1. Introduction to the Minimal Model Program

1.1. Overview of the program.
   MMP1: Pick “good” representative \( \{X_i\}_{i \in I} \) in each birational equivalence class.
   MMP2: Study the good properties of these \( X_i \).
   MMP3: Study the possible (birational) relations between the \( X_i \).
   MMP4: Construct the moduli space

1.2. MMP for curves.
   MMP1: In each class, there exists a unique nonsingular projective curve \((k = \mathbb{K}, \text{char} \ k = 0)\). A curve/surface is a projective \(k\)-variety of dimension 1/2).
   We may find \( C \) via normalisation.
   Example 1. \( \text{Spec} \ k[x^{1/2}] \rightarrow \text{Spec} \ k[x,y]/(x^3 - y^2) \)
   The integral closure of \( k[x,y]/(x^3 - y^2) \) in \( k(x,x^{3/2}) = k[x^{1/2}] \) is isomorphic to \( k[x,x^{3/2}] \) since \( (x^{1/2})^2 - x = 0 \).
   MMP2: Study the properties of \( C \) using \( K_C \) via \( \sigma = p_g = h^0(C, \mathcal{O}_C(K_C)) \).
   MMP3: \( C \) is unique.
   MMP4: Fix \( g \).
   \( g = 0 \): There exist a unique birational equivalence class: \( \mathbb{P}^1 \). Moduli space =pt.
   \( g = 1 \): elliptic curves. We have the \( j \)-invariant and \( j(C) \in \mathbb{A}^1_k \). The (coarse) moduli space is \( \mathbb{A}^1_k \).
   \( g \geq 2 \): The moduli space has dimension \( 3g - 3 \).

1.3. Outline of MMP for surfaces.
   MMP1: First we retrieve a nonsingular projective surface via resolution of singularities. Then blow down \((-1)\)-curves using Castelnuovo’s contractibility criterion.
   \[
   S \quad \downarrow \\
   \text{ruled surface}
   \]
   \[
   \text{minimal model} \quad \leftarrow \quad \text{MMP} \quad \rightarrow \quad \text{ruled surface}
   \]
   MMP2: Properties are dictated by \( K_S \), e.g. cohomology, Kodaira dimension \( \kappa(S) \), genus etc.
   MMP3: If \( \kappa(S) \geq 0 \), then there exists a unique “good representative” which is the minimal model. If \( \kappa(S) = -\infty \), then there are many choices and we study the birational relations via the Sarkisov program.
   MMP4: Look at the pluricanonical system and...

1.4. Explanations.
Definition 1. A surface \( S \) is a minimal model iff \( K_S \) is nef, that is, for all curves \( C \subset S \) we have \( K_S \cdot C \geq 0 \).

The previous definition of minimal models (\( S \) is a minimal model if for all birational morphisms \( S \rightarrow S' \) to a surface are isomorphisms) included cases where \( K_S \) is not nef, for the remainder of the seminar series, we will refer to minimal models only when there exists a unique minimal model.

Proposition 1. \( K_S \) is nef iff \( S \) is not birationally ruled, and for all birational morphisms \( \sigma : S \rightarrow S' \), \( \sigma \) is an isomorphism

Proof. If there exists a birational map \( \sigma : S \rightarrow S' \) which is not an isomorphism, then it is a sequence of blow downs of \((-1)\)-curves. Since \((-1)\)-curves satisfy \( K_S \cdot E < 0 \), \( K_S \) is not nef.
If \( K_S \) is not nef, then there exists an extremal contraction (as yet undefined) \( \phi : S \rightarrow W \) such that \( \phi \) contracts a \((-1)\)-curve, or \( S \) is birationally ruled. We will prove this in the following seminars. \( \square \)

Theorem 1. (Abundance theorem) If \( S \) is a minimal model, then the pluricanonical system, \( |mK_S| \), is base point free for \( m \gg 0 \) and \( m \) sufficiently divisible.

This gives us a morphism \( \phi_{|mK_S|} : S \rightarrow S_{\text{can}} \) called the Iitaka fibration. The dimension of \( S_{\text{can}} \) is equal to the Kodaira dimension of \( S \).
Definition 2.

\[ \kappa(S) = \begin{cases} -\infty & \text{if } H^0(S, mK_S) = 0 \text{ for all } m > 0 \\ \text{tr. deg}_k \bigoplus_{m \geq 0} H^0(S, mK_S) & \text{otherwise} \end{cases} \]

Example 2. Let \( S = \mathbb{P}^2 \). Then \( H^0(S, \mathcal{O}_S(mK_S)) = H^0(S, \mathcal{O}_S(-3m)) = 0 \) for all \( m > 0 \), hence \( \kappa(\mathbb{P}^2) = -\infty \).

This is an example of a del Pezzo surface which are characterised by the property that the anticanonical divisor \((-K_S)\) is ample.

Definition 3. A surface \( S \) is \( K3 \) if \( K_S \sim 0 \) and \( h^1(S, \mathcal{O}_S) = 0 \).

Note that this implies \( \kappa(S) = 0 \).

