Workshop on
Algebraic, Number Theoretic
and Graph Theoretic Aspects
of Dynamical Systems
UNSW, Sydney, Australia
2-6 February, 2015
Room M-032, Mezzanine Level
Red Centre Building (East Wing)

Program

Monday 2 Feb

08:30–09:20  Registration
09:20–09:30  Introductory Remarks
10:30–11:00  Coffee Break
11:00–11:30  Alina Ostafe, On Intersections of Orbits in Polynomial Dynamical Systems over
Finite Fields
11:30–12:00  Min Sha, On the Quantitative Dynamical Mordell-Lang Conjecture
12:00–13:00  Research Discussions
13:00–14:30  Lunch
14:30–15:30  Cheryl E. Praeger, Cayley Graphs of Finite Groups
15:30–16:00  Coffee Break
16:00–16:30  Sawian Jaidee, Mertens’ Theorem for Arithmetical Dynamical System
16:30–17:00  Seiken Saito, Mertens Theorem for Galois Coverings (in the number field case
and the graph case)
18:00–20:00  Reception (Red Centre, 3rd Floor, Staff Common Room)

Tuesday 3 Feb

09:30–10:30  Klaus Schmidt, Sandpiles and Harmonic Systems
10:30–11:00  Coffee Break
11:00–11:30  Shahrina Ismail, Perfect Triangles and Rational Points on Elliptic Curves
11:30–12:00  Michael Coons, A Function Related to Mahler’s Method
12:00–13:00  Research Discussions
13:00–14:30  Lunch
14:30–15:30  Tony Guttmann, Recent Results in Pattern-avoiding Permutations
15:30–15:45  Group Photo
15:45–16:15  Coffee Break
16:15–16:45  Ana Zumalacárregui, The Least Common Multiple of Sets of Positive Integers
16:45–17:15  Peter Van Der Kamp, How Integrable Systems Give Rise to Recurrences Exhib-
iting the Laurent Phenomenon
18:00–20:00  Franco Vivaldi, The Arithmetic of Chaos, Public Lecture, Colombo Theater C

1All coffee breaks will be in the Red Centre, 3rd Floor, Staff Common Room
Wednesday 4 Feb

09:30–10:30  Nick Wormald, Random graphs, Random Regular Graphs and Combs
10:30–11:00  Coffee Break
11:00–11:30  Catherine Greenhill, The Switch Markov Chain for Sampling Irregular Graphs
11:30–12:00  David Pask, Graphs, Dynamics and C*-algebras
12:00–12:10  Break
12:10–12:40  Konrad Pilch, Generalisations of Bost-Connes Systems and Hilbert’s 12th Problem
12:40–        Trip to the City and Dinner, More Information will be distributed via email

Thursday 5 Feb

09:30–10:30  Franco Vivaldi, Renormalization in Two-dimensional Piecewise Isometries
10:30–11:00  Coffee Break
11:00–11:30  Grant Cairns, Piecewise Linear Periodic Maps of the Plane with Integer Coefficients
11:30–12:00  Timothy Siu, Universal Period Distributions for Piecewise Cat Maps over Rational Lattices
12:00–13:00  Research Discussions
13:00–14:30  Lunch
14:30–15:30  Alexander Gamburd, Expander Graphs, Strong Ergodicity, and Superstrong Approximation
15:30–16:00  Coffee Break
16:00–16:30  Michael Baake, Polynomial Iteration, Symbolic Dynamics, and Long-range Order
16:30–17:00  Christian Huck, Ergodic Properties of Visible Lattice Points and Related Systems

Friday 6 Feb

09:30–10:30  Thomas Ward, Rigid and Flexible Invariants for Group Automorphisms
10:30–11:00  Coffee Break
11:00–11:30  George A. Willis, Subgroups of Totally Disconnected Groups that are Ergodic for an Automorphism
11:30–12:00  Kamil Bulinski, Plünnecke inequalities for measure preserving actions
12:00–13:00  Concluding Discussions
13:00–14:30  Lunch
ABSTRACTS OF INVITED TALKS

Expander Graphs, Strong Ergodicity, and Superstrong Approximation

Alexander Gamburd
CUNY Graduate Center

Abstract. After introducing expander graphs and briefly discussing classical results and constructions, I will talk about recent developments pertaining to establishing expansion property for congruence quotients of thin groups—discrete subgroups of semi-simple groups which are Zariski dense but of infinite index. This expansion property is intimately related to strong ergodicity of the associated group action and can be viewed as a far-reaching generalization of the strong approximation theorem.

