1 SO WHAT THE HELL IS COHOMOLOGY?

Cohomology encodes information, all kinds of information. The first examples you will see in your mathematical career will be in algebraic topology: (co)homology encodes the number of n-dimensional holes in a manifold. For example, if \( n > 0 \), \( H_n(M, \mathbb{R}) \cong \mathbb{R} \) if \( n \in \{0, n\} \), while \( H_i(\mathbb{R}^n, \mathbb{Z}) \cong \mathbb{Z} \) if \( i = 0 \), \( 0 \) otherwise.

So then, how does cohomology help us in algebraic geometry?

1. For nonsingular algebraic curves, the genus \( g(C) = \dim_k H^1(C, \mathcal{O}_C) \) is a very useful invariant. If \( g(C) = 0 \), then \( C \cong \mathbb{P}^1 \) and if \( g(C) = 1 \), then \( C \) is an elliptic curve.

Now let us look at the affine line \( \mathbb{A}^1 \):

**Q**: What functions are defined on this variety?

**A**: All polynomials in the variable \( t \). In the language of algebraic geometry, the ring of regular functions on the affine line is \( k[t] \).

Now look at the projective line \( \mathbb{P}^1_{X,Y} \):

**Q**: What functions are defined on this variety?

**A**: Well, on the open patch \( \mathbb{P}^1 \setminus \{\infty\} \), the ring of regular functions is the polynomial ring \( k[X] \). On the open patch \( \mathbb{P}^1 \setminus \{0\} \), the ring of regular functions is \( k[Y] \). Since a regular function on the whole variety must be regular on both patches, \( \mathcal{O}_{\mathbb{P}^1} \) is contained in both \( k[X] \) and \( k[Y] \). Thus \( \mathcal{O}_{\mathbb{P}^1} \cong k \). We also write \( \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong k \).

Note: \( \Gamma \) is, in fact, a functor from the category of varieties to the category of commutative rings w/ identity.

**Exercise 1.** Show that \( \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong k \) actually implies \( \mathbb{P}^1 \) is not affine.

What does this have to do w/ cohomology? Well, \( \Gamma(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X) \) is an isomorphism of functors.
2 Varieties and sheaves

What is a variety? It is a topological space $X$ with a ring of functions defined on each open set, that is, a sheaf of rings.

So what is a sheaf? First, let us define a presheaf:

- $\forall U \subseteq X$ open, we need a ring $\mathcal{F}(U)$
- $\forall V \subseteq U$, a restriction map $\rho_{U,V}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ s.t.
  - i. $\rho_{U,U}$ is an isomorphism;
  - ii. $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$ (equality as maps)

A presheaf is a sheaf if

$\{U_i\}_{i \in I}$ is an open covering of $U \subseteq X$ implies the following two conditions:

I. $f_1, f_2 \in \mathcal{F}(U)$ such that $f_1|_{U_i} = f_2|_{U_i}$, $\forall i, j \in I \implies f_1 = f_2$.

II. $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ $\forall i, j \in I$, then there exists an $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

These conditions are equivalent to saying that the following sequence is exact:

$$0 \rightarrow \mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightarrow \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

Exercise 2. What are the maps here?

So, if the classification problem is what we’re interested in, how do sheaves help? We have already seen that $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{k}$ whilst $\Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) = \mathbb{k}[t]$.

Exercise 3. Use sheaves in a similar manner to show that $\mathbb{A}^2 - (0,0)$ is not affine.

3 Cohomology of sheaves

We can set up cohomology in two ways:

1. Via the right derived functor of $\Gamma$.
2. Via Cech cohomology.

1 Cech cohomology

Let $\{U_i\}_{i \in I}$ be an affine open covering of $U$ and $\mathcal{F}$ a sheaf of rings on $U$(with indexing set $I$ totally ordered). To reduce cluttered notation, we set $U_{i_0 \ldots i_p} := U_{i_0} \cap \ldots \cap U_{i_p}$.

First let

$$C^p(U, \mathcal{F}) = \prod_{i_0 < \ldots < i_p} \mathcal{F}(U_{i_0 \ldots i_p})$$
We then define maps

\[ d_p : C^p \rightarrow C^{p+1} \]

\[ \alpha_{i_0, \ldots, i_p} \mapsto \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, i_k, \ldots, i_{p+1}, i_{p+1} \cap U_i, \ldots, i_{p+1}} \]

Note that this is indeed a complex, i.e. \( d_{p+1} \circ d_p = 0 \). We then define the \( i \)th Čech cohomology ring

\[ H^n(U, \mathcal{F}) := \ker d_n / \text{im} d_{n-1} \text{ for all } n > 0 \text{ and} \]

\[ H^0(U, \mathcal{F}) := \ker d_0 \]

Then \( 0 \rightarrow C^0 \rightarrow C^1 \rightarrow \ldots \) is

\[ 0 \rightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightarrow \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \]

and since \( \mathcal{F} \) is a sheaf, \( \ker d_0 \cong \mathcal{F}(U) \cong \Gamma(U, \mathcal{F}) \). This proves that

\[ H^0(U, \mathcal{F}) \cong \Gamma(U, \mathcal{F}) \].

