1 Basic Revision

1.1 Derivatives and integrals

Suppose \( f : \mathbb{C} \rightarrow \mathbb{C} \). Then \( f \) is differentiable at \( z_0 \in \mathbb{C} \) if

\[
f'(z_0) := \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}
\]

exists, that is if the limit exists.

If \( f'(z_0) \) exists and \( w = f(z) = u(x, y) + iv(x, y) \) then the equations \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \) hold at \( (x_0, y_0) \). These equations are called the Cauchy–Riemann equations.

It turns out to be much more useful to study functions that are differentiable not just at a single point but on open sets. We say that \( f \) is analytic at \( z_0 \) if there is an \( r > 0 \) such that \( f'(z) \) exists whenever \( |z - z_0| < r \).

A central tool of complex analysis is the contour integral. Recall that a curve \( C \) in the complex plane is a contour if \( C \) can be parametrized by a piecewise smooth map \( z : [a, b] \rightarrow \mathbb{C} \). That is \( z \) has a continuous nonzero derivative except at possibly a finite number of points in \( [a, b] \).

If \( f \) is a complex-valued function defined on \( \mathbb{C} \), then the contour integral of \( f \) on \( C \) is defined to be

\[
\int_C f(z) \, dz := \int_a^b f(z(t)) \, z'(t) \, dt.
\]

The value of the contour integral is dependent of the orientation of the curve, but is otherwise independent of the parametrization.

A closed contour is one for which \( z(a) = z(b) \). A simple closed contour is a closed contour which does not intersect itself. The Jordan curve theorem says that such curves have a well-defined inside.
1.2 The main results

1.2.1 Cauchy’s theorem, aka the Cauchy–Goursat Theorem

Theorem 1. If $f$ is analytic on a simply-connected region $\Omega$ then

$$\int_{\gamma} f(z) \, dz = 0$$

for all closed $\gamma$ in $\Omega$.

Recall: a region is an open, connected subset of $\mathbb{C}$. Simply-connected means that the inside of any simple closed curve in $\Omega$ contains only points of $\Omega$.

This result is deep and is the central result in the subject. It is easy to show that on a region $\Omega$ the condition $\int_{\gamma} f(z) \, dz = 0$ for all closed $\gamma$ in $\Omega$ is equivalent to $f$ having a primitive $F$ on $\Omega$, that is, to the existence of a function $F$ such that $F' = f$ on $\Omega$. Cauchy’s theorem is saying that on a simply-connected region the analyticity of $f$ implies that $f$ has a primitive.

(Note also Morera’s Theorem: if $\int_{\gamma} f(z) \, dz = 0$ for all closed $\gamma$ in $\Omega$, then $f$ is analytic. This follows easily once we know that if $F$ is differentiable on $\Omega$, then it is infinitely differentiable, so in particular, $f$ is differentiable too!)

1.2.2 Cauchy’s integral formula

Theorem 2. Suppose that $f$ is analytic on and inside a simple closed contour $\gamma$ which is positively oriented. Then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} \, dz$$

for any point $z_0$ in the inside of $\gamma$.

Recall: positively oriented means that the the parametrization is chosen so that inside of the curve is on left as you go around the courve. That is it is parametrized in an anticlockwise direction.

This is a relatively straightforward consequence of Cauchy’s theorem. From the integral formula it follows that $f$ is infinitely differentiable at $z_0$ and that

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{k+1}} \, dz, \quad k = 0, 1, 2, \ldots \quad (1)$$

These are called Cauchy’s formulae for the derivatives.
1.2.3 Liouville’s theorem.

**Theorem 3.** If $f$ is entire and bounded then $f$ is a constant function.

**Recall:** *Entire* means that $f$ is analytic at every point in $\mathbb{C}$. *Bounded* means that there exists a constant $M$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

This is a simple application of (1) with $k = 1$. Fix any $z_0 \in \mathbb{C}$. Choose a really big circle $C$ centred at $z_0$ of radius $R$. Then standard integral inequalities show

$$|f'(z_0)| \leq \frac{1}{2\pi} \int_C \frac{|f(z)|}{|z - z_0|^2} |dz| \leq \frac{2\pi RM}{R^2},$$

and so $f'(z_0) = 0$. The fundamental theorem of calculus

$$f(z_1) - f(z_2) = \int_{z_1}^{z_2} f'(z) \, dz = 0$$

now completes the proof.