Recent Results in Pattern-avoiding Permutations

Tony Guttmann
The University of Melbourne

Abstract. In the late 1960s Don Knuth asked which permutations could be sorted by various data structures. For a simple stack, he proved that permutations avoiding the pattern 312 can be so sorted, and further, that the number of such permutations of length \( n \) is given by \( C_n \), the \( n^{th} \) Catalan number. Knuth went on to pose the same question about other data structures that are widely used in sorting operations, namely deques, and two stacks, both in parallel and series. All these problems remain unsolved to this day, though there have been important developments. However while it is known that the number of permutations sortable by these structures grows exponentially, with a structure-dependent growth constant, the value of this constant is unknown in each case. We will discuss our efforts to estimate the constant in a number of cases, and also discuss the sub-leading asymptotic behaviour.

Knuth’s work and subsequent developments saw the area of pattern-avoiding permutation develop into an interesting area of combinatorics in its own right. The fundamental question asked is “how many permutations of length \( n \) avoid a given pattern?” Here the pattern is a sub-permutation. The answer to this question is known for the \( 3! = 6 \) patterns of length three. In every case the answer is given by the \( n^{th} \) Catalan number. For the \( 4! = 24 \) permutations of length four, it is known that these fall into 3 distinct classes. Two of these have been completely solved, and it is known that the growth constants in the two solved cases are 8 and 9 exactly. For the third class, those avoiding the pattern 1324, very little is exactly known. By extensive computer enumeration, and careful numerical work, we show that the growth constant is \( 11.60 \pm 0.01 \) and obtain a good estimate of the asymptotic behaviour, which is quite different to that for the two solved cases.

Finally, we comment on permutations avoiding the vincular pattern \( abcd - e \). Several cases of such patterns remain unsolved. From extensive enumerations we are able to make conjectures for the exact value of the growth constants for several of the unsolved cases. These turn out to be transcendental numbers. We then prove that the corresponding generating functions cannot be D-finite, contrary to earlier conjectures that all such patterns are D-finite.

This is joint work with Andrew Conway.
Cayley Graphs of Finite Groups

Cheryl E Praeger
University of Western Australia

Abstract. A Cayley graph on a finite group can be viewed as a dynamical system on the group, with a given specified generating set. Such graphs have been used as models for complexity analyses of randomized group computations, and contribute to the problem of analyzing the complexity of graph isomorphism. The diameters of Cayley graphs give a measure of the ‘mixing time’ needed for randomly traversing or sampling from the group, and a long-standing conjecture of Babai specifies how large the diameter may be in terms of the group order (especially for simple groups). Results on “growth in groups”, independently by Breuillard-Green-Tao and Pyber-Szabo yield proof of Babai’s conjecture for bounded rank Lie type groups, and the best result for finite alternating and symmetric groups come very recently from Helfgott-Seress. I will discuss these results and how they yield the best current results towards proof of other conjectures on non-regular group actions. If there is time, I’ll also discuss an approach to describing the structure of edge-transitive Cayley graphs — the role played by simple groups, and how much additional symmetry is possible.

Sandpiles and Harmonic Systems

Klaus Schmidt
University of Vienna

Abstract. The \(d\)-dimensional abelian sandpile model is a lattice model introduced in 1987 by Bak, Tang and Wiesenfeld as an example of what they called ‘self-organized criticality’. Although this deceptively simple model has been studied quite intensively both in the physics and mathematics literature, some very basic questions about it are still open, like its properties under two different kinds of dynamics: ‘addition’ (of grains of sand), and the shift-action.

By extending an algebraic construction originally introduced by A. Vershik for Markov partitions of hyperbolic automorphisms of the 2-torus one can show that the sandpile model is closely related to a certain \(\mathbb{Z}^d\)-action by automorphisms of a compact abelian group, the ‘harmonic model’.