2 Problems w/ the Zariski topology

We have seen that cohomology is pretty much looking at the types of functions we allow on smaller and smaller open subsets of \( X \).

The problem w/ the Zariski topology is that our open subsets are so large we don’t get a good cohomology theory.

**Example 1.** Let’s look at \( \mathbb{A}^1 \): the Zariski closed sets of \( \mathbb{A}^1 \) are solutions to polynomials in one variable; thus closed sets are finite collections of points \( \Rightarrow \) Zariski open sets are complements of finite sets of points. Now these are huge. Setting \( k = \mathbb{C} \), compare the Zariski topology w/ the complex topology. It is interesting to note that

- any Zariski open set is unbounded in the complex plane;
- there are uncountably many complex open sets which are not Zariski open.

How does this problem manifest itself in terms of cohomology theory?

- If \( \dim(X) = n \), then \( H^i(X, \mathcal{F}) = 0, \forall i > n \). However, \( X \) is actually a \( 2n \) - dimensional real manifold and should have higher cohomology.

- (Grothendieck’s Theorem) If \( X \) is an irreducible topological space, then \( \Lambda \) a constant sheaf implies that for \( i > 0 \),

\[ H^i(X, \Lambda) = 0 \]

- The inverse mapping theorem fails.

We say that the Zariski topology is too coarse. If, then, cohomology is given to us by looking at smaller and smaller open subsets via their intersections, we need to generalise the concept of an open subset!
When looking at open subsets of a topological space \( X \) in terms of sheaf theory, what we are really doing is looking at a category with

- objects: open subsets of \( X \);
- morphisms: inclusions.

Viewing things this way, we readily that the intersection of two open subsets of \( U \) is just their fibered product over \( U \)!

Let’s remind ourselves what an open covering is:

**Definition 2.** A covering of a topological space \( X \) is a collection of open subsets \( \{U_i\}_{i \in I} \) of \( X \) such that

\[
X = \bigcup_{i \in I} U_i
\]

Note that

- i. \( X = X \) is the trivial covering of \( X \).
- ii. If \( X = \bigcup_{i \in I} U_i \) is a covering and for each \( i, U_i = \bigcup_{j \in J} V_{ij} \) is a covering, then
  
  \[
  X = \bigcup_{i \in I, j \in J} V_{ij}
  \]
  
  is also a covering.
- iii. If \( \{U_i\}_{i \in I} \) is a covering of \( X \) and \( V \subset X \), then \( \{U_i \cap V\}_{i \in I} \) is a covering of \( V \).

Now we are able to define a Grothendieck topology:

**Definition 3.** A Grothendieck topology is a category \( \mathcal{C} \) and a set \( \text{CovT} \) of families of maps \( \{\phi_i: U_i \rightarrow U\}_{i \in I} \) such that

1. \( \{U \rightarrow U\} \in \text{CovT} \).
2. \( \{U_i \rightarrow U\}_{i \in I} \in \text{CovT} \) and \( \forall i, \{V_{ij} \rightarrow U_i\}_{j \in J} \in \text{CovT} \), then \( \{V_{ij} \rightarrow U\}_{i \in I, j \in J} \in \text{CovT} \).
3. \( \{U_i \rightarrow U\}_{i \in I} \in \text{CovT} \) and \( V \rightarrow U \) an arbitrary morphism in \( \text{CatT} \) implies
   
   \[
   \{U_i \times_U V \rightarrow V\}_{i \in I} \in \text{CovT}
   \]

**Example 4.** The category \( \text{Et}/X \) of schemes etale over \( X \) with coverings surjective families of etale morphisms \( \{U_i \rightarrow U\} \) in \( \text{Et}/X \) is a Grothendieck topology.

Such a Grothendieck topology gives us a good cohomology theory:

**Theorem 5.** (Comparison theorem) For any finite abelian group \( \Lambda \),

\[
H^r(X_{\text{et}}, \Lambda) \simeq H^r(X(\mathbb{C}), \Lambda).
\]