$AC(\sigma)$ operators, algebras of functions and algebras of operators

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What the talk is about

Structure theorems, integral representations, diagonalization of operators, functional calculus for operators, . . .

. . . for operators whose eigenvector expansions may only converge conditionally.

A new theory which removes many of the restrictions of the earlier work. But many open questions remain.

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On a Hilbert space

$$T^* T = T T^* \iff \|f(T)\| = \|f\|_\infty, \qquad f \in C(\sigma(T))$$

$$\iff T = \int_{\sigma(T)} \lambda \, \mathcal{E}(d\lambda)$$

for some countably additive spectral measure $\mathcal{E}$.

If T is also compact then $T = \sum \lambda_j E_j$ and the sum converges **unconditionally** (that is, in any order) in norm.

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Conditionally convergent expansions

Many expansions of this form which correspond to normal operators on L^2 only converge conditionally on other L^p spaces (eg Fourier series).

D.R. Smart/J.R. Ringrose (1960s): theory of **well-bounded** operators on Banach spaces, based on a weaker type of spectral integral, and a functional calculus for the **absolutely continuous** functions on an interval $[a, b] \subset \mathbb{R}$:

$$\|f(T)\| \leq K \|f\|_{BV} := K(|f(a)| + \operatorname*{var}_{[a,b]} f)$$

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Limitations of the theory

1. functional calculus based on an interval containing $\sigma(T)$ rather than $\sigma(T)$ itself $\rightsquigarrow$ define $AC(\sigma(T)) \subset BV(\sigma(T))$.
2. only covers case where $\sigma(T) \subset \mathbb{R}$.
3. theory is rather complicated if X is not reflexive.

For compact operators, one still gets a representation of the form

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Berkson and Gillespie (1980s): defined $AC(\mathbb{T})$, giving a theory of **trigonometrically well-bounded operators** and $AC(R)$ for a rectangle $R = \{x + iy \,:\, x \in [a, b],\ y \in [c, d]\} \subset \mathbb{C}$ giving a theory of ***AC*-operators**.

Theorem

Let X be reflexive. Then

1. *T is trigonometrically well-bounded if and only if $T = e^{iA}$ for some well-bounded operator A.*
2. *T is an AC-operator if and only if $T = A + iB$ for a pair of commuting well-bounded operators A, B.*

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Does the analogy work?

self-adjoint	$\leftrightsquigarrow$	well-bounded
unitary	$\leftrightsquigarrow$	trigonometrically well-bounded
normal	$\leftrightsquigarrow$	AC-operator

Problems:

1. The functional calculus still isn't based on functions defined on $\sigma(T)$.
2. The class of AC-operators is not stable under affine transformations. There are examples where T is an AC-operator, but $(1+i)T$ is not.
   The functional calculus and integral representation theory for $\alpha T + \beta I$ **should** be the same as those for T!

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The challenge

Question

Let $\sigma \subset \mathbb{C}$ be compact. Is it possible to define an algebra $BV(\sigma)$ of functions of 'bounded variation on σ' in such a way that

1. *it should agree with the 'usual definition' if $\sigma \subset \mathbb{R}$;*
2. *it should contain all sufficiently well-behaved functions (polynomials, C^∞ functions, characteristic functions of polygons and so forth);*
3. *for all $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, we should have $BV(\alpha\sigma + \beta) \cong BV(\sigma)$.*

Outline of the definition

Suppose $\emptyset \neq \sigma \subset \mathbb{C}$ and that $f : \sigma \to \mathbb{C}$.

Let $\gamma = \{\gamma(t)\}$ denote a 'nice' curve in $\mathbb{C}$. Define

- Wiggliness of the curve:

$$\mathrm{vf}(\gamma) = \text{max number of times any line crosses } \gamma.$$

- Variation of the function along γ in σ:

$$\mathrm{cvar}(f, \gamma, \sigma) = \sup\{\sum_j |f(\gamma(t_j)) - f(\gamma(t_{j-1}))|\}$$

  taken over increasing t_j with $\gamma(t_j) \in \sigma$.