1.2.4 Taylor series expansion.

A very important tool in complex analysis is the theory of *power series*. A power series (around $z_0$) is a function of the form

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots = \sum_{k=0}^{\infty} a_k(z - z_0)^k. \quad (2)$$

Every power series has a radius of convergence $R$ such that the infinite sum in (2) converges if $|z - z_0| < R$ and diverges if $|z - z_0| > R$. Such a series always converges uniformly on any closed disk inside the radius of convergence which allows one to show that you can differentiate a power series term by term:

$$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots, \quad |z - z_0| < R.$$ 

In particular, a power series is always analytic on this disk.

Conversely, it follows from Cauchy’s integral formula and formulae for the derivatives that if $f$ is analytic on $|z - z_0| < R$, then

$$f(z) = f(z_0) + \frac{1}{1!} f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

whenever $|z - z_0| < R$. This is called the *Taylor series expansion* of $f$. (The proof also requires the high school result that $1/(1 - w) = 1 + w + w^2 + w^3 + \cdots$ provided $|w| < 1$.)

Thus, analytic functions are precisely those defined by power series expansions on *disks* of positive radii. (There is no such unifying result for real analytic functions.)
1.2.5 Laurent’s theorem.

The function \( f(z) = (1 - z)^{-1} \) is analytic on the open unit disk with Taylor series
\[
f(z) = 1 + z + z^2 + z^3 + \cdots, \quad |z| < 1
\]
around 0. This series obviously does not converge for \( |z| > 1 \), but here we can still write \( f(z) \) in terms of the powers of \( z \), as long as we allow negative powers. That is
\[
\frac{1}{1-z} = \left( \frac{-1}{z} \right) \left( \frac{1}{1-\frac{1}{z}} \right) = -z^{-1} \left( 1 + z^{-1} + z^{-2} + \cdots \right) = -z^{-1} - z^{-2} - z^{-3} - \cdots
\]
for \( |z| > 1 \).

If we allow both positive and negative powers we can get expansions of this type on any annulus on which a function is analytic.

**Theorem 4.** Let \( f \) be analytic on an annulus \( r < |z - z_0| < R \) (where possibly \( r = 0 \) or \( R = \infty \)). Then on this annulus \( f \) can admits a ‘Laurent series expansion’
\[
f(z) = \sum_{k \in \mathbb{Z}} c_k(z - z_0)^k = \ldots + c_{-2}(z - z_0)^{-2} + c_{-1}(z - z_0)^{-1} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots
\]

In this expression we cannot write the coefficients in terms of derivatives of \( f \) at \( z_0 \) (after all the derivatives may not exist at \( z_0 \).) Nevertheless we can express them in terms of integrals:
\[
c_k = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z - z_0)^{k+1}} \, dz, \quad r < \rho < R, \quad \text{for } k \in \mathbb{Z}.
\]

In the case where \( r = 0 \) (so that the Laurent expansion is valid on \( 0 < |z - z_0| < R \), that is, on a punctured disk) the coefficient \( c_{-1} \) is called the residue of \( f \) at \( z_0 \). Note that substituting \( k = -1 \) into the previous formula shows that if \( 0 < \rho < R \), then
\[
c_{-1} = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} f(z) \, dz
\]
or, if you are actually interested in the integral of \( f \) around the circle \( |z - z_0| = \rho,0 \)
\[
\int_{|z-z_0|=\rho} f(z) \, dz = 2\pi i \cdot c_{-1}.
\]

