As hinted at last week, the MMP produces either a minimal model or a Mori fibre space. The latter is the end result of an extremal contraction \(S \rightarrow W\) where \(\dim W = 0, 1\).

We know that this must be the contraction of some extremal ray.

1.1. Dichotomy. Given a surface \(S\), how do we know which one we get without running the MMP? We have the Easy dichotomy theorem

**Theorem 1.** A surface \(S\) produces a Mori fibre space iff there exists a nonempty Zariski open \(U \subset S\) such that for all \(p \in U\) there is an irreducible curve \(D\) through \(p\) with \(K_S \cdot D < 0\).

There is a much better answer to the above question. The Hard dichotomy theorem says that the Kodaira dimension determines whether a surface \(S\) produces a Mori fibre space of minimal model. This is much harder to prove, we will prove the easy direction:

**Theorem 2.** A surface \(S\) produces a Mori fibre space iff \(\kappa(S) = -\infty\). This simply means \(h^0(S, O(nK_S)) = 0\) for all \(m > 0\).

**Proof.** Suppose \(\phi : S \rightarrow W\) is a Mori fibre space, and \(h^0(S, O(nK_S)) \neq 0\) for some positive integer \(n\). Let \(D \in |nK_S|\). Pick a curve \(C\) in a fibre of \(\phi\) but not in \(D\), then \(0 > nK_S \cdot C = D \cdot C \geq 0\), (since \(C\) is contracted, it intersects \(K_S\) negatively by def. of extremal contraction) which is a contradiction. □

1.2. Characterisation of Mori fibre spaces.

**Theorem 3.** Let \(\phi : S \rightarrow W\) be a Mori fibre space. If \(\dim W = 1\) then \(S\) is a birationally ruled surface. If \(\dim W = 0\), then \(S\) is \(\mathbb{P}^2\).

**Note 1.** We can verify the above using the easy dichotomy theorem. If \(S\) is birationally ruled, there there exists an isomorphism \(U \rightarrow \mathbb{P}^1 \times C\) for some curve \(C\) and open set \(U\). Let \(p \in U\), pick a curve \(F\) through \(p\) contained in a fibre, then \(F \cdot K_S = F \cdot \pi_1^*(K_{\mathbb{P}^1}) + F \cdot \pi_2^*(K_C) < 0\). [Hartshorne II Ex. 8.3].

It is easy to show that \(\mathbb{P}^2\) produces a Mori fibre space.

Recall that the contraction theorem last week identified extremal contractions with extremal rays. The above theorem can be used to characterise extremal rays.

**Theorem 4.** Let \(S\) be a surface, \(K_S\) not nef (recall that extremal rays live in \(NE(S)_{K_S < 0}\), \(K_S\) not nef guarantees this is nonempty). An extremal ray \(\mathbb{R}_+[C]\) is one of the following:

1. \(C\) is a \((-1)\)-curve with \(K_S \cdot C = -1\) and \(\phi_C : S \rightarrow W\) is the contraction of the \((-1)\) curve \(C\).
2. \(C\) is a fibre with \(K_S \cdot C = -2\), of the algebraic \(\mathbb{P}^1\)-bundle \(\phi_C : S \rightarrow W\).
3. \(C\) is a line in \(S \simeq \mathbb{P}^2\), with \(K_S \cdot C = -3\) and \(\phi_C : S \rightarrow W\) is the structure morphism \(\mathbb{P}^2 \rightarrow \text{Spec} k\).

Conversely, any curve \(C\) satisfying the criteria of 1, 2, or 3 span an extremal ray \(\mathbb{R}_+[C]\). It has minimal intersection with \(-K_S\) among the curves in its numerical class.
1.3. Castelnuovo’s rationality criterion. As an application of the above, we prove

**Theorem 5.** Let $S$ be a surface, it is rational iff $h^1(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(2K_S)) = 0$.

**Proof.** Since these numbers are both birational invariants, it is easy to check that they hold for $\mathbb{P}^2$.

The strategy to prove the converse:

1. show the numerical criteria forces $K_S$ to be not nef,
2. run MMP and show we end up with a Mori fibre space
3. use the characterisation of Mori fibre spaces to check the few cases to show $S$ is rational.

Note that very ample divisors have positive self intersection, hence ample divisors do too. Kleiman’s criterion means that nef divisors have nonnegative self intersection.

Suppose $K_S$ is nef, then $K_S^2 \geq 0$. Recall Riemann-Roch for surfaces:

$$\chi(\mathcal{O}_S(D)) - \chi(\mathcal{O}_S) = \frac{1}{2}D \cdot (D - K_S)$$

where $D \in \text{Div} S$. Applying this for $D = -K_S$, we get

$$h^0(S, \mathcal{O}_S(-K_S)) \geq \chi(\mathcal{O}_S(-K_S)) = K_S^2 + \chi(\mathcal{O}_S) = K_S^2 + 1 \geq 1$$

This means there is an effective $D \in |-K_S|$, so for every very ample $A$ (recall that by Bertinis theorem, for very ample $A$, almost all $A' \in |A|$ is irreducible nonsingular curves) we have

$$0 \leq A \cdot D = A \cdot (-K_S) \leq 0 \text{ (since } K_S \text{ nef)}$$

giving $-K_S \sim 0 = D$. Hence $h^0(S, \mathcal{O}_S(2K_S)) = 1$ contradicting hypothesis.

Running the MMP leaves the numerical criteria invariant, so the end result cannot have nef $K_S$. This means we get a Mori fibre space, either birationally ruled $\mathbb{P}^1 \times C$ or $\mathbb{P}^2$ by characterisation of Mori fibre spaces. It suffices to compute the genus of $C$ to show it is $\mathbb{P}^1$.

This follows from the following computation:

$$0 = h^1(S, \mathcal{O}_S) = h^1(\mathbb{P}^1 \times C, \mathcal{O}_{\mathbb{P}^1 \times C})$$
$$= h^1(C, \mathcal{O}_C)$$
$$= g_C$$

[EGA, III, 6.7.8]:

$$H^1(\mathbb{P}^1 \times C, \pi_1^* \mathcal{O}_{\mathbb{P}^1} \otimes \pi_2^* \mathcal{O}_C) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \otimes H^1(C, \mathcal{O}_C) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \otimes H^0(C, \mathcal{O}_C)$$

$\square$