1. The Grothendieck Spectral Sequence

Let \( G : A \rightarrow B \) and \( F : B \rightarrow C \) be left exact functors between abelian categories with enough injectives. With some hypothesis on \( F \) and \( G \), there is a Grothendieck spectral sequence relating the right derived functors of \( F, G \) and \( F \circ G \). The statement is as follows:

**Theorem 1.** Suppose that \( G \) sends \( A \)-injectives to \( F \)-acyclics (that is, \( R^pF(GI) = 0 \) for all \( A \)-injectives \( I \) and \( p > 0 \)). Then there is a stage 2 spectral sequence

\[
E_2^{p,q} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(F \circ G)(A)
\]

The proof of this follows readily from the spectral sequences associated to a double complex. The main technical tool is the so-called fully injective resolution (or Cartan-Eilenberg resolution) of a complex.

Let \( C^\bullet \) be a complex, and \( 0 \rightarrow C^\bullet \rightarrow R^\bullet \) be an injective resolution, that is

\[
\begin{array}{cccc}
0 & \rightarrow & R^0 & \rightarrow & R^1 & \rightarrow & R^2 & \rightarrow \\
& \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & R^0,0 & \rightarrow & R^1,1 & \rightarrow & R^2,1 & \rightarrow \\
& \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & C^0 & \rightarrow & C^1 & \rightarrow & C^2 & \rightarrow \\
& \uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & & \\
\end{array}
\]

such that each “column” is an injective resolution of \( C^i \), and each row is a complex. A fully injective resolution is a resolution satisfying the following properties:

\[
\begin{align*}
0 & \rightarrow Z^p(C^\bullet) \rightarrow Z^p(J^\bullet,0) \rightarrow Z^0(J^\bullet,1) \rightarrow \ldots \\
0 & \rightarrow B^p(C^\bullet) \rightarrow B^p(J^\bullet,0) \rightarrow B^0(J^\bullet,1) \rightarrow \ldots \\
0 & \rightarrow H^p(C^\bullet) \rightarrow H^p(J^\bullet,0) \rightarrow H^0(J^\bullet,1) \rightarrow \ldots
\end{align*}
\]

are injective resolutions of \( Z^p(C^\bullet), B^p(C^\bullet) \) and \( H^p(C^\bullet) \) resp., where

\[
\begin{align*}
Z^p(J^\bullet,p) &= \ker d^p, \\
B^p(J^\bullet,p) &= \text{im } d^{p-1}, \\
H^p(J^\bullet,p) &= Z^p(J^\bullet,p)/B^p(J^\bullet,p)
\end{align*}
\]

Fully injective resolutions exist (c.f. [Lang]).

**Proof.** (of theorem 1.) Let \( X \in \text{ob}(A) \), and \( 0 \rightarrow X \rightarrow I^\bullet \) an injective resolution. Consider the complex \( GI^\bullet \) in \( B \) and let \( J^\bullet \) be a fully injective resolution of \( GI^\bullet \). Consider the spectral sequences \( E_r \) and \( E_r' \) associated to the double complex \( FJ^\bullet \).

We have

\[
\begin{align*}
E_0^{p,q} &= F(J^p,q) \\
E_1^{p,q} &= (R^qF)(GI^p) \\
&= \begin{cases} 
(F \circ G)(I^p) & \text{if } q = 0 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]
since $G$ sends injectives to $F$-acyclics. Hence $E_{2,0}^p = H^p((F \circ G)(I^p)) = R^p(F \circ G)(X) \simeq E_{2,0}^p \simeq H^p(Tot J)$. It suffices to show $E_2^{p,q} \simeq (R^p F)(R^q G)(X)$ to finish the proof. We have

$$E_{0}^{p,q} = FJ^q,p$$

$$E_{1}^{p,q} \simeq H^q(FJ^q,p)$$

$$= \frac{\ker(FJ^q,p \rightarrow FJ^{q+1,p})}{\text{im}(FJ^{q-1,p} \rightarrow FJ^q,p)}$$

$$\simeq \frac{F(Z^q(J^{\bullet,p}))}{F(B^q(J^{\bullet,p}))}$$

$$\simeq F(H^q(J^{\bullet,p}))$$

The last few lines follow from the fact that $Z^q(J^{\bullet,p}), B^q(J^{\bullet,p}), H^q(J^{\bullet,p})$ are injective for all $p, q$. So

$$E_2^{p,q} \simeq H^p(0 \rightarrow FH^q(J^{\bullet,0}) \rightarrow FH^q(J^{\bullet,1}) \rightarrow \ldots)$$

$$\simeq R^p F(H^q(GI^{\bullet}))$$

$$\simeq (R^p F)(R^q G)(X)$$

$\square$