Algebraic Geometry Seminar: Etale Cohomology III

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1 All things etale: degrees, sites and sheaves...

1.1 The degree of an etale map

Definition 1. *Let* $\varphi\colon Y \longrightarrow X$ *be an etale map. This morphism induces an homomorphism of function fields*

$$K(X) \hookrightarrow K(Y)$$

and we define the degree of φ *to be the degree of the field extension* $[K(X)\colon K(Y)]$.

Remark 2. If the degree of φ is n, geometrically this means that above each $y \in Y$ there sit n points of X mapping to y. Draw this!

1.2 Grothendieck topologies and sites

Recall that a Grothendieck topology is a category $\mathcal{C}$ and a set CovT of families of maps $\{\phi_i\colon U_i \longrightarrow U\}_{i\in I}$ such that

I. $\{U \xrightarrow{\sim} U\} \in \mathrm{CovT}$.

II. $\{U_i \longrightarrow U\}_{i\in I} \in \mathrm{CovT}\ \text{and}\ \forall i, \{V_{ij} \longrightarrow U_i\}_{j\in J} \in \mathrm{CovT}, \text{then}\ \{V_{ij} \longrightarrow U\}_{i\in I, j\in J} \in \mathrm{CovT}$.

III. $\{U_i \longrightarrow U\}_{i\in I} \in \mathrm{CovT}\ \text{and}\ V \longrightarrow U\ \text{an arbitrary morphism in CatT implies}$

$$\{U_i \times_U V \longrightarrow V\}_{i\in I} \in \mathrm{CovT}.$$

This system of coverings is called a *topology* and the category $\mathcal{C}$ equipped w/ the topology is called a site.

Example 3.

- The site X_{zar} consists of the category $\mathcal{C}$ of open subsets of X w/ morphisms inclusions.
- The site X_{et} consists of the category Et/X of schemes etale over X w/ morphisms X-morphisms $\varphi\colon U \longrightarrow V$. The coverings are surjective families of etale morphisms $\{U_i \longrightarrow U\}$ in Et/X.
- The site X_{Fl} has underlying category the category of all X-schemes, Sch/X. The coverings are surjective families of X-morphisms $\{\varphi\colon U_i \longrightarrow U\}$ with each φ_i flat and of finite-type.
- Let G be a profinite group. The site $\mathbf{T}_G$ has underlying category the set of all finite discrete G-sets. Coverings are surjective families of G-maps.

1.3 Sheaves for the etale topology

A *presheaf of sets* on a site $\mathbf{T}$ is a contravariant functor $\mathcal{F}\colon Cat(\mathbf{T}) \longrightarrow Sets$. A *sheaf* is a presheaf $\mathcal{F}$ that satisfies the sheaf condition, i.e.

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

is exact for every covering $\{U_i \longrightarrow U\}_{i \in I}$.

Example 4.

- For $U \longrightarrow X$ etale, define $\mathcal{O}_{X_{et}}(U) := \Gamma(U, \mathcal{O}_U)$. This is a sheaf and is called the structure sheaf on X_{et}.
- Let $\mathcal{M}$ be a sheaf of coherent $\mathcal{O}_X$ − modules. For any etale map $\varphi: U \longrightarrow X$, we obtain a coherent $\mathcal{O}_U$ − module $\varphi^*\mathcal{M}$ on U_{zar}. Then $U \mapsto \Gamma(U, \varphi^*\mathcal{M})$ is a presheaf, denoted $\mathcal{M}^{et}$.

Exercise 1. Show that $\mathcal{M}^{et}$ is a sheaf.

Exercise 2. Show that $(\mathcal{O}_{X_{zar}})^{et} = \mathcal{O}_{X_{et}}$.

1.3.1 The sheaves on Spec(k)

Let $\mathcal{F}$ be a presheaf of abelian groups on $(\text{Spec } k)_{et}$, k a field. We may view this as a covariant functor $Et/X \longrightarrow Ab$.

Proposition 5. *Then $\mathcal{F}$ is a sheaf $\iff \mathcal{F}(\oplus \mathcal{A}_i) = \oplus \mathcal{F}(\mathcal{A}_i)$ and $\mathcal{F}(k') \simeq \mathcal{F}(K)^{\mathrm{Gal}(K/k')}$ for every finite Galois extension K/k' of fields w/ k' of finite degree over k.*

Now let k^{sep} be a seperable closure of k and let $G = \mathrm{Gal}(k^{sep}/k)$. For $\mathcal{F}$ a sheaf on $(\text{Spec}\, k)_{et}$, define

$$M_{\mathcal{F}} = \varinjlim \mathcal{F}(k'),$$

k' running through the subfields of k^{sep} that are finite and Galois over k. $M_{\mathcal{F}}$ is then a discrete G-module. There is, in fact, an inverse functor...

Exercise 3. Show that the category of sheaves on $(\text{Spec } k)_{et}$ and the category of discrete G-modules are equivalent.