The purpose of this lecture (which is based on joint work with Evgeny Verbitskiy) is a discussion of this construction and of the conclusions that can be drawn from the connection between these systems.

The Arithmetic Complexity of Orbits

Joseph H. Silverman
Brown University

Abstract. A naive measure of the arithmetic complexity of a number, or more generally of a point \(P = (x_1, ..., x_N)\) with algebraic coordinates, is the number of bits required to store \(P\) on a computer. This quantity is called the height of \(P\) and is denoted \(h(P)\). Let \(X\) be a quasi-projective variety, let \(P\) be a point of \(X\), and let \(f: X \to X\) be a map given by rational functions, with everything defined over a number field. Then the growth rate of \(h(f^n(P))\) provides an interesting measure of the arithmetic complexity of the forward orbit of \(P\) under iteration of \(f\). In favorable situations, the canonical height limit \(\hat{h}_f(P) = \lim_{n \to \infty} h(f^n(P))/\delta^*_n\) exists and has
good properties, where the dynamical degree $\delta_f = \lim_{n \to \infty} (\deg f^n)^{1/n}$ measures the geometric complexity of the iterates $f^n$. Even if $\hat{h}_f(P)$ does not exist, we may consider a coarser growth invariant, the arithmetic degree $\alpha_f(P) = \lim_{n \to \infty} h(f^n(P))^{1/n}$. \textbf{Conjecture 1}: The limit $\alpha_f(P)$ exists, is an algebraic integer, and takes on only finitely many values as $P$ varies. \textbf{Conjecture 2}: If the forward orbit of $P$ is Zariski dense in $X$, then $\alpha_f(P) = \delta_f$. We will discuss these definitions and conjectures and the recent proof of Conjecture 1 when $X$ is projective and $f$ is a morphism and of Conjecture 2 when $X$ is a torus or an abelian variety and $f$ is an isogeny. (Joint work with Shu Kawaguchi).

Renormalization in Two-dimensional Piecewise Isometries

Franco Vivaldi
Queen Mary, University of London

\textbf{Abstract}. Piecewise-isometries are zero-entropy dynamical systems with very complex behaviour. In one-dimension they are interval-exchange transformations (IET), which are connected to diophantine arithmetic by the Boshernitzan and Carrol theorem: in any IET defined over a quadratic number field, the process of induction results, after scaling, in an eventually periodic sequence of IETs. This can be viewed as a generalization of Lagrange’s theorem on the eventual periodicity of the continued fractions of quadratic irrationals.

In two dimensions we have polygon-exchange transformations (PET). Until recently, their study has been limited to specific systems defined over quadratic fields, which, invariably, have been found to exhibit eventual periodicity. I’ll describe some recent results in parametrized families of PETs, where the (fixed) rotational component identifies a quadratic field. We show that a suitable induction eventually generates a scaled version of the original map, re-parametrized by a Lüroth-type function —a piecewise affine version of Gauss’ map. The parameter values corresponding to exact scaling are found to be precisely the elements of the underlying quadratic field. The proofs required computer-assistance. (Joint work with J. H. Lowenstein).

Rigid and Flexible Invariants for Group Automorphisms

Thomas Ward
Durham University

\textbf{Abstract}. The structure of the space of all pairs $(G,T)$, where $G$ is a compact metric group and $T : G \to G$ a continuous ergodic automorphism can be studied via dynamical invariants. We will discuss some of the issues that come up in trying to understand which invariants can vary flexibly and which exhibit rigidity in moving through this space.

In particular, we will describe how many of these questions reduce to issues like Lehmer’s problem or constructions of sets of primes with prescribed properties.
Abstract. When a network grows randomly, the point at which it achieves a given property can often be pinpointed in advance with high probability. In their early work on random graphs in the late 50’s, Erdős and Rényi considered the threshold of appearance of a giant component, and of various other subgraphs.

The models of random graphs introduced at that time have received much attention since then, and have found many applications, particularly in computer science. Many interesting results can be stated in terms of the random graph process. In this, the random graph grows in time by the addition of random edges, making the graph ever denser as time goes on. The threshold of appearance of a given subgraph $H$ can be defined as the time at which the random graph process contains a copy of $H$ with probability at least $1/2$.