Example 3. Let \( S \) be a hypersurface of degree 4 in \( \mathbb{P}^3 \). Note that as a divisor on \( \mathbb{P}^3 \), \( S \sim 4H \) where \( H \) is a hyperplane in \( \mathbb{P}^3 \).

The adjunction formula says

\[ K_S = (S + K_{\mathbb{P}^3})|_S = (4H - 4H)|_S = 0. \]

To show \( h^1(S, \mathcal{O}_S) = 0 \), consider the restriction sequence

\[
\begin{align*}
0 & \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-S) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_S \longrightarrow 0 \\
& \quad \cdots \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \cong H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow \cdots
\end{align*}
\]

since \( H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0 \) by earlier cohomology computations, \( h^1(S, \mathcal{O}_S) = 0 \). This shows \( S \) is \( K3 \).

A surface \( S \) with Kodaira dimension equal to 1 is an elliptic surface, that is, there exists a surjection \( S \longrightarrow C \) where \( C \) is a curve and the generic fibre is an elliptic curve. (Explicit example given in seminar 5).

Surfaces with Kodaira dimension equal to 2 are called surfaces of general type. Kodaira’s lemma characterises surfaces of general type:

Lemma 1. (Kodaira) \( \kappa(S) = 2 \) iff \( K_S^2 > 0 \).

Example 4. Take a hypersurface \( S \) with degree \( d \geq 5 \), then \( \mathcal{O}_S(K_S) = \mathcal{O}_S(d - 4) \). Since \( d - 4 > 0 \), \( K_S \) is very ample on \( S \), hence \( K_S > 0 \).

2. What happens if \( K_S \) is not nef

Let \( k = \overline{k} \), char \( k = 0 \). A surface (resp. curve) is a nonsingular projective variety of dimension 2 (resp. 1).

2.1. Statement of purpose.

- To given an overview of the MMP
- To be constantly reminded by examples (surfaces, invariants of surfaces).
- To show instructure proofs, that is, proofs which demonstrate the importance of the theory.

Last week we saw that a nonsingular irreducible hypersurface of degree 4 in \( \mathbb{P}^3 \) is \( K3 \), we will give another example of a \( K3 \) surface.

Example 5. Consider \( T = \mathbb{P}^1_{x_0,x_1} \times \mathbb{P}^2_{y_0,y_1,y_2} \), recall that closed subsets of \( T \) are given by bihomogeneous polynomials in \( x_i \)'s and \( y_i \)'s. Let \( S = X_{2,3} \) be the zero set of a polynomial in the \( x_i \)'s and \( y_i \)'s of bihomogeneous degree (2, 3).

The Picard group of \( T \) is isomorphic to \( \mathbb{Z}H_1 \oplus \mathbb{Z}H_2 \) where \( H_1 \) and \( H_2 \) are pullbacks of the hyperplane generators of Pic \( \mathbb{P}^1 \) and Pic \( \mathbb{P}^2 \) respectively. Let \( \pi_1, \pi_2 \) be the first and second projection maps of \( T \), then

\[
K_T = \pi_1^*(K_{\mathbb{P}^1}) + \pi_2^*(K_{\mathbb{P}^2}) = -2H_1 - 3H_2
\]

and by the adjunction formula

\[
K_S = (K_T + S)|_S = (-2H_1 - 3H_2 + 2H_1 + 3H_2)|_{X_{2,3}} = 0.
\]

To show that \( h^1(S, \mathcal{O}_S) = 0 \), we do the same thing as before

\[
\begin{align*}
0 & \longrightarrow \mathcal{O}_T(-S) \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_S \longrightarrow 0 \\
& \quad \cdots \longrightarrow H^1(T, \mathcal{O}_T) \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow H^2(T, \mathcal{O}_T(-S)) \cong H^1(T, \mathcal{O}_T) \longrightarrow \cdots
\end{align*}
\]

\[
\begin{align*}
H^1(T, \mathcal{O}_T) &= H^1(T, \pi_1^*(\mathcal{O}_{\mathbb{P}^1}) \oplus \pi_2^*(\mathcal{O}_{\mathbb{P}^2})) \\
&= H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \\
&= 0
\end{align*}
\]
hence $h^1(S, \mathcal{O}_S) = 0$.

**Theorem 2.** If $K_S$ is not nef, then there exists an extremal contraction, that is, a morphism $\phi : S \rightarrow W$ such that

1. $\phi$ is not an isomorphism
2. if $C$ is a curve and $\phi(C) = \text{pt}$ then $K_S \cdot C < 0$
3. if $C$ and $D$ are curves and $\phi(C) = \text{pt}$, $\phi(D) = \text{pt}$, then $C$ and $D$ are numerically equivalent, that is, $C \cdot A = D \cdot A$ for all $A \in \text{Pic} S$.
4. $\phi$ has connected fibres, and $W$ is normal projective.

**Note 1.** Suppose $C, D$ are numerically equivalent, and $\psi : X \rightarrow Y$ be a projective morphism with very ample divisor $L$ on $X$. Then a curve $C \subset X$ is contracted by $\psi$ iff $\psi^*(L) \cdot C = 0$.