Recall that if the Laurent expansion has only finitely many terms with negative powers, then it is not too hard to calculate the residue.
Example: Suppose that
\[
 f(z) = \frac{c_{-2}}{(z - z_0)^2} + \frac{c_{-1}}{(z - z_0)^{-1}} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots.
\]
Then
\[
 (z - z_0)^2 f(z) = c_{-2} + c_{-1}(z - z_0) + c_0(z - z_0)^2 + \ldots
\]
and so
\[
 \frac{d}{dz} \left( (z - z_0)^2 f(z) \right) = c_{-1} + 2c_0(z - z_0) + 3c_1(z - z_0)^2 + \ldots.
\]
Heuristically you just now want to plug in \( z = z_0 \). You can’t do this as the left-hand-side is generally not defined there, but you can take limits:
\[
 c_{-1} = \lim_{z \to z_0} \frac{d}{dz} \left( (z - z_0)^2 f(z) \right)
\]
If \( f \) has more terms with negative powers you would have to multiply by a higher power and take more derivatives.

It is worth noting that the formula for the residue is easy to deduce for functions with a finite Laurent series
\[
 f(z) = \sum_{k=-N}^{N} c_k (z - z_0)^k.
\]
Let \( C_\rho \) denote the circle of radius \( \rho \) centred at \( z_0 \). The fact that \( (z - z_0)^k \) has a primitive if \( k \neq -1 \) implies that \( \int_{C_\rho} (z - z_0)^k \, dz = 0 \). The integral \( \int_{C_\rho} (z - z_0)^{-1} \, dz = 2\pi i \) can easily be done directly from the definition. Thus, in this case
\[
 \int_{C_\rho} f(z) \, dz = \sum_{k=-N}^{N} c_k \int_{C_\rho} (z - z_0)^k \, dz = 2\pi i c_{-1}.
\]
The general case depends on showing the the Laurent series converges uniformly on \( C_\rho \), and so we can swap the integral with the infinite sum.

1.2.6 Homotopy invariance of integrals

One important consequence of Cauchy-Goursat is that one can deform contours without changing the value of corresponding contour integral. That is, if \( \gamma_1 \) and \( \gamma_2 \) are closed contours in the plane, and you can deform \( \gamma_1 \) to \( \gamma_2 \) without passing over any point where the function \( f \) fails to be analytic, then
\[
 \int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz.
\]
More formally, let $\Omega$ denote an open subset of $\mathbb{C}$, and let

$$C(\mathbb{T}, \Omega) = \{ \gamma : \mathbb{T} \to \Omega \mid \gamma \text{ is a contour} \} \subseteq C(\mathbb{T}).$$

The set $C(\mathbb{T}, \Omega)$ becomes a metric space with

$$d(\gamma_1, \gamma_2) = \sup_{z \in \mathbb{T}} |\gamma_1(z) - \gamma_2(z)|.$$

A closed contour is then an element of the set $C(\mathbb{T}, \Omega)$. We say that $\gamma_1$ and $\gamma_2$ are homotopic if there exists a continuous map $\Phi : [0, 1] \to C(\mathbb{T}, \Omega)$ such that $\Phi(0) = \gamma_1$ and $\Phi(1) = \gamma_2$.

Thus, if $f$ is analytic on $\Omega$ and $\gamma_1$ and $\gamma_2$ are homotopic, then $\int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz$.

### 1.2.7 Winding numbers.

The winding number of a closed curve in $\mathbb{C}$ around a given point $z_0$ is an integer representing the total number of times that curve travels anticlockwise around the point.

If $\gamma$ is a closed contour, then the **winding number** or **index** $I(\gamma, z_0)$ of $\gamma$ around $z_0$ can be defined by

$$I(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}. \quad (3)$$

(It is of course a theorem that this is always an integer!)

