THE BRAUER GROUP

KENNETH CHAN

0.1. Number theory. Let $X$ be a $\mathbb{Q}$-variety, the Hasse principle says that

$$X(\mathbb{Q}) \neq \emptyset \iff X(\mathbb{R}) \neq \emptyset \text{ and } X(\mathbb{Q}_p) \neq 0 \text{ for all primes } p$$

This doesn’t work all the time, e.g. when $X$ is a smooth cubic hypersurface in $\mathbb{P}^3$ (see [2]). Manin came up with a way to explain the failure of the Hasse principle, based on the Brauer group. This became known as the Brauer-Manin obstruction.

0.2. The Brauer group of a field. Let $k$ be a field, $A$ be a central simple $k$-algebra, that is, $A$ is a finite $k$-vector space such that its centre is $k$ and it has no nontrivial two sided ideals. (Elementary examples are matrix algebras and generalised quaternion algebras.) The Wedderburn-Artin theorem says the following: every central simple $k$-algebra $A$ is of the form $A \cong M_n(D)$ where $D$ is some central division $k$-algebra. We say that $A \cong M_n(D)$ and $A' \cong M_m(D')$ are equivalent if $D \cong D'$, and let $B(k)$ denote the set of equivalence classes of central simple $k$-algebras. To put a group structure on $B(k)$, we have

(1) If $A$ and $B$ are both central simple $k$-algebras, then so is $A \otimes B$.

(2) Let $A^\circ$ denote the opposite algebra of $A$, then $A \otimes k A^\circ \cong M_n(k)$.

Define $[A] \cdot [A'] = [A \otimes A']$ This is well defined since if $A = M_n(D)$ and $A' = M_m(D')$ then $A \otimes A' = M_{mn}(D \otimes D') \sim D \otimes D'$. The identity element is $[k] = [M_n(k)]$ and $[A]^{-1} = [A^\circ]$. This makes $B(k)$ into a group, which we call the Brauer group of $k$.

0.3. Functoriality. The assignment of a field to its Brauer group is (covariantly) functorial, let $i : k \rightarrow K$ be an inclusion of fields, there is a map $B(i) : B(k) \rightarrow B(K)$ given by $A \mapsto A \otimes_k K = A_K$. The relative Brauer group $B(k/K)$ is defined by

$$0 \rightarrow B(k/K) \rightarrow B(k) \rightarrow B(K)$$

If $A_K \cong M_n(K)$ we say that $A$ is split by $K$; so $B(k/K)$ consists of the equivalence classes $[A] \in B(k)$ where $A$ is split by $K$.

Example 0.1. Examples where the Brauer group is small.

- (Wedderburn) $B(\mathbb{F}_n) = 0$, $B(K) = 0$ for any $K$ which is an algebraic extension of a finite field.
- (Frobenius) $B(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ (representatives: $\mathbb{R}, \mathbb{H}$)
- $B(\mathbb{C}) = 0$, $B(k) = 0$ for any (separably) closed field $k$.
- (Tsen-Lang) $B(\mathbb{C}[x]) = 0$, $B(k(X)) = 0$ for any curve $X$ defined over an algebraically closed field.
These examples are atypical. In general, Brauer groups can be quite large

- (Class field theory)

\[ 0 \longrightarrow \text{Br}(\mathbb{Q}) \longrightarrow \bigoplus_p \text{Br}(\mathbb{Q}_p) \overset{\Sigma}{\longrightarrow} \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \]

Note that \( B(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z} \) for finite \( p \). There is a similar exact sequence for number fields.

0.4. Splitting fields. Every division \( k \)-algebra \( D \) can be split by a finite Galois extension \( K/k \). The degree of \( D \) divides \([K:k]\). Hence

\[ B(k) = \lim_{\kappa} B(K/k) \]

where the limit is taken over all finite Galois extensions of \( k \). This limit is actually a union.

0.5. Cohomology. Let \( K \) be a Galois extension of \( k \), and \( G = \text{Gal}(K/k) \). The cohomology set \( H^1(G, \text{PGL}_n(K)) \) classifies central simple \( k \)-algebras of degree \( n \) which are \( K \)-split up to isomorphism. The exact sequence of groups

\[ 1 \longrightarrow K^* \longrightarrow GL_n(K) \longrightarrow \text{PGL}_n(K) \longrightarrow 1 \]

gives a coboundary map of pointed sets

\[ \delta_n : H^1(G, \text{PGL}_n(K)) \longrightarrow H^2(G, K^*) \]

which is injective due to the vanishing of \( H^1(G, \text{GL}_n(K)) \).

Let \( B_n(K/k) = \{ A \mid [A] \in B(K/k), \deg A = n \} \), so \( B(K/k) = \bigcup B_n(K/k) \). Since two central simple algebras of the same degree are equivalent iff they are isomorphic, \( B_n(K/k) \) is isomorphic to \( H^1(G, \text{PGL}_n(K)) \) as pointed sets (have to do something to show that the distinguished elements are the same under the set isomorphism). We will abuse notation and define

\[ \delta_n : B_n(K/k) \longrightarrow H^2(G, K^*) \]

The maps \( \delta_n \) are compatible with each other, that is, \( \delta_n(A \otimes A') = \delta_n(A) + \delta_n(A') \) (have to do something to show this); and since \( B_n(K/k) \) and \( H^1(G, \text{PGL}_n(K)) \) are isomorphic as pointed sets, \( \delta_n(A) = 0 \) iff \( A \) is \( k \)-split by exactness. Putting the \( \delta_n \)'s together gives an injective homomorphism of groups

\[ B(K/k) \longrightarrow H^2(G, K^*). \]

Proposition 0.2. There is an isomorphism of groups

\[ B(K/k) \cong H^2(G, K^*) \]

Proof. It suffices to show \( \delta_n : B_n(K/k) \longrightarrow H^2(G, K^*) \) is surjective for some \( n \).

Let \( n = [K:k] = |G| \). Recall that to coboundary map is given by

\[ \delta : H^1(G, \text{PGL}_n) \longrightarrow H^2(G, K^*) \]

\[ \Phi \mapsto (g, h) \mapsto \varphi_g \cdot g \varphi_h \cdot \varphi_{gh}^{-1} \]

where \( \varphi \in \text{GL}_n \) such that \( \varphi(g) \mapsto \Phi(g) \) for all \( g \in G \). The cocycle condition on \( a(s,t) \in H^2(G, K^*) \) is

\[ a(s, t)a(st, u) = a(s, tu) \cdot s(a(t, u)) \]

for all \( s, t, u \in G \). Given \( a(s, t) \in H^2(G, K^*) \) we will produce a \( \varphi : G \longrightarrow \text{GL}_n(K) \) of the required form.
To this end, let $V$ be a $K$-vector space of dimension $n$, with basis $e_g, g \in G$. Define
\[
\varphi_g : V \rightarrow V \\
\varphi_g(e_h) = a(g, h)e_{gh}
\]
Then
\[
\varphi_s(s(p_t)(e_u)) = a(s, tu) \cdot s(a(t, u)) \cdot e_{stu} \\
a(s, t) \cdot \varphi_{st}(e_u) = a(s, t) \cdot a(st, u) \cdot e_{stu}
\]
together with the cocycle condition shows that
\[
a(s, t) = \varphi_s(s(p_t))\varphi_{st}^{-1}.
\]
Taking the limit over all finite Galois extensions give
\[
B(k) = \lim_{\longrightarrow} H^2(\text{Gal}(K, k^*), K^*) = H^2(\text{Gal}(k^s, k^*), k^{s,*}).
\]

0.6. **Brauer group is torsion.** From the cohomological interpretation of the Brauer group, we conclude immediately that it is torsion. This follows from the following elementary group cohomological fact

**Proposition 0.3.** If $G$ is a finite group, then $|G|H^2(G, M) = 0$.

**Proof.** Exercise. \hfill \square

0.7. **Period and index.** Define the period of $[A] \in B(k)$ to be the order of $[A]$ in $B(k)$. If $A \simeq M_n(D)$, define the index of $A$ to be $\deg D$.

In general the period of $[A]$ divides its index. A recent result (c.f. [1]) of de Jong is the following

**Theorem 0.4.** (de Jong, 2004) Let $k$ be a separably closed field and let $K/k$ be a finite extension of transcendence degree 2. Let $[A]$ have period prime to the characteristic of $k$. Then the period of $[A]$ is equal to its index.

In particular, this result holds for functions fields of surfaces over a field of characteristic zero.

0.8. **Further reading.**
