1 ETALE MAPS

In order to construct our desired etale cohomology, the morphisms of primary interest to us are etale morphisms. We begin by defining etale maps in the category of differentiable manifolds:

**Definition 1.** A $C^\infty$ map $\phi: N \rightarrow M$ of differentiable manifolds is called etale at $n \in N$ if the map induced map on tangent spaces

$$d\phi: \text{Tgt}_n(N) \rightarrow \text{Tgt}_{\phi(n)}(M)$$

is an isomorphism.

We will now state a theorem of complex analysis which tells us a great deal about etale maps. It is important to note that this theorem does NOT hold in the Zariski topology.

**Theorem 2.** *(Inverse Mapping Theorem)* A $C^1$ map of differentiable manifolds is a local isomorphism at any point at which it is etale.

1.1 Etale maps between varieties

**Definition 3.** A regular map of nonsingular varieties $\varphi: X \rightarrow Y$ is called etale at $x \in X$ if

$$d\varphi: \text{Tgt}_x(X) \rightarrow \text{Tgt}_{\varphi(y)}(Y)$$

is an isomorphism.

The inverse mapping theorem does NOT hold in the Zariski topology because the Zariski open sets are just too big.

**Example 4.**

$$\varphi: \mathbb{A}^1_k \rightarrow \mathbb{A}^1_k \quad x \mapsto x^n$$

is etale except at the origin (we will prove this soon). However, this is not a local isomorphism at any point on the affine line. If it were, it would induce an isomorphism on the function fields but it does not!

**Exercise 1.** Why not?!

**Exercise 2.** Let $k = \mathbb{C}$ and show that $\varphi$ does indeed obey the Inverse Mapping Theorem.

**Proposition 5.** Let $\varphi: X \rightarrow Y$ be a regular map w/ $X = Y = \mathbb{A}^n$ *(regular here means that $\varphi$ can be written as $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ with each $\varphi_i$ a polynomial)*. $\varphi$ is etale at $(a_1, \ldots, a_n)$ if and only if the Jacobian matrix

$$\left(\frac{\partial \varphi_i}{\partial X_j}(a_1, \ldots, a_n)\right)$$

is invertible.

**Example 6.** In example 4, $\frac{dX^n}{dX} = nX^{n-1} \Rightarrow \begin{cases} \varphi \text{ etale at all } x \neq 0 \text{ if char } k \neq n; \\ \varphi \text{ etale nowhere otherwise.} \end{cases}$

Noe we define etale for arbitrary varieties, possibly singular.
**Definition 7.** A regular map of varieties $\varphi: X \rightarrow Y$ is called etale at $x \in X$ if

$$d\varphi: C_x(X) \rightarrow C_{\varphi(x)}(Y)$$

is an isomorphism. Here, $C_x(X)$ is the tangent cone to $X$ at $x$.

**Disclaimer**—I will not define tangent cone here. The example hints at the definition. Knowledge-mongers should see Shafarevich, *Basic Algebraic Geometry I*.

**Example 8.** The tangent cone at the origin to the curve $V$ defined by $Y^2 - X^3 = 0$ is given by $Y^2 = 0$.

The map $\varphi: \mathbb{A}^1 \rightarrow V: t \mapsto (t^2, t^3)$ is not etale at the origin because the corresponding map of tangent cones

$$k[Y]/(Y^2) \rightarrow K[T]$$

is not an isomorphism.

1.1.1 Interlude—Everything you always wanted to know about schemes but were too afraid to ask.

Recall what $\mathbb{A}^1$ is: it’s just a copy of $k$. The functions defined on $\mathbb{A}^1$ are exactly those from the polynomial ring $k[x]$. Notice that the maximal ideals of $k[x]$ are in 1-1 correspondence with points of $\mathbb{A}^1$: they are all of the form $(x - a)$ for some some $a \in k$. Grothendieck’s idea was to look at the topological space which has these maximal ideals of points but this did not quite work SO he looked at the topological space which has the prime ideals of $k[x]$ as points.

**POINTS**—As a point set, $\text{Spec } R := \{p \subseteq R| p $ a prime ideal of $R\}$. We write the point corresponding to $p$ as $[p]$.

**TOPOLOGY**—The closed sets of $R$ are $V(f) := \{[p] \in \text{Spec } R| f \notin p \}$.

**SHEAF**—The sheaf of regular functions $\mathcal{O}_{\text{Spec } R}$ is given by

$$\mathcal{O}_{\text{Spec } R}(D(f)) = R_{f \notin p}.$$  

It is enough to describe the sheaf on these distinguished open sets because they form a base for the topology, i.e. any open set is a union of distinguished open sets.

**Example 9.** Looking at the open set of $\mathbb{A}^1 := \text{Spec } k[x]$ which does not contain the point corresponding to the prime ideal $(x)$, we can invert $x$. In the language of varieties, $\frac{1}{x}$ is a regular function on $\mathbb{A}^1 \setminus \{0\}$.

**Remark 10.** $\mathcal{O}_{\text{Spec } R}(\text{Spec } R) = \mathcal{O}_{\text{Spec } R}(D(1)) = R_{f \notin p} = R$.

The above remark gives us a 1-1 correspondence between commutative rings with identity and affine schemes! To get other types of schemes, all we have do is glue together copies of affine schemes. For example, the projective line works in exactly the same way as it did with varieties.

1.2 Etale maps between schemes

**Definition 11.** A morphism $f: X \rightarrow Y$ of schemes is etale if it is flat and unramified.
1.2.1 Flat morphisms

**Definition 12.** $f: X \to Y$ is called flat if $\mathcal{O}_{Y, f(x)}$ is a flat $\mathcal{O}_{X, x}$-algebra, $\forall x \in X$.

From the definition it is quite hard to see what is going on but all that flatness really means is that the fibres (i.e. preimages of points) vary continuously, i.e. above each point of $Y$ we get schemes of the same dimension. Thus, if $f$ is flat,

$$\dim Y_x = \dim Y - \dim X.$$ 

**Example 13.**

- The projection $\text{Spec } k[x, y]/(xy) \to \text{Spec } k[y]$ is not flat.
- The projection $\pi: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is flat since above each point we have a projective line.
- More generally, the projection $\pi: \mathbb{P}^1 \times C \to C$ is flat for any curve $C$. These are called ruled surfaces and the projection is called a ruling.
- $\text{Bl}(\mathbb{P}^2) \to \mathbb{P}^2$ is not flat but $\text{Bl}(\mathbb{P}^2) \to \mathbb{P}^1$ is!
- A Pencil of cubics is flat above $\mathbb{P}^1$.

1.2.2 Ramification

First, recall the definition of a local ring: a ring with a unique maximal ideal.

**Example 14.** Let $X$ be a variety. The stalk $\mathcal{O}_{X, x}$ of germs of functions at a point $x \in X$ has a unique maximal ideal $m_x$ containing all functions that vanish at $x$. This is clearly an ideal and is the unique maximal ideal since any function which does not vanish at $x$ is invertible in the stalk.

**Definition 15.** A local homomorphism $g: A \to B$ of local rings is unramified if $B/\mathfrak{m}_AB$ is a finite separable field extension of $A/\mathfrak{m}_A$. Equivalently,

- $g(\mathfrak{m}_A) = \mathfrak{m}_B$ and
- $B/\mathfrak{m}_B$ is finite and separable over $A/\mathfrak{m}_A$.

**Example 16.** We have a map $g: k[x] \to k[x]$ s.t. $g(x) = x^2$. Localising at $(x)$ and setting $A = B = k[x]_{(x)}$, we see $g(\mathfrak{m}_A) = x^2B$. Thus $g$ is NOT unramified.

**Definition 17.** A morphism $f: X \to Y$ of schemes is unramified if it is of finite type and the maps $\mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X, x}$ are unramified for all $x \in X$.

So, finally, etale means something!

**Example 18.** What are then the etale maps to the spectrum of a field? Well, let $f: X \to \text{Spec } k$ be an etale map. Firstly, note that $f$ unramified $\implies X$ is the disjoint union of a finite no. of 0-dimensional schemes Spec $K_i$. Each $K_i$ is necessarily a ramified $k$-algebra, i.e. $K_i$ is a finite separable field extension of $k$.

- If $k = \overline{k}$, there are no nontrivial etale morphisms to Spec$(k)$.
- Spec$(\mathbb{C}) \to \text{Spec}(\mathbb{R})$ is the only nontrivial etale morphism from a connected scheme to Spec$(\mathbb{R})$.
- What can we say about Spec$(\mathbb{Q})$?

**Exercise 4.** In this talk, etale has been defined in three different ways, in increasing order of generality. Show that the definitions coincide when dealing w/ the same objects.