Kahn and Kalai made a simple-sounding but deep conjecture relating this threshold to another one: the threshold of expectation for a subgraph $H$ is the time at which the expected number of copies of $H$ in the random graph process exceeds 1. One very special case of this conjecture gained some notoriety as the comb conjecture, made about 15 years ago by Kahn.

Random regular graphs are a different but commonly used model of random graphs with low density. This model enters somewhat surprisingly into a solution of the comb conjecture, recently obtained jointly with Jeff Kahn and Eyal Lubetzky. In this talk I will give an exposition of results on random graphs and random regular graphs, with the proof of the comb conjecture as a focus.
1. **Michael Baake** (*University of Bielefeld*)

   **Polynomial iteration, symbolic dynamics, and long-range order**

   **Abstract:** It has long been known that concrete dynamical systems, such as the iteration of polynomial maps on the unit interval, reveal a substantial part of their structure via kneading theory and symbolic dynamics. In this talk, I plan to revisit this setting and discuss some of the emerging symbolic sequences with methods from ergodic theory, and spectral theory in particular. One example will be the period-doubling sequence, which is a Toeplitz sequence with pure point spectrum that can be understood in a systematic way via methods from the theory of aperiodic order.

2. **Kamil Bulinski** (*University of Sydney*)

   **Plünnecke inequalities for measure preserving actions**

   **Abstract:** Given a measure preserving action of a countable abelian group G on a probability space X, we estimate the the measure of subsets of the form AB where A is a subset of G and B is a subset of X with positive measure. These generalize the classical Plunnecke inequalities in Additive Combinatorics. We obtain, via a Furstenberg correspondence principle, new inequalities for the Banach density of sumsets in abelian groups, which extend those given by Jin.

   Joint work with Alexander Fish

3. **Grant Cairns** (*La Trobe University*)

   **Piecewise Linear Periodic Maps of the Plane with Integer Coefficients**

   **Abstract:** In 1993, Morton Brown published an interesting piecewise linear of the plane, of periodic 9, with integer coefficients, that has found applications in dynamics, topology and combinatorics. We give four maps, of order 5, 7, 8 and 12, which are very similar to Brown’s map in that they are piecewise linear in two pieces; the right half plane and the left half plane. We show that there are no other possible orders for such maps. We then show that every integer \( n > 1 \) appears as the period of a piecewise linear, periodic map of the plane with integer coefficients, provided we allow enough piecewise linear pieces. Finally, we explain how the collection of these maps can be classified in terms of rooted binary trees of fixed height.

   Joint work with Yuri Nikolayevsky and Gavin Rossiter

4. **Michael Coons** (*University of Newcastle*)

   **A Function Related to Mahler’s Method**

   **Abstract:** I will talk about the radial asymptotics of a Mahler function of degree two. If time permits, I will use the asymptotic information to give some algebraic independence results concerning this function.

   Joint work with Richard Brent and Wadim Zudilin

5. **Catherine Greenhill** (*University of New South Wales*)

   **The Switch Markov Chain for Sampling Irregular Graphs**

   **Abstract:** The problem of efficiently sampling from a set of (undirected) graphs with a given degree sequence has many applications. One approach to this problem uses a simple
Markov chain, which we call the switch chain, to perform the sampling. The switch chain is known to be rapidly mixing for regular degree sequences. We prove that the switch chain is rapidly mixing for any degree sequence with minimum degree at least 1 and with maximum degree \( d_{\text{max}} \) which satisfies \( 3 \leq d_{\text{max}} \leq \frac{1}{4} \sqrt{M} \), where \( M \) is the sum of the degrees. The mixing time bound obtained is only an order of \( n \) larger than that established in the regular case, where \( n \) is the number of vertices.

6. Christian Huck (University of Bielefeld)

Ergodic Properties of Visible Lattice Points and Related Systems

Abstract: Recently, the dynamical and spectral properties of square-free integers, visible lattice points and various generalisations have received increased attention. One reason is the connection with Sarnak’s conjecture on the ‘randomness’ of the Möbius function, another the explicit computability of correlation functions as well as eigenfunctions for these systems. We summarise some of the results, with focus on spectral and dynamical aspects, and expand a little on the implications for mathematical diffraction theory.

Joint work with Michael Baake

7. Shahrina Ismail (University of Queensland)

Perfect Triangles and Rational Points on Elliptic Curves

Abstract: The existence and parameterization of rational-sided triangles with additional conditions has a long history. For every triangle, we consider seven parameters which are the three sides, three medians and an area. A Perfect triangle is a triangle with all the 7 parameters are rational. To date, no one has found a Perfect triangle nor has anyone proved its nonexistence. Buchholz & Rathbun (1997) introduced a curve, \( C_4 \), whose points parameterize triangles with 6/7 parameters to be rational. We give a strategy for establishing that no such points will give a Perfect triangle.

Joint work with Victor Scharaschkin

8. Sawian Jaidee (Khon kaen University)

Mertens’ Theorem for Arithmetical Dynamical System

Abstract: We find an analogue of Mertens’ Theorem of analytic number theory for S-integer dynamical systems. We have found such a theorem for i) \( S \) is finite and ii)\( S \) is co-finite. Here I will talk about this theorem when \( S \) and its complement are infinite.

9. Peter Van Der Kamp (La Trobe University)

How Integrable Systems Give Rise to Recurrences Exhibiting the Laurent Phenomenon

Abstract: The talk consists of two parts:
1. Based on a recursive factorisation technique, we show how integrable difference equations give rise to recurrences which possess the Laurent property. We demonstrate that the same technique also provides a proof of the Laurent property. We illustrate this idea by deriving novel Somos 4 sequences whose coefficients are periodic functions in 4 variables with period 8, as well as Somos 5 sequences with periodic coefficients, in 5 variables, with period 7, and we prove they are Laurent phenomenon sequences.
2. Due to the Laurent property, starting from initial values 1 the terms of the Somos 4 & 5 sequences are integers, or polynomials if parameters are present. We show that each term \( \tau_n \) divides an arithmetic sequence of terms, \( \tau_m \), with \( m = n + d(n)N \), where the common difference \( d(n) \) is a polynomial function of \( n \).
10. **Alina Ostafe** *(University of New South Wales)*

*On Intersections of Orbits in Polynomial Dynamical Systems over Finite Fields*

**Abstract:** Motivated by results on intersections of orbits of D. Ghioca, T. J. Tucker, and M. E. Zieve in characteristic zero, we use recent explicit versions of Hilbert’s Nullstellensatz to answer several natural questions about reductions of orbits modulo a prime $p$ of polynomial dynamical systems defined over $\mathbb{Z}$. In particular, we show that under certain restrictions on the intersections of such orbits over $\mathbb{C}$ corresponding to two distinct polynomial systems, the intersections modulo $p$ are rare. Our approach is based on a new result about the reduction modulo prime numbers of systems of multivariate polynomials over the integers.

Joint work with Carlos D’Andrea, Igor E. Shparlinski and Martín Sombra

11. **David Pask** *(University of Wollongong)*

*Graphs, Dynamics and C*-algebras*

**Abstract:** I shall give a quick overview, for an non-specialist audience, of the results from the last 19 years on graph C*-algbras. I shall focus on these results from the point of view of dynamics, as many of the hypotheses of our results have a dynamical interpretation.

Joint work with Teresa Bates, Alex Kumjian, Iain Raeburn, Jean Renault

12. **Konrad Pilch** *(University of Adelaide)*

*Generalisations of Bost-Connes Systems and Hilbert’s 12th Problem*

**Abstract:** In 1995 Alain Connes and Benoit Bost created what is now known as a Bost-Connes system, a C*-algebra dynamical system with a significant number theoretic flavour, in their search for techniques that can be used against the Riemann Hypothesis. Turns out, the systems are not as heavy machinery as Alain Connes would like, nevertheless, Bost-Connes systems do have some very interesting properties and applications I will explain. In particular, I will aim to explain what improvements and generalisations have been wrought over the last twenty years and complete the talk with a discussion of my own work in generalising this fascinating dynamical system and the possibilities that I envision for the future of this field, including pushing the boundaries of Hilbert’s 12th Problem.

