Infinite Tensor Products of $C_0(\mathbb{R})$: 
Towards a Group Algebra for $\mathbb{R}^{(\mathbb{N})}$.

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Abstract

The construction of an infinite tensor product of the C*-algebra $C_0(\mathbb{R})$ is not obvious, because it is nonunital, and it has no nonzero projection. Based on a choice of an approximate identity, we construct here an infinite tensor product of $C_0(\mathbb{R})$, denoted $L_V$, and use it to find (partial) group algebras for the full continuous representation theory of $\mathbb{R}^{(\mathbb{N})}$. We obtain an interpretation of the Bochner–Minlos theorem in $\mathbb{R}^{(\mathbb{N})}$ as the pure state space decomposition of the partial group algebras which generate $L_V$. We analyze the representation theory of $L_V$, and show that there is a bijection between a natural set of representations of $L_V$ and $\text{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H})$, but that there is an extra part which essentially consists of the representation theory of a multiplicative semigroup $Q$ which depends on the initial choice of approximate identity.

Keywords: C*-algebra, group algebra, infinite tensor product, topological group, Bochner–Minlos theorem, state space decomposition, continuous representation.

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1 Introduction

The class of locally compact groups has a rich structure theory with a great many tools developed to analyze the representation theory of such groups, e.g., group C*-algebras, induction, integral decompositions etc. Unfortunately there are many non-locally compact groups which naturally arise in analysis or physics applications, e.g. mapping groups or inductive limit groups, and for such groups these tools fail, and one has to do the analysis on a case-by-case basis, with no systematic theory to draw on.

Here we want to consider the question of how to generalize the notion of a (twisted) group algebra to topological groups which are not locally compact (hence have no Haar measure). Such a generalization, called a full host algebra, has been proposed in [Gr05]. Briefly, it is a $C^*$-algebra $\mathcal{A}$ whose multiplier algebra $M(\mathcal{A})$ admits a homomorphism $\eta: G \to U(M(\mathcal{A}))$, such that the (unique) extension of the
representation theory of \( A \) to \( M(A) \) pulls back via \( \eta \) to the continuous unitary representation theory of \( G \). There is also an analogous concept for unitary \( \sigma \)-representations, where \( \sigma \) is a continuous \( \mathbb{T} \)-valued 2-cocycle on \( G \). Thus, given a full host algebra \( A \), the continuous unitary representation theory of \( G \) can be analyzed on \( A \) with a large arsenal of \( C^* \)-algebraic tools. Such a host algebra need not exist for a general topological group because there exist topological groups with faithful unitary representations but without non-trivial irreducible ones (cf. [GN01]). One example of a full host algebra for a group which is not locally compact, has been constructed explicitly for the \( \sigma \)-representations of an infinite dimensional topological linear space \( S \), considered as a group cf. [GrN09].

Probably the simplest infinