, V_1, V_2, V_3, V_4; p, q) L^*(V_0, V_1, V_2, V_3, V_4; \alpha, \beta, \gamma, \delta), \]  

(27)

where

\[ L(V_0, V_1, V_2, V_3, V_4; p, q) = \left[ \left( q \frac{V_3}{V_1} - p \right) - \frac{V_0}{V_3} \left( q - p \frac{V_3}{V_1} \right) \right] \]  

(28)

\(^6\) Note that the four integrals are not independent, i.e. \( I_{\alpha}, I_{\delta} = 3 q^2 + p^2 + q l_{\delta} \). However, since \( I_{\beta} \) is linear in \( p \) and \( q \) we still use this integral in the construction of a dual mapping.
\[ L^*(V_0, V_1, V_2, V_3, V_4; \alpha, \beta, \gamma, \delta) = \left[ \frac{1}{V_0} (\alpha V_1 V_3 + \beta V_2 V_3 + \beta V_1 V_2 + \delta) \right. \]
\[ \left. - \frac{V_4}{V_2} (\alpha V_2 + \beta (V_3 + V_1) + \gamma V_1 V_2 V_3) \right] \frac{1}{V_3}. \]  

(29)

The equation
\[ L(V_0, V_1, V_2, V_3, V_4; p, q) = 0, \]

(30)
solved for \( V_4 \) gives precisely (8). With (27), this reminds us that \( I \) of (22) is an integral of this map. On the other hand, the equation
\[ L^*(V_0, V_1, V_2, V_3, V_4; \alpha, \beta, \gamma, \delta) = 0, \]

(31)
defines a different fourth-order difference equation:
\[ V_4 = \frac{V_2 V_1 V_3 + \beta (V_1 V_2 + V_2 V_3) + \delta}{V_0 \alpha V_2 + \beta (V_1 + V_3) + \gamma V_1 V_2 V_3}. \]

(32)

We call (32) the dual map corresponding to (8). It follows from (27) that \( I \) is also an integral of the dual. But significantly in equation (27), the parameter sets \( \{ p, q \} \) and \( \{ \alpha, \beta, \gamma, \delta \} \) dissociate from one another on the right-hand side. Since \( p \) and \( q \) do not appear in the dual map (32), we can conclude that their coefficients in \( I \) are separately integrals of (32). More precisely, we can use (10)–(13) to rewrite the expression for \( I \) of (22) as
\[ I(V_0, V_1, V_2, V_3; p, q; \alpha, \beta, \gamma, \delta) = q I_q - p I_p, \]

(33)

where \( I_q \) and \( I_p \) given by
\[ I_q = (\alpha I_{\alpha,q} + \beta I_{\beta,q} + \gamma I_{\gamma,q} + \delta I_{\delta,q}) \]
\[ = \alpha \left( \frac{V_3}{V_0} + \frac{V_0}{V_3} \right) + \beta \left( \frac{V_2}{V_0} + \frac{V_3}{V_2} + \frac{V_0}{V_1} + \frac{V_1}{V_0} + \frac{V_2 V_3}{V_1} + \frac{V_0 V_2}{V_1} \right) \]
\[ + \gamma (V_0 V_1 + V_1 V_2 + V_2 V_3) + \delta (V_0 V_2^{-1} + V_1 V_2^{-1} + V_2 V_3^{-1}) \]

(34)
\[ I_p = (\alpha I_{\alpha,p} + \beta I_{\beta,p} + \gamma I_{\gamma,p} + \delta I_{\delta,p}) \]
\[ = \alpha \left( \frac{V_1}{V_0} + \frac{V_2}{V_1} + \frac{V_3}{V_2} \right) + \beta \left( \frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_0 V_2}{V_1} \right) + \gamma V_0 V_3 + \delta V_0^{-1} V_3^{-1} \]

(35)
are integrals of the dual.

Let us make some remarks about this process.

**Remark.**

1. It has been convenient that the parameters \( p \) and \( q \) entered the integrals of (10)–(13) in a linear way. One sees that the number of integrals of the dual is equal to the number of parameters appearing in the original map and vice versa.

2. The original map is actually degenerate, and can be reduced to a second-order difference equation. This is achieved by introducing the reduced variables: \( W_0 = \frac{V_0}{V_1}, W_1 = \frac{V_1}{V_2} \).

Then (8) reduces to
\[ W_2 = \frac{1}{W_0} \frac{q - p W_1}{W_1 - p}. \]

(36)
Of the four integrals \( I_\alpha, I_\beta, I_\gamma, I_\delta \) of the original map, only \( I_\beta \) can be expressed in terms of the reduced variables. However, differencing just \( I_\beta \) in (22)–(27) would lead to a completely trivial dual equation determined by \( L^*(V_0, V_1, V_2, V_3, V_4; 0, \beta, 0, 0, 0) = 0 \).

(3) For general parameters \( \alpha, \beta, \gamma \) and \( \delta \), the dual map (32) has two integrals. It can also be checked to be measure preserving. But this is not enough to ensure integrability. In fact, no symplectic structure and no additional integrals have been found so far for the general case. Numerically, some special cases of (32) have been shown to have zero algebraic entropy [20]. It may also be interesting to investigate the possible integrability of the dual by extending to four dimensions the arithmetic integrability tests of [16, 17].

We mention some special cases of the dual map that we have proved to be integrable.

- For \( \beta = 0 \), (32) reduces to the four-dimensional sine-Gordon (SG) mapping [1]. It can be derived from the SG difference equation using a periodicity constraint [6, 1].
- For \( \gamma = \delta = 0 \), (32) reduces to a three-dimensional mapping in terms of the variables \( W_0 = V_1, W_1 = V_2, W_2 = V_3 \):

\[
W_3 = \frac{W_1 W_0 \alpha W_2 + \beta (1 + W_1 W_2)}{W_2 \alpha W_1 + \beta (1 + W_1 W_2)}
\]  

(37)

For this reduced mapping, there are two integrals \( I_q \) and \( I_p \) with \( \gamma = \delta = 0 \), both of which can be expressed in terms of \( W_0, W_1 \) and \( W_2 \). In combination with measure preservation this implies integrability.

- For \( \alpha = 0 \), (32) reduces to a three-dimensional mapping in terms of the variables \( W_0 = V_1, V_0, W_1 = V_2 V_1, W_2 = V_3 V_2 \):

\[
W_3 = \frac{W_1 W_2 - \beta (W_1 + W_2) + \gamma W_1 W_2}{W_0 - \beta (W_1 + W_2) + \gamma W_1 W_2}
\]  

(38)

Again, for \( \alpha = 0 \), both of the integrals \( I_q \) and \( I_p \) of (32) can be expressed in terms of \( W_0, W_1 \) and \( W_2 \).

(4) Finally, it is worthwhile to note that we can achieve four normal forms for (32) with \( \gamma = \delta = 0 \); \( \gamma = \delta = 1 \); \( \gamma = 1, \delta = 0 \); resp \( \gamma = 1, \delta = -1 \). This can be done using the rescaling \( V_1 \mapsto \lambda V_1 \) and the symmetry \( V_1 \mapsto V_1^{-1} \), \( \gamma \leftrightarrow \delta \).

The exercise of constructing the dual of (8) is not without some mystery! From (36) it follows that (8) is in a sense a trivial four-dimensional mapping since it can be reduced to a two-dimensional one. And yet, ignoring this fact and taking its four non-independent integrals \( I_\alpha, I_\beta, I_\gamma, I_\delta \) produces, in special cases, duals (37) and (38) which appear to be genuinely new and nontrivial integrable three-dimensional mappings (see [18] for some other examples of three-dimensional integrable mappings).

3. Integrable ODEs derived from soliton equations have a dual

In this section and in section 4, we present a method for obtaining the dual equations of a general class of higher-dimensional mappings arising from the MKdV partial difference equation [6]. However, we first use a general formulation on the basis of a given Lax representation for general integrable partial difference equations. This is done to make the treatment directly applicable to other integrable dynamical systems as well. The remaining sections 5 and 6 will specialize to the MKdV case.

A (scalar) PDE on a two-dimensional lattice \( f_{\ell,m} = 0, \ell, m \in \mathbb{Z} \) has a Lax representation if there are matrices \( L_{\ell,m}(k), M_{\ell,m}^{-1}(k), N_{\ell,m}(k) \) depending on a spectral parameter \( k \) such that

\[
L_{\ell,m}(k) M_{\ell,m}^{-1}(k) - M_{\ell+1,m}^{-1}(k) L_{\ell,m+1}(k) = f_{\ell,m} N_{\ell,m}(k)
\]  

(39)

in which \( f_{\ell,m} \) does not depend on \( k \), and \( N_{\ell,m} \) is nonsingular on \( f_{\ell,m} = 0 \).
We restrict ourselves to the PΔE

\[ f_{\ell,m} = 0 \]

with \( f_{\ell,m} \) of the form

\[ f_{\ell,m} = f(u_{\ell,m}, u_{\ell+1,m}, u_{\ell,m+1}, u_{\ell+1,m+1}, p_{\ell,m}) \]  \( (40) \)

where the \( p_{\ell,m} \) denote any additional parameters arising from the matrices \( L \) and \( M \), and the fields \( u_{\ell,m} \), for simplicity, are taken to be scalars. The subscripts \( \ell, m \) allow for the possibility that the parameters \( p \) depend on the lattice sites \((\ell, m)\). This general setting is investigated in order to obtain dual equations and their integrals with a sufficient amount of generality.

A PΔE can be reduced to an ordinary difference equation (OΔE) through travelling-wave reductions \([6]\). This can be done considering two integers \(z_1 \) and \( z_2 > z_1 \) which are relatively prime. In the \((z_1, z_2)\) travelling-wave reduction the parameters \( p_{\ell,m} \) in the matrices \( L_{\ell,m}(k) \) and \( M_{\ell,m}^{-1}(k) \) depend on the sites \((\ell, m)\) via the similarity variable \( n = z_1 \ell + z_2 m \) and we consider periodic solutions of \( f_{\ell,m} = 0 \) satisfying \( u_{\ell,m} = u_{\ell-z_2+m+z_1} \), i.e.

\[ p_{\ell,m} = p_n, \quad u_{\ell,m} = u_n, \quad n = z_1 \ell + z_2 m \]  \( (41) \)

and these solutions can be obtained from the OΔE

\[ f_n = f(u_n, u_{n+z_1}, u_{n+z_2}, u_{n+z_1+z_2}, p_n) = 0. \]  \( (42) \)

The OΔE can be solved specifying initial values on a standard staircase \([6]\) consisting of points \((\ell_i, m_i)\), \( i = 0, 1, \ldots, z_1 + z_2 - 1 \) with \( n(\ell_i, m_i) = n + n_i \) such that every value \( n_i \) with \( n \geq z_1 + z_2 \) occurs exactly once among the \( n_i \) values on the staircase. In fact, the \( u_n \) with \( n \geq z_1 + z_2 \) can be obtained from the OΔE \( f_{n+v} = 0 (v \geq 0) \), expressing \( u_{n+v+z_1+z_2} \) in terms of \( u_{n+v}, u_{n+v+z_1}, u_{n+v+z_2} \).