Joint work with Mathai Varghese

13. **Seiken Saito** *(Waseda University)*

*Mertens Theorem for Galois Coverings (in the number field case and the graph case)*

**Abstract:** The famous Mertens theorem is the asymptotic formula for the product of $1 - 1/p$ over all prime numbers $p$ which do not exceed a real number $x$. In 1974, K. S. Williams gave an generalization of the Mertens theorem for the primes which are congruent to $a$ modulo $q$. Here $a$ and $q$ are natural numbers with $(a, q) = 1$. By the class field theory, Williams theorem is the generalization of the Mertens theorem for an abelian Galois extension of the rational number field. We obtain generalizations of Mertens theorem for non-abelian Galois coverings in the following two cases: (1) We give a generalization of Mertens theorem for a Galois extension of a number field; (2) Prime cycles in a graph are analogues of prime numbers. For example, the Ihara-Selberg zeta-function for a graph is defined as the Euler product over all prime cycles in it. We also obtain an analogue of (1) for a Galois covering (i.e., a regular covering) of a finite graph. This talk is based on my joint work with Takehiro Hasegawa (Shiga Univ.) and Iwao Sato (Oyama Nat. Coll. of Tech.).

Joint work with Takehiro Hasegawa (Shiga University) and Iwao Sato (Oyama National College of Technology)
14. Min Sha (University of New South Wales)

On the Quantitative Dynamical Mordell-Lang Conjecture

Abstract: The dynamical Mordell-Lang conjecture concerns the structure of the intersection of an orbit in an algebraic dynamical system with an algebraic variety. In this talk, I will present some upper bounds on the size of this intersection for various cases when it is finite and the variety is a hypersurface. This continues the work of Silverman and Viray on the intersections of orbits and linear spaces.

Joint work with Alina Ostafe

15. Timothy Siu (UNSW)

Universal Period Distributions for Piecewise Cat Maps over Rational Lattices

Abstract: We look at specific reversible piecewise linear maps of the 2-torus on their invariant rational lattices and consider the distribution of their periodic orbits. It has been conjectured by Roberts and Vivaldi (2005, 2009) that the distribution follows the Gamma distribution. We will provide an alternative way of obtaining this distribution for specific reversible maps by breaking down the periodic orbits and looking at them on a smaller scale.

Joint work with John Roberts

16. George A. Willis (University of Newcastle)

Subgroups of totally disconnected groups that are ergodic for an automorphism

Abstract: The nub subgroup of an automorphism \( \alpha \) of a totally disconnected, locally compact group \( G \) is the largest subgroup on which \( \alpha \) acts ergodically. This subgroup, which is necessarily compact, has other, quite different, characterizations that give it significant role in the structure theory of t.d.l.c. groups. Its structure was described originally in work on topological dynamics by B. Kitchens and W. Schmidt but may also be given a more algebraic flavour.

Going beyond a single automorphism, there is a notion of flatness of a group of automorphisms of \( G \) and a corresponding nub subgroup for a flat group that is expected to play a similar role in the general theory as does the nub of a single automorphism. All finitely generated abelian or nilpotent groups of automorphisms are flat, as are polycyclic groups. Results in the topological dynamic literature on ergodic actions of such groups can also be expected to play a role.

17. Ana Zumalacárregui (UNSW)

The Least Common Multiple of Sets of Positive Integers

Abstract: It is well known that the classical Chebyshev’s function \( \psi(n) = \sum_{m<n} \Lambda(m) \) has an alternative expression in terms of the least common multiple of the first \( n \) integers: \( \psi(n) = \log \text{lcm}(1, 2, \ldots, n) \).

Here we generalize this function by considering, for a set \( \mathcal{A} \subseteq [1, n] \), the quantity \( \psi(\mathcal{A}) := \log \text{lcm}\{a : a \in \mathcal{A}\} \) and we ask ourselves about its asymptotic behavior.

We will focus on sets given by \( \mathcal{A}_f = \{f(1), f(2), \ldots, f(n)\} \) for some polynomial with integer coefficients and also discuss the case where the set is chosen at random in \([1, n]\) with prescribed size.

Joint work with J. Cilleruelo, J. Rué and P. Šarka