**Definition 4.** Define

$$
N^1(S) = (\text{Pic} S/ \sim_{\text{num}}) \otimes \mathbb{R}
$$

$$
Z_1(S) = \text{free abelian group generated by integral curves}
$$

$$
N_1(S) = (Z_1(S)/ \sim_{\text{num}}) \otimes \mathbb{R}
$$

$$
NE(S) = \left\{ \sum a_i [C_i] \mid C_i \subset X \text{a proper curve}, a_i \in \mathbb{R}_{+} \right\} \subseteq N_1(S)
$$

**Example 6.** $\mathbb{P}^1 \times \mathbb{P}^1$

This is important because

$L$ nef \iff $L \cdot C \geq 0$ for all $C \subset S$

$L \cdot C > 0$ for all $[C] \in \overline{NE}(S)$

Kleiman’s criterion below says that nefness is in some sense “close” to ampleness.

**Theorem 3.** (Kleiman) $L$ is ample on $S$ iff $L \cdot C > 0$ for all $[C] \in \overline{NE}(S)$.

Note that for $L$ to be ample it is not enough that $L \cdot C \geq 0$ for all $[C] \in \overline{NE}(S)$. Kleiman’s criterion says that ample divisors are nef. We will prove theorem 2 below.

**Proof.** Let $A$ be ample on $S$, define $A^\perp = \{ L \in \text{Pic} S \mid A \cdot L = 0 \}$. Then by Kleiman’s criterion, $A^\perp \cap \overline{NE}(S) = \emptyset$.

We want to perturb $A$ slightly, let $A' = A + rK_S$, $r \in \mathbb{Q}_{>0}$. If $K_S$ is nef, then there exists $[C] \in \overline{NE}(S)$ such that $K_S \cdot C < 0$. So there is some $r$ such that $A' \cdot C = 0$.

The rationality theorem asserts that $r = \sup \{ t \mid A + tK_S \text{is not nef} \} \in \mathbb{Q}$. Then for some sufficiently large and divisible $n, [nL]$ gives a map $\Phi_{[nL]} : S \rightarrow V$. The base point freeness theorem asserts that this map is base point free.

Why does $\Phi$ give an extremal contraction?

**Case 1:** $\dim V = 2$: Since $\Phi$ is not an isomorphism, there exists a curve $E$ contracted by $\Phi$. So $0 = E \cdot L = E \cdot (A + rK_S) = E \cdot A + rE \cdot K_S$, but $A$ is ample, so $E \cdot A > 0$. Therefore $E \cdot K_S < 0$. By the Hodge index theorem (If $D \cdot E = 0$ and $D^2 > 0$ then $E^2 < 0$) $E^2 < 0$, so $E$ is a $(-1)$-curve. Let $\Phi : S \rightarrow W$ be the contraction of this $(-1)$-curve, this is the extremal contraction.

**Case 2:** $\dim V = 1$: We use the Stein factorisation theorem.

1. There exists a reducible fibre. See Matsuki.
2. If all fibres are irreducible, then they are algebraically equivalent (parameterised by points on a base curve). But algebraic equivalence implies numerical equivalence, so $\Phi_C : S \rightarrow W = V$ is the extremal contraction, with $C$ a fibre. The fibre has genus $g_C = C \cdot (C + K_S)/2 + 1 = 0$ so is rational.

**Case 3:** $\dim V = 0$: This occurs iff $mL \sim_{\text{num}} 0$ iff $L = A + rK_S = 0$. Define the Picard number $\rho(S) = \dim_{\mathbb{R}} N^1(S)$.

- If $\rho(S) = 1$ then every element of $\text{Pic} S$ is numerically equivalent to a multiple of some curve $C \subset S$. So $\Phi$ is the extremal contraction.
- If $\rho(S) > 1$, pick an ample divisor $A'$ not numerically equivalent to some multiple of $A$, then $L' = A' + r'K_S$ is not numerically equivalent to $L$, hence not numerically equivalent to zero. Hence $L'$ gives case 1 or 2.
Why is $|nL|$ base point free? Assume the contrary and blow up a base point $p$: $\sigma: \tilde{S} \to S$ with exceptional divisor $E$, then the restriction sequence can be written as

$$0 \to \mathcal{O}_{\tilde{S}}(\sigma^*(nL) - E) \to \mathcal{O}_{\tilde{S}}(\sigma^*(nL)) \to \mathcal{O}_E \to 0$$

since $\sigma^*(nL) \cdot E = 0$. The long exact cohomology sequence gives, in part,

$$\ldots \to H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\sigma^*(nL))) \to H^0(\tilde{S}, \mathcal{O}_E) \to H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\sigma^*(nL) - E)) \to \ldots$$

The rightmost term vanishes by Kawamata-Viehweg, so the first map is surjective. But this is impossible since $p$ is a base point of $|nL|$ so all sections of $\mathcal{O}_{\tilde{S}}(nL)$ restrict to zero on $p$. So all sections of $\mathcal{O}_{\tilde{S}}(\sigma^*(nL))$ restrict to zero on $E$. $\square$