The monodromy matrix \( L_n \) is defined to be the ordered product of Lax matrices along a standard staircase (more detail will follow in the next section). From (39) it can be shown that

\[ \text{Trace } L_n^v = \text{Trace } L_{n+1}^v, \quad v = 1, 2, \ldots. \]  \( (43) \)

In the special case of \( 2 \times 2 \) matrices \( L_n, M_n^{-1} \), such that \( \text{det } L_n \) is trivial, we can restrict ourselves to \( v = 1 \).

Equation (43) is satisfied independently of the value of the spectral parameter \( k \), and the coefficients of the various powers of \( k \) appearing in Trace \( L_n \) give integrals of the OΔE \( f_n = 0 \). (Note that it is not generally true that all integrals of the OΔE can be obtained that way. In fact of the four integrals \( I_{\phi}, I_{\psi}, I_{\phi}, I_{\psi} \) of the mapping (8) in the example of section 2, only the integral \( I_{\phi} \) follows from the Lax representation of the MKdV reduction associated with (8)). On the other hand, taking the difference between Trace \( L_n \) and its shifted version we obtain a relation

\[ \text{Trace } L_n - \text{Trace } L_{n+1} = f_n f_n^* \]  \( (44) \)

containing a factor \( f_n^* \), ensuring that \( f_n^* = 0 \) implies the vanishing of the lhs as well. By analogy with (6), the equation \( f_n^* = 0 \) will be called the dual OΔE.

4. Standard staircase and dual OΔE

We now give a prescription for the dual OΔE in the case that \( z_1 = 1, z_2 = z, n = l + zm \). In this case, the standard staircase as introduced in [6] can be constructed in the following steps, see, e.g., figure 1.
To construct a standard staircase for $z_1 = 1, z_2 = z$:

(a) we start with the point $(\ell_0, m_0)$ with $n(\ell_0, m_0) = n + n_0, n_0 = 1$;
(b) we do a step to the left, to the point $(\ell_1, m_1) = (\ell_0 - 1, m_0)$ with $n(\ell_1, m_1) = n + n_1, n_1 = 0$;
(c) then we do a step upward to the point $(\ell_2, m_2) = (\ell_1, m_1 + 1)$ with $n(\ell_2, m_2) = n + n_2, n_2 = z$;
(d) next we do $z - 2$ steps to the left via points $(\ell_i, m_i) = (\ell_0 - i + 1, m_0 + 1)$ with $n(\ell_i, m_i) = n + n_i, n_i = z - i + 2, i = 3, 4, \ldots, z - 1$ to reach the point $(\ell_z, m_z) = (\ell_0 - z + 1, m_0 + 1)$ with $n(\ell_z, m_z) = n + n_z, n_z = 2$;
(e) a final step to the left brings us to $(\ell_{z+1}, m_{z+1}) = (\ell_0 - z, m_0 + 1)$ with $n(\ell_{z+1}, m_{z+1}) = 1$.

The staircase is thus completed. See figure 1 for an example.

To obtain the monodromy matrix $L_n$ we associate Lax matrices with the steps of the staircase in the following way:

(i) we associate with the first step from $(\ell_0, m_0)$ to $(\ell_1, m_1)$ the Lax matrix $S(n_0, n_1) = L_n$;
(ii) with the second step from $(\ell_1, m_1)$ to $(\ell_2, m_2)$ we associated the Lax matrix $S(n_1, n_2) = L_n^{-1}$;
(iii) with the $z - 2$ steps to the left from $(\ell_i, m_i)$ to $(\ell_{i+1}, m_{i+1})$ with $i = 2, \ldots, z - 1$ we associate the Lax matrices $S(n_i, n_{i+1}) = L_n^{-i-1}$;
(iv) with the final step from $(\ell_z, m_z)$ to $(\ell_{z+1}, m_{z+1})$ we associate the Lax matrix $S(n_z, n_{z+1}) = L_{n+1}$.

The monodromy matrix $L_n$ is the ordered product of Lax matrices along the standard staircase

$$L_n = \prod_{i=0}^{z} S(n_i, n_{i+1}).$$

From the explicit $S(n_i, n_{i+1})$ we have the factorization property

$$L_n = L_n M_n^{-1} A_n L_{n+1}$$

with

$$A_n = L_{n+2} L_{n+3} \ldots L_{n+2}.$$  

The shifted monodromy matrix

$$L_{n+1} = \prod_{i=0}^{z} S(n_i + 1, n_{i+1} + 1),$$

cf figure 1, can be expressed as

$$L_{n+1} = L_{n+1} M_{n+1}^{-1} L_{n+2} A_n$$

and therefore, cf (39)

$$L_n L_n^{-1} = L_{n+1} L_{n+1}^{-1} = (L_n M_n^{-1} - M_{n+1}^{-1} L_n) A_n = f_n N_n A_n.$$
Hence
\[ \text{Trace } \mathcal{L}_n - \text{Trace } \mathcal{L}_{n+1} = f_n \text{Trace } N_n T_n \] (49a)

with
\[ T_n = \Lambda_n L_{n+1} = L_{n+1} - L_{n+2} - \ldots - L_{n+1}. \] (49b)

The dual equation is given by
\[ f_n^* = \text{Trace } N_n T_n = 0. \] (50)

It is completely determined by the matrix \( N_n(k) \) in equation (39) and the matrices \( L \) in equation (49b). Since Trace \( \mathcal{L}_n \) is also an integral of the dual equation, we can use the difference to define a dual–dual equation \( f_n^{**} = 0 \) via Trace \( \mathcal{L}_n - \text{Trace } \mathcal{L}_{n+1} = f_n^* f_n^{**} \). In comparing this to equation (44) it is clear that the dual–dual equation \( f_n^{**} = 0 \) is just the original one \( f_n = 0 \), provided that the same integrals are used in the construction of the dual and the dual–dual equation.

5. Dual MKdV OΔE in the case \( z_1 = 1, z_2 = z \)

We now specialize the discussion of the previous two sections to consider the dual OΔE associated with the MKdV PΔE \( f_{\ell,m} = 0 \), in the case that \( z_1 = 1, z_2 = z, n = \ell + zm \). The dual equation associated with the more general reduction of the MKdV PΔE with \( z_2 > z_1 > 1 \) can also be investigated. However, this is more complicated and will not be explicitly pursued in this paper.

In the investigation of the dual equation of the \( z_1 = 1, z_2 = z \) reduction of the MKdV PΔE, one may anticipate that the number of parameters appearing in the dual equation will depend on the number of parameters in the original MKdV PΔE, or in the Lax matrices occurring in the associated Lax representation. We therefore choose a rather general inhomogeneous setting in which the Lax matrices \( L_{\ell,m}(k), M_{\ell,m}^{-1}(k) \) contain parameters \( p_{\ell,m}, r_{\ell,m}, a_{\ell,m}, b_{\ell,m}, q_{\ell,m}, s_{\ell,m}, c_{\ell,m}, d_{\ell,m} \). The subscripts indicate a possible dependence of the parameters on the sites \( (\ell, m) \) of the two-dimensional lattice; albeit with such a dependence satisfying compatibility conditions to ensure that a consistent difference equation \( f_{\ell,m} = 0 \) follows from the Lax equation
\[ L_{\ell,m}(k) M_{\ell,m}^{-1}(k) - M_{\ell+1,m}^{-1}(k) L_{\ell+1,m}(k) = 0, \] (51)
i.e. equation (39) with the right-hand side replaced by 0.

To investigate the \( z_1 = 1, z_2 = z \) reduction of the MKdV PΔE with \( u_{\ell,m} \) satisfying the periodicity constraint \( u_{1,m} = u_{1+zm+1} = u_n, n = \ell + zm \), we impose the conditions \( p_{\ell,m} = p_{\ell+m,n} = p_n \) for \( p = (p, r, a, b, q, s, c, d) \). That is, all parameters occurring in the Lax matrices satisfy the periodicity constraint also.

Hence we consider the Lax matrices
\[ L_{\ell,m}(k) = \begin{pmatrix} p_n & -a_n u_{\ell+1,m} \\ -k^2 b_n & r_n u_{\ell+1,m} + a_n \\ \end{pmatrix}, \quad M_{\ell,m}^{-1}(k) = \begin{pmatrix} s_n & d_n u_{\ell,m} \\ k^2 c_n & q_n u_{\ell,m} + d_n \\ \end{pmatrix}. \] (52)

The special case of (52): \( p_{\ell,m} = p, r_{\ell,m} = r, q_{\ell,m} = q, s_{\ell,m} = s, a_{\ell,m} = a, b_{\ell,m} = c_{\ell,m} = d_{\ell,m} = 1 \), was investigated in [1] in an investigation of four-dimensional mappings satisfying a specific symplectic structure (as a generalization of [6]). Working out the (1, 1) and (2, 2) elements of condition (51) and insisting that the (1, 2) and (2, 1) elements of this condition lead to the same difference equation, we are led to the compatibility conditions
\[ s_{n+1} = \frac{p_n}{p_{n+z}}, \quad q_{n+1} = \frac{r_n}{r_{n+z}} q_n, \quad d_{n+1} = \frac{a_n}{b_{n+z}} c_n, \quad c_{n+1} = \frac{b_n}{a_{n+z}} d_n \] (53)
and

\[ k_n := k \frac{a_n b_n}{p_n r_n} = k_{n+z}, \]

or equivalently with (53)

\[ l_n := c_n d_n \]

\[ q_n b_n = l_{n+1} = 1. \]

Under these conditions, we have equation (39) with

\[ f_{\ell,m} = p_n d_n - a_n q_n \frac{u_{\ell+1,m}}{u_{\ell,m+1}} + \frac{u_{\ell+1,m+1}}{u_{\ell,m}} \left( a_{n+z} r_{n+1} - a_{n+1} r_{n+z} \right). \]

The equation \( f_{\ell,m} = 0 \) with \( f_{\ell,m} \) given by (55a) is a general inhomogeneous version of the MKdV partial-difference equation of [1, 6] with parameters \( p \) depending on the sites \((\ell, m)\) via the similarity variable \( n \). Under conditions (53), (54) for the parameters, it has the Lax representation (39) with the matrices \( L_{\ell,m}(k) \) and \( M_{\ell,m}(k) \) given by (52) and the matrix \( N_{\ell,m}(k) \) given by

\[ N_{\ell,m}(k) = \left( \begin{array}{cc} 0 & u_{\ell,m} \\ -k^2 & 0 \end{array} \right). \]

In the travelling-wave reduction \( z_1 = 1, z_2 = z \), (55) reduces to the primary MKdV ODE

\[ f_n = 0 \]

with

\[ f_n = p_n d_n - a_n q_n \frac{u_{n+1}}{u_{n+z}} + \frac{u_{n+1+z}}{u_n} \left( a_{n+z} r_{n+1} - a_{n+1} r_{n+z} \right). \]

The dual equation follows from (50) and (55b) and is given by

\[ f_n^* = 0, \]

where

\[ f_n^* = t_{12} u_n - b_n \frac{p_{n+z}}{a_{n+z}} \frac{k^2}{p_n} u_{n+z+1} \]

and \( t_{12} \) and \( t_{21} \) are the off-diagonal elements of the matrix \( T_n \), defined by (49b).

Evaluating the off-diagonal elements of the matrix \( T_n \) it can be shown that the dual MKdV ODE (57) can be expressed as a \((z-1)\)-dimensional mapping in terms of the reduced variables

\[ W_{n+v} = \frac{r_{n+v}, a_{n+z+1}, u_{n+z+2}}{a_{n+v}, p_{n+v}, u_{n+v}}. \]

We have

\[ \frac{W_{n+z-1}}{W_n} = k_0 Z_{12}(W_{n+1}, W_{n+2}, \ldots, W_{n+z-2}) \]

in which \( Z_{12} \) and \( Z_{21} \) are the off-diagonal elements of the matrix

\[ Z(W_{n+1}, W_{n+2}, \ldots, W_{n+z-2}) = \left( \begin{array}{cccc} 1 & 1 & \ldots & 1 \\ k_{n+z-1} & k_{n+z-2} & \ldots & k_{n+1} \end{array} \right) \]

\[ \times \left( \begin{array}{cccc} 1 & 1 & \ldots & 1 \\ W_{n+z-3} & W_{n+z-2} & \ldots & W_{n+1} \end{array} \right). \]

cf appendix A for some details of the derivation.

In equation (59) the numerator \( Z_{12} \) and denominator \( Z_{21} \) are multilinear functions containing all \( 2^{z-2} \) terms \( W_{n+1}^{\mu_1}, W_{n+2}^{\mu_2}, \ldots, W_{n+z-2}^{\mu_{z-2}}, \mu_1, \mu_2, \ldots, \mu_{z-2} = 0, 1 \), with coefficients depending on \( k_{n+1}, k_{n+2}, \ldots, k_{n+z-1} \).
For example for $z = 3, 4, 5$, we have respectively

$z = 3$: \[ Z_{12}(W_{n+1}) = 1 + W_{n+1} \\
Z_{21}(W_{n+1}) = k_{n+2} + k_{n+1}W_{n+1} \]

$z = 4$: \[ Z_{12}(W_{n+1}, W_{n+2}) = 1 + W_{n+1} + W_{n+2}(k_{n+2} + W_{n+1}) \\
Z_{21}(W_{n+1}, W_{n+2}) = k_{n+3}(1 + k_{n+1}W_{n+1}) + W_{n+2}(k_{n+2} + k_{n+1}W_{n+1}) \]

$z = 5$: \[ Z_{12}(W_{n+1}, W_{n+2}, W_{n+3}) = (1 + k_{n+3}W_{n+1})(1 + W_{n+1}) \\
+ W_{n+2}(1 + W_{n+3})(k_{n+2} + W_{n+1}) \\
Z_{21}(W_{n+1}, W_{n+2}, W_{n+3}) = (k_{n+4} + k_{n+3}W_{n+1})W_{n+1} \\
+ W_{n+2}(k_{n+4} + W_{n+1})(k_{n+2} + k_{n+1}W_{n+1}) \]

(61)

6. Integrability of the dual MKdV

To prove that the dual MKdV OΔE is integrable we first construct two integrals $J_s, J_q$ and two 2-integrals $J_{s_2}, J_{d_2}$ satisfying $J_{s_2} = J_{d_2}, J_{d_0} = J_{s_0}$.

From equations (46a), (49b), (50), it is clear that the parameters $\{s_n, e_n, d_n, q_n\}$ in the matrix $M_n^{-1}$ in Trace $L_n$ dissociate from the parameters $p_n, d_n, b_n, r_n$ occurring in the matrices $L$ and the dual equation. To investigate the integrals of the dual equation it is therefore worthwhile to consider the decomposition

\[ \text{Trace } L_n = J_{s_0} + J_{q_0} + J_{d_0} + J_{s_0} \]

(62)

in which $J_{s_0}, J_{q_0}, J_{d_0}, J_{q_0}$ are terms arising from elements $s_n, e_n, d_n, q_n$ of the matrix $M_n^{-1}$ in Trace $L_n = \text{Trace } L_n M_n^{-1} T_n$.

Here using (52), and involving the matrix elements $t_{ij}$ of $T_n$, we have

\[ \left( \begin{array}{cc} \frac{\partial}{\partial s_n} & \frac{\partial}{\partial q_n} \\ \frac{\partial}{\partial e_n} & \frac{\partial}{\partial d_n} \end{array} \right) = \left( \begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array} \right) \left( \begin{array}{cc} p_n & -a_n \frac{u_{n+1}}{u_{n+2}} \\ -k^2 b_n & r_n \frac{a_n}{u_{n+2}} \end{array} \right) \]

(63)

Introducing

\[ J_s = J_{s_0}/s_n, \quad J_q = J_{q_0}/e_n, \quad J_d = J_{d_0}/d_n, \quad J_q = J_{q_0}/q_n \]

(64)

and using equation (53) to express $s_{n+1}, q_{n+1}, d_{n+1}, c_{n+1}$ in terms of $s_n, e_n, d_n, q_n$, we find from (49a) and (50) that if the dual equation is satisfied then

\[ 0 = \text{Trace } L_n - \text{Trace } L_{n+1} = s_n \left\{ J_s = \frac{p_n}{p_{n+1}} J_{s_0} + q_n \left\{ J_q = \frac{r_n}{r_{n+1}} J_{q_0} \right\} \right\} \]

(65)

Since the matrices $L_{n+1}, L_{n+2}, \ldots, L_{n+1}$ and $N_p$ occurring in the dual equation do not contain any of the parameters $s_n, q_n, e_n, d_n$, equation (65) must be valid independently of the choice of $s_n, q_n, e_n, d_n$. This means that

\[ J_s = \frac{p_n}{p_{n+1}} J_{s_0}, \quad J_q = \frac{r_n}{r_{n+1}} J_{q_0}, \quad J_c = \frac{a_n}{a_{n+1}} J_{c_0}, \quad J_d = \frac{b_n}{b_{n+1}} J_{d_0} \]

(66)

and hence, again using (53) and (64),

\[ J_s = J_{s_0}, \quad J_q = J_{q_0}, \quad J_c = J_{c_0}, \quad J_d = J_{d_0} \]

(67)

More generally, $k$-integrals of a mapping are defined as integrals of the $k$th iterate of the mapping $[19]$, here corresponding to the shift $n \to n + k$. 

From equation (63) we can solve the off-diagonal elements $t_{12}, t_{21}$ of the matrix $T_n$ to obtain

\[
\frac{1}{u_n^2} \frac{t_{12}}{t_{21}} = \frac{a_n \frac{\partial}{\partial x} u_{n+1} + p_n \frac{\partial}{\partial x} u_n}{k^2 b_n \frac{\partial}{\partial x} u_n + r_n \frac{\partial}{\partial x} u_{n+1}}.
\]  

(68)

Inserting this in the dual MKdV (57), including the factor $a_{n+1}/b_n$ in the denominator and the factor $p_{n+1}/p_n$ in the numerator and using (53) yields

\[
\frac{u_{n+1}}{u_n} = \frac{k^2 a_n \frac{\partial}{\partial x} u_{n+1} + p_n \frac{\partial}{\partial x} u_n}{k^2 a_n \frac{\partial}{\partial x} u_n + p_n \frac{\partial}{\partial x} u_{n+1}}.
\]  

(69)

Equation (69) is very similar to the MKdV ODE (56). To prove its integrability we rewrite it as follows, recalling (54b):

\[
\frac{u_{n+1}}{u_n} = \frac{k^2 a_n q_n \partial_x u_n + p_n d_n \lambda^{-1} \partial_x c_n}{k^2 a_n \partial_x u_n + p_n d_n \lambda^{-1} \partial_x c_n}.
\]  

(70)

Equation (70) is the travelling-wave reduction $u_{\ell,m} \to u_{\ell+1,m} = u_n$ of the PDE

\[
\frac{u_{\ell+1,m}}{u_{\ell,m}} = \frac{a_n q_n \partial_x u_{\ell+1,m} + p_n d_n \lambda^{-1} \partial_x c_n}{a_n \partial_x u_{\ell+1,m} + p_n d_n \lambda^{-1} \partial_x c_n}.
\]  

(71a)

with

\[
q_n^* = k^2 q_n \partial_x, \quad d_n^* = -d_n \lambda^{-1} \partial_x c_n,
\]

\[
s_{n+1}^* = k^2 s_{n+1} \partial_x, \quad d_{n+1}^* = -d_{n+1} \lambda^{-1} \partial_x c_n.
\]  

(71b)

Since (71a) is identical to (55a), it follows that in order for (71a) to be integrable via a Lax representation, we must impose the compatibility conditions

\[
\frac{c_n}{d_n} = -c_n \partial_x \lambda^{-1}
\]  

(73)

and using (53) and (67) it follows with (71b) that this condition is indeed satisfied. Comparing (71) and (72) with (55a) and (53) it follows that the matrices

\[
L_{\ell,m}(k^*) = \left( \begin{array}{cc}
p_n & -a_n u_{\ell+1,m} \\
\frac{-k^2 p_n}{u_{\ell,m}} & r_n u_{\ell+1,m} \end{array} \right)
\]  

(74a)

and

\[
M_{\ell,m}^{-1}(k^*) = \left( \begin{array}{cc}
s_{n+1}^* & d_n^* u_{\ell,m} \\
\frac{k^2 s_{n+1}^*}{u_{\ell,m}} & q_n^* \partial_x \end{array} \right),
\]  

(74b)

in which $k^*$ is a new spectral parameter, satisfy the relation

\[
L_{\ell,m}(k^*) M_{\ell,m}^{-1}(k^*) - M_{\ell+1,m}^{-1}(k^*) L_{\ell,m+1}(k^*)
\]

\[
= \left( p_n d_n - a_n q_n \frac{u_{\ell+1,m}}{u_{\ell,m+1}} + \frac{u_{\ell+1,m}}{u_{\ell,m}} \left( a_n s_{n+1}^* - d_n r_n \frac{u_{\ell+1,m}}{u_{\ell,m+1}} \right) \right) N_{\ell,m}(k^*). \]

(75)

This means that equation (71) has a Lax representation in terms of the matrices $L_{\ell,m}(k^*)$ and $M_{\ell,m}^{-1}(k^*)$ and this can be used to evaluate the monodromy matrix $L_n^*$ for the travelling-wave
reduction (70) built up from these matrices. From Trace $\mathcal{L}_n^*$, one can obtain additional integrals of the dual MKdV OΔE for larger values of $\varepsilon$.