More generally, any closed curve (or more precisely its parametrization) can be thought of as an element of $C(\mathbb{T})$, the continuous functions from the unit circle to $\mathbb{C}$. One can show that if $z_0$ does not lie on $\gamma$ it is possible to write

$$\gamma(e^{it}) = z_0 + r(t)e^{i\theta(t)}, \quad t \in [0, 2\pi]$$

with $r(t) > 0$ and $\theta(t) \in \mathbb{R}$ a **continuous** real-valued function on $[0, 2\pi]$. Since $\gamma(e^{0i}) = \gamma(e^{2\pi i})$ this implies that $\theta(2\pi) - \theta(0)$ must be a multiple of $2\pi$, and we can set

$$I(\gamma, z_0) = \frac{\theta(2\pi) - \theta(0)}{2\pi}.$$

Again, there is work to be done to check that this all makes sense (and agrees with the formula (3)).

**Warning:** the image of $\gamma \in C(\mathbb{T})$ in the complex plane might not look like a nice curve!
1.2.8 The Residue Theorem.

**Theorem 5.** Suppose $f$ is defined and is analytic on some region $\Omega$ except for a finite number of points $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\Omega$; and suppose that $\gamma$ is a closed contour in $\Omega$ which does not wind about any points in the complement of $\Omega$. Let $\text{Res}(f, \alpha_k)$ be the residue of $f$ at $\alpha_k$. Then

$$\int_{\gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{n} I(\gamma, \alpha_k) \cdot \text{Res}(f, \alpha_k)$$

A special case occurs if $\gamma$ is a positively oriented *simple* closed contour then

$$\int_{\gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{m} \text{Res}(f, \alpha_k) \quad (= 2\pi i \times \text{the sum of the residues})$$

where the singularities $\alpha_1, \alpha_2, \ldots, \alpha_m$ are those inside $\gamma$. Of course, we always only consider a $\gamma$ with no singularity of $f$ on $\gamma$.

1.2.9 Zeros, poles and singularities

Suppose first that $f$ is analytic at $z_0$ and that $f(z_0) = 0$. Then the Taylor series for $f$ about $z_0$ looks like

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots = (z - z_0)^m (a_m + a_{m+1}(z - z_0) + \cdots)$$

where $m$ is chosen so that $a_m \neq 0$. That is $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$. In this case $z_0$ is called a zero of order, or multiplicity, $m$.

Suppose that $f : D \subseteq \mathbb{C} \to \mathbb{C}$. Let $\Omega_f \subset D$ denote the set of points on which $f$ is analytic. Then $\Omega_f$ is obviously an open subset of $\mathbb{C}$. A *singular point*, or *singularity*, of $f$ is
a point in the boundary of $\Omega_f$. Thus if $z_0$ is a singular point of $f$, then $f$ is not analytic at $z_0$, but it is analytic at some point in every disk around $z_0$.

If $z_0$ is a singular point $z_0$ but $f$ is analytic on some punctured disk $0 < |z - z_0| < \epsilon$, then we say that $z_0$ is an isolated singularity. Thus $z_0 = 0$ is an isolated singularity of cosec$z$, but not of Log$z$. In particular $f$ has a convergent Laurent series in a punctured disk around any isolated singularity. Depending on the nature of this series we can classify such singularities further:

- If the coefficients of all the negative powers of $(z - z_0)$ are zero then $z_0$ is a removable singularity. An example would be $z_0 = 0$ for the function $f(z) = \frac{\sin z}{z}$. One could redefine $f$ at $z_0$ to make $f$ analytic at this point.

- If only finitely many of the coefficients of the negative powers of $(z - z_0)$ are nonzero, i.e.

  \[
  f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \cdots + \frac{b_1}{z - z_0} + \sum_{k=0}^{\infty} a_k(z - z_0)^k, \quad 0 < |z - z_0| < \epsilon
  \]

  with $b_m \neq 0$, then we say that $z_0$ is a pole of order, or multiplicity, $m$.

- If infinitely many of the coefficients of the negative powers of $(z - z_0)$ are nonzero then we say that $z_0$ is an essential singularity. An example would be

  \[
  f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k} = e^{1/z}.
  \]

Zeros and poles are of course closely related: if $f$ has a zero of order $m$ at $z_0$, then $\frac{1}{f}$ has a pole of order $m$ at $z_0$. 