1. Spectral sequences

A spectral sequence consists of the following data

(1) bigraded objects $E_r = \bigoplus_{p,q\in \mathbb{Z}} E_{p,q}^r$ for $r \geq 0$.

(2) differentials $d : E_r \rightarrow E_r$ such that

(a) $d_r(E_{p,q}^r) \subset E_{p+r,q-r+1}^{r+1}$ and

(b) $E_{r+1} = H(E_r)$ that is

$$E_{r+1}^{p,q} = \text{cohomology at } E_r^{p,q} \text{ w.r.t. the differentials } d_r$$

$$= \ker(d_r : E_{p,q}^r \rightarrow E_{p+1,q-r+1}^{r+1})$$

$$= \operatorname{im}(d_r : E_{p-r,q+r-2}^{r-2} \rightarrow E_{p,q}^r)$$

In applications, we often have $E_{p,q}^r = 0$ unless both $p$ and $q$ are nonnegative. In that case, we say that $\{E_r\}$ is a first quadrant spectral sequence. We will only consider first quadrant spectral sequences.

For each pair of nonnegative integers $p, q$, there exists an integer $r$ such that

$$E_{p,q}^r = E_{p,q}^{r+1} = \ldots$$

We say that the $(p,q)$-th term stabilises at stage $r$, and denote

$$E_{p,q}^\infty := E_{p,q}^r.$$

Let $H^s$, for $s \geq 0$ be a collection of objects equipped with finite filtrations $0 \subset F_1 H^s \subset \ldots \subset F_t H^s = H^s$. The spectral sequence $E_r$ abuts to $H^s$ if there are isomorphisms

$$E_{p,q}^\infty = \frac{F_p H^{p+q}}{F_{p+1} H^{p+q}}$$

This is denoted

$$E_{p,q}^r \Rightarrow H^{p+q}$$

1.1. Spectral sequence associated to a double complex. Let $M = M^{p,q}$ be a double complex, with vertical differentials $D$ and horizontal differentials $d$. We can define a complex $\operatorname{Tot} M$ associated to $M$ by

$$\operatorname{Tot}^n M = \bigoplus_{p+q=n} M^{p,q}$$

with differential $D + d$. Note that $(D + d)^2 = D^2 + d^2 + Dd + DD = 0$ so $D + d$ is a differential.

**Theorem 1.** There are spectral sequences $E_{0}^{p,q} = M^{p,q}$ and $E_{0}^{q,p} = M^{q,p}$ both abutting to $H^{p+q}(\operatorname{Tot} M)$.

To illustrate how we use this spectral sequence, we will prove the following result.

**Proposition 1.** Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be homomorphisms in an abelian category. Then there is an exact sequence

$$0 \rightarrow \ker(f) \rightarrow \ker(g \circ f) \rightarrow \ker(g) \rightarrow \operatorname{coker}(f) \rightarrow \operatorname{coker}(g \circ f) \rightarrow \operatorname{coker}(g) \rightarrow 0.$$

**Proof.** Consider the spectral sequence associated to the double complex

$$0 \rightarrow C \rightarrow C \rightarrow 0$$

$$\downarrow g \circ f \downarrow 1_g \uparrow$$

$$A \xrightarrow{f} B \xrightarrow{g} \operatorname{coker}(f) \rightarrow 0$$
We will use the following numbering convention,

\[
E_r^{0,2} 
\vdots 
E_r^{0,1} \quad E_r^{1,1} 
E_r^{0,0} \quad E_r^{1,0} \quad E_r^{2,0}
\]

with the \((0, 0)\)-th term at the bottom left corner, and zeroes at all unlabelled entries. Then

\[
E_1^{p,q} = \coker(g \circ f) \xrightarrow{b} \coker(g) \quad \ker(g) \xrightarrow{c} \coker(f),
\]

\[
E_2^{p,q} = \ker(b) \quad \coker(b) 
\ker(a) \quad \frac{\ker(c)}{\im(a)} \quad \coker(f)
\]

The yellow terms stabilise, and denote \(d : \ker(b) \longrightarrow \coker(f)\). Continuing, we have

\[
E_3^{p,q} = \ker(d) \quad \coker(b) 
\ker(a) \quad \frac{\ker(c)}{\im(a)} \quad \coker(d)
\]

and the spectral sequence stabilises at page 3. Let \(H^r\) denote the total \(r\)-th homology of the double complex. By taking the spectral sequence the other way we get \(H^0 = \ker(f)\) and \(H^i = 0\) for all \(i > 0\).

Since both spectral sequences abut to the total cohomology \(H^\bullet\), we can say the following.

1. The only nonzero \(E_3^{p,q}\) term is \(E_3^{0,0} = \ker(a)\). This is equal to \(H^0 = \ker(f)\).
2. Vanishing of \(E_3^{1,0} = \ker(c)/\im(a)\), together with 1., means the following sequence is exact

\[
0 \longrightarrow \ker(f) \longrightarrow \ker(g \circ f) \xrightarrow{a} \ker(g) \xrightarrow{c} \coker(f)
\]

3. Vanishing of \(E_3^{0,1} = \coker(d)\) and \(E_3^{1,0} = \ker(d)\) means \(d : \ker(b) \longrightarrow \coker(f)\) is an isomorphism, so we can extend the exact sequence of 2. to

\[
0 \longrightarrow \ker(f) \longrightarrow \ker(g \circ f) \xrightarrow{a} \ker(g) \xrightarrow{c} \coker(f) \longrightarrow \coker(g \circ f) \longrightarrow \coker(g) \longrightarrow \coker(b)
\]

4. Vanishing of \(E_3^{1,1} = \coker(b)\) means the last term in the above sequence is zero, thus proving the result.

\(\square\)