The parameters $\mathcal{J}_r$, $\mathcal{J}_s$, $\mathcal{J}_d$, and $\mathcal{J}_q$ can be determined specifying the initial conditions for $u_n, u_{n+1}, \ldots, u_{n+\varepsilon}$ on a standard staircase. Because of this, equation (29) will be called the level-set-dependent (LSD) MKdV OΔE.

The term level-set-dependent relates to a concept introduced in [3, 4]. Given two OΔEs, with OΔE (2) possessing integrals, we will say

$$\text{OΔE}(2) = \text{LSD OΔE}(1),$$

if OΔE (2) acts on the level sets of its integrals as OΔE (1). The idea is that an initial condition for OΔE (2) determines the values of the integrals of OΔE (2). Then the ensuing motion can be described by a simpler (and possibly integrable) OΔE (1). This is completely analogous to the two-dimensional setting of [3, 4] in which it was shown that each QRT map actually acts as a generalized McMillan map on the level sets of the QRT integral. Here we have shown that the dual MKdV map acts as a MKdV map on the level sets of the integrals $\mathcal{J}_r, \mathcal{J}_q$ and the 2-integrals $\mathcal{J}_s = \mathcal{J}_{d_{n+1}}$ and $\mathcal{J}_d = \mathcal{J}_{c_{n+1}}$.

The explicit expressions for $\mathcal{J}_r, \mathcal{J}_q, \mathcal{J}_s$ and $\mathcal{J}_d$ are given by

$$\mathcal{J}_r = S(Z_{11} + k_n W_n Z_{12}), \quad \mathcal{J}_q = \frac{Q(Z_{21} + W_n Z_{22})}{W_n W_{n+1} \ldots W_{n+\varepsilon-2}} \quad (77)$$

$$\mathcal{J}_s = -k^l C_n S(Z_{11} + W_n Z_{12}) u_{n+1} \frac{u_{n+2}}{u_{n+\varepsilon}}, \quad \mathcal{J}_d = -C_n^{-1} Q(Z_{21} + k_n W_n Z_{22}) \frac{u_{n+1}}{W_n W_{n+1} \ldots W_{n+\varepsilon-2}}$$

in which the $W_{n+1}$ have been defined by (58) and $Z_{\alpha\beta} = Z_{\alpha\beta}(W_{n+1}, W_{n+2}, \ldots, W_{n+\varepsilon-2})$ are matrix elements of the matrix $Z$ defined by (60).

The parameter $C_n$ is defined by

$$C_n = \frac{c_n a_n}{s_n p_n} \quad (78)$$

and the constants $Q$ and $S$ by

$$S_n = p_n p_{n+1} \ldots p_{n+\varepsilon-1} s_n = S \quad (S_{n+1} / S_n = 1)$$

$$Q_n = r_n r_{n+1} \ldots r_{n+\varepsilon-1} q_n = Q \quad (Q_{n+1} / Q_n = 1). \quad (79)$$

Some details of the derivation of (77) are presented in appendix B.

In summary, we have considered the dual MKdV OΔE associated with the $z_1 = 1, z_2 = \varepsilon$ similarity reduction of the MKdV PΔE. These considerations can be extended to more general similarity reductions $z_1 > 1$. Also there, the dual MKdV OΔE can be expressed as a generalized MKdV with parameters $\mathcal{J}$ depending on the initial conditions but in the general case we do not anticipate relations like (67) ensuring the existence of a Lax representation and consequently the integrability of the dual MKdV OΔE. The present treatment can also be adapted to the SG case with only minor variations.

7. Concluding remarks

We have studied the dual MKdV OΔE associated with the $z_1 = 1, z_2 = \varepsilon$ similarity reduction of the MKdV PΔE. This dual equation is an integrable system, since it is the LSD version of the MKdV OΔE. In this LSD version, there appear the integrals $\mathcal{J}_r, \mathcal{J}_q$ and the 2-integrals $\mathcal{J}_s = \mathcal{J}_{d_{n+1}}$ and $\mathcal{J}_d = \mathcal{J}_{c_{n+1}}$ of the dual MKdV which follow from the trace of the monodromy matrix. The dual MKdV has a Lax representation that is obtained from the Lax representation (52) of the MKdV PΔE using the substitutions (71b) which involve the integrals $\mathcal{J}_r, \mathcal{J}_q$ and $\mathcal{J}_d$. Note that $\mathcal{J}_r$ and $\mathcal{J}_q$ depend only on the variables $W_n$.8
the 2-integrals $J_{cn}, J_{dn}$, and introduces a new spectral parameter. The (reduced variable) dual MKdV (59), (60) contains the parameters $k_n, \ldots, k_{n+z-1}$, defined by (54a). In the homogeneous case, this set of parameters reduces only to the single parameter $k$. However, it is worthwhile to note that the dual equation has been obtained using the trace of the monodromy matrix

$$\text{Trace } T_n = \sum_m k^{2m} I_m$$  

(80)

in which the integrals $I_m$ of the original MKdV OΔE are the coefficients of different powers of the spectral parameter. Rather than considering the special case of (80), one can also start from an arbitrary linear combination of integrals

$$I = \sum_m \alpha_m I_m$$  

(81)

and the dual MKdV in the homogeneous case can be inferred from (59)–(60) by replacing $k^{2m}$ by the parameter $\alpha_m$. In this way, the family of dual MKdV mappings for increasing $z$ will contain an increasing number of parameters $\alpha_m$, but the existence of a Lax representation is not obvious in this more general case.