1.3.2 Skyscraper sheaves

Let X be a variety and $x \in X$ a point. For any etale map $\varphi: U \longrightarrow X$, define

$$\Lambda^x(U) := \oplus_{u \in \varphi^{-1}(x)} \Lambda.$$

This means that $\Lambda^x(U) = 0$ unless $x \in \varphi(U)$. However, if $x \in \varphi(U)$, $\Lambda^x(U) = \oplus_{u \in \varphi^{-1}(x) \cap U} \Lambda$.

1.4 Direct images of sheaves

Let $\varphi: X \longrightarrow Y$ be a morphism of varieties and let $\mathcal{F}$ be a sheaf on X_{et}. For $U \longrightarrow Y$ etale, set

$$\varphi_*\mathcal{F}(U) = \mathcal{F}(U \times_Y X).$$

This makes sense since $U \longrightarrow Y$ etale $\Longrightarrow$ $U \times_Y X \longrightarrow X$ etale.

Proposition 6. *If $\mathcal{F}$ is a sheaf on X_{et}, then $\varphi_*\mathcal{F}$ is a sheaf on Y_{et} and is called the direct image of φ.*

Exercise 4. Prove the above proposition.

Remark 7. φ_*, when viewed as a functor from the category of sheaves on X_{et} to the category of sheaves on Y_{et}, is not exact. For example. if φ is a map from X to a point, then φ_* is the global sections functor Γ, which we know not to be exact. We will see an example of this shortly.

Example 8. $O \longrightarrow \mathcal{I}_{0,\infty} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}/\mathcal{I}_{o,\infty} \longrightarrow 0$ is an exact sequence.

1.5 In which we concern ourselves with the category of sheaves

Definition 9. *Working in the category PreSh(X_{et}), a sequence of presheaves* $0 \longrightarrow \mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}'' \longrightarrow 0$ *is called exact if*

$$0 \longrightarrow \mathcal{P}'(U) \longrightarrow \mathcal{P}(U) \longrightarrow \mathcal{P}''(U) \longrightarrow 0$$

is exact for all $U \longrightarrow X$ *etale.*

Definition 10. *A sequence of sheaves* $0 \longrightarrow \mathcal{S}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}'' \longrightarrow 0$ *is exact if and only if it is exact at the stalks.*

Remark 11. If a sequence of sheaves is exact, THIS DOES NOT IMPLY that the sequence will be exact as a sequence of presheaves. However, when we sheafify an exact sequence of presheaves, we DO get an exact sequence of sheaves! That is, the functor

$$i\colon Sh(X_{et}) \longrightarrow PreSh(X_{et})$$

is not exact BUT

$$a\colon PreSh(X_{et}) \longrightarrow Sh(X_{et})$$

is.

Example 12. $O \longrightarrow \mathcal{I}_{0,\infty} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}/\mathcal{I}_{o,\infty} \longrightarrow 0$ is exact at the stalks but not globally.

2 Derived functor cohomology

First recall that the functor taking a sheaf $\mathcal{F}$ to its global sections $\Gamma(X,\ \mathcal{F})$ is a left exact functor but NOT necessarily exact. What sheaf cohomology does is measure how far from being exact this functor really is:

Let $\mathcal{F}$ be a sheaf and choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \ldots$$

and, applying the functor $\Gamma(X, -)$, obtain

$$\Gamma(X, \mathcal{I}^0) \longrightarrow \Gamma(X, \mathcal{I}^1) \longrightarrow \Gamma(X, \mathcal{I}^2) \longrightarrow \ldots$$

Now this sequence is not necessarily exact. It is, however, a complex and thus we are able to define $H^r(X_{et}, \mathcal{F})$ to be its r^{th} cohomology group.

Remark 13. We have the following:

i. $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$;

ii. If $\mathcal{F}$ is injective, then $H^i(X, \mathcal{F}) = 0, \forall i > 0$;

iii. A short exact sequence $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ gives rise to a long exact sequence

$$0 \longrightarrow H^0(X_{et}, \mathcal{F}') \longrightarrow H^0(X_{et}, \mathcal{F}) \longrightarrow H^0(X_{et}, \mathcal{F}'') \longrightarrow H^1(X_{et}, \mathcal{F}') \longrightarrow \ldots$$

ALSO, these three properties determine the functor $H^r(X, -)$ up to unique isomorphism.

Remark 14. Letting $x = \mathrm{Spec}\, k$ and noting that $(M_{\mathcal{F}})^G = \Gamma(x, \mathcal{F})$, one sees the derived functors of $M \mapsto M^G$ and $\mathcal{F} \mapsto \Gamma(x, \mathcal{F})$ correspond, implying that

$$H^r(\, x, \mathcal{F}) \;=\; H^r(G, \mathcal{M}_F).$$

Thus etale cohomology is a generalisation of Galois cohomology.