- Variation of f over σ:

$$\mathrm{var}(f, \sigma) = \sup_\gamma \frac{\mathrm{cvar}(f, \gamma, \sigma)}{\mathrm{vf}(\gamma)}$$

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Functions of bounded variation

Define the set of functions of bounded variation over σ as

$$BV(\sigma) = \{f : \sigma \to \mathbb{C} \,:\, \|f\|_{BV} := \sup_{z \in \sigma} |f(z)| + \mathrm{var}(f, \sigma) < \infty\}.$$

It is true, although not obvious, that

- $BV(\sigma)$ is always a Banach algebra (under pointwise operations);
- our $BV([0,1])$ agrees with the usual one;
- our $BV(\mathbb{T})$ agrees with the natural one considered by Berkson and Gillespie.

It is true (and obvious) that the definition has the affine invariance property required.

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Say that p is a **polynomial in two variables** on $\mathbb{C}$ if it is of the form

$$p(x+iy) = \sum_{n,m=0}^{N} c_{n,m} x^n y^m, \qquad c_{n,m} \in \mathbb{C}.$$

Let $\mathcal{P}$ denote the set of all such polynomials.

Define $AC(\sigma)$, the set of **absolutely continuous functions** on σ, to be the norm closure of $\mathcal{P}$ in $BV(\sigma)$.

Theorem

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$AC(\sigma)$ operators

$T \in B(X)$ is an $AC(\sigma)$ **operator** if it has an $AC(\sigma)$ functional calculus. That is, if there exists a bounded algebra homomorphism $\Psi : AC(\sigma) \to B(X)$ such that $\Psi(z \mapsto z^k) = T^k$ for $k = 0, 1, 2, \dots$.

Theorem

Suppose that $\sigma \subset \mathbb{R}$. Then

$$T \text{ is an } AC(\sigma) \text{ operator} \iff T \text{ is an } AC(\sigma(T)) \text{ operator} \iff T \text{ is a well-bounded operator.}$$

T is an $AC(\mathbb{T})$ operator $\iff$ T is trigonometrically well-bounded.

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More generally

The new definition is restrictive than Berkson and Gillespie's.

$$T \text{ is an } AC(\sigma) \text{ operator} \quad \underset{\not\Leftarrow}{\Rightarrow} \quad T = A + iB$$

with A, B commuting well-bounded operators.

Clearly if T is an $AC(\sigma)$ operator then $\alpha T + \beta$ is an $AC(\alpha\sigma + \beta)$ operator.

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Integral representations?

If X is reflexive, and T is an $AC(\sigma)$ operator, then there exists a spectral projection $E(H)$ for every closed half-plane $H \subset \mathbb{C}$. Since it is possible to reconstruct T from $\{E(H)\}$, one may formally write

$$T = \int_\sigma z\, dE.$$

There are however many open problems remaining:

- What properties on a family $\{E(H)\}$ guarantee that $T = \int_\sigma z\, dE$ is an $AC(\sigma)$ operator?
- Is there a good integration theory? For which functions f can one make sense of $\int_\sigma f(\lambda)\, dE(\lambda)$?
- Does every $AC(\sigma)$ functional calculus extend to a $BV(\sigma)$ functional calculus?

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Other questions

1. If T admits an $AC(\sigma)$ functional calculus for some σ, does it admit an $AC(\sigma(T))$ functional calculus?
2. Is $\{f(T) : f \in BV(\sigma)\}$ the SOT closure of $\{f(T) : f \in AC(\sigma)\}$?
3. What relationships exist between σ, and the algebras $AC(\sigma)$ and $BV(\sigma)$. If $AC(\sigma) \cong AC(\sigma')$ what can be said about σ and σ'?

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References

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2. Ashton, B., and Doust, I., A comparison of algebras of functions of bounded variation, *Proc. Edin. Math. Soc.*, **49** (2006), 575-591.

(available from `www.maths.unsw.edu.au/~iand`)