Finally, it may be noted that the dual equations (59), (60) have been obtained taking into account only the integrals of the MKdV OΔE that appear in the trace of the monodromy matrix. However, in special cases, the MKdV OΔE $f_n = 0$ of (56) may have additional integrals which can be inferred from the work of Hydon [2] or by considering homogeneous expressions in the variables $u$ of various degrees. This would then lead to a dual equation in the variables $u$ with more parameters, similar to the case of the motivating example of section 2. But in that case the Lax representation (74) and LSD description (70) are no longer valid and the integrability of the dual equation is open.

We conclude with some remarks:

- The considerations in this paper can be applied to the $z_1 = 1, z_2 = z$ reductions of other PΔE as well. For instance, for the reduction of the integrable SG PΔE, the dual equation can be expressed as a LSD SG OΔE and a Lax representation is obtained by simple substitutions from the Lax representation of the SG OΔE. Also the dual of the $z_1 = 1, z_2 = z$ reduction of the KdV PΔE can be investigated in a similar way.

- The treatment in this paper can also be extended to more general reductions with $z_1 > 1$ of the MKdV PΔE. The dual equation can be obtained from the trace of the monodromy matrix and one can show that the dual equation can be regarded as an extended version of the MKdV OΔE as given by (70) with the coefficients $J_{cn}, J_{dn}, J_{cn}, J_{dn}$ explicitly given in terms of the variables $u_n$. In this general case, however, the reasoning leading to a condition of the type (66) is no longer applicable and there is no direct relation between the coefficients $J_{cn}, J_{dn}, J_{cn}, J_{dn}$ and the integrals of the dual MKdV. So in this case the extended MKdV equation (70) cannot be interpreted as an LSD MKdV equation and also a Lax representation like (75) does not apply. Hence, at this stage, the integrability of the dual MKdV equation for $z_1 > 1$ has not been proved. It is interesting to note, however, that the $z_1 = 2, z_2 = 3$ reduction of the MKdV PΔE leads to a dual equation which is a four-dimensional mapping that is superintegrable by the existence of two integrals $I_n = I_{n+1}, J_n = J_{n+1}$ and two coupled 2-integrals $G_n = H_{n+1}, H_n = G_{n+1}$.

We hope to investigate the integrability of the more general cases in the near future.

- From (76), it follows that each orbit of OΔE (2) corresponds to an orbit of OΔE (1) restricted to a particular intersection of level sets of the integrals of OΔE (2). This shows that considered ‘orbit-by-orbit’ OΔE (2) is solvable, and in that sense integrable, if OΔE
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(1) is. For example, if \( O_{\Delta E} (2) \) is a rational map, each of its orbits will have vanishing algebraic entropy, thus passing this test. However, on the face of it, LSD integrability does not imply symplecticity or Liouville integrability of \( O_{\Delta E} (2) \), as these refer to global structures of \( O_{\Delta E} (2) \) by which orbits are assembled together.

Appendix A

To prove equation (59) we first express the Lax matrix \( L_{n+1}(k) \), cf (52), as

\[
L_{n+1}(k) = p_{n+1} \begin{pmatrix} 1 & \frac{p_{n+1}}{a_{n+1}} \frac{1}{a_{n+2}} \\ 0 & -\frac{a_{n+1}}{p_{n+1}} u_{n+1} \end{pmatrix} \begin{pmatrix} k_n W_{n+1} & 1 \\ W_{n+1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{a_{n+1}}{p_{n+1}} u_{n+1} \end{pmatrix} \tag{A1}
\]

Then from (49b) we have

\[
T_n = p_{n+1} p_{n+2} \cdots p_{n+z-1} \begin{pmatrix} 1 & 0 \\ -\frac{p_{n+z-1}}{a_{n+1}} \frac{1}{a_{n+2}} \end{pmatrix} \begin{pmatrix} k_{n+z-1} W_{n+z-1} & 1 \\ W_{n+z-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{a_{n+1}}{p_{n+1}} u_{n+2} \end{pmatrix} \tag{A2}
\]

which can be expressed as

\[
T_n = p_{n+1} \cdots p_{n+z-1} \begin{pmatrix} 1 & 0 \\ -\frac{p_{n+z-1}}{a_{n+1}} \frac{1}{a_{n+2}} \end{pmatrix} Z \begin{pmatrix} 1 & 0 \\ 0 & -\frac{a_{n+2}}{p_{n+1}} u_{n+2} \end{pmatrix} \tag{A3}
\]

in which the matrix \( Z \) has been defined by (60).

From (A3) and (57) we then obtain

\[
\frac{u_{n+z+1}}{a_{n+2}} = \frac{k^2 p_n}{a_{n+2}} \frac{p_{n+z}}{Z_{12}} a_{n+1} u_{n+z-1} \frac{u_{n+z+1} u_{n+2} u_{n+1}}{u_n} \tag{A4}
\]

which with (54a) and (58) yields (59).

Appendix B

From equations (63) and (A3) we obtain the relations

\[
J_a = S \left( Z_{11} + k_a \frac{a_{n+1}}{a_n} u_{n+2} a_n \frac{p_{n+1}}{u_n} \right) \tag{B1}
\]

and equations (77) can be derived using (58) and the relation

\[
\frac{u_{n+z+1} u_{n+1}}{u_{n+z-1} a_{n+2}} = \frac{a_{n+1}}{a_n} r_n^2 \frac{a_{n+1}}{a_n} p_n \frac{p_{n+1}}{u_n} \frac{1}{W_{n+1} \cdots W_{n+z-2}}. \tag{B2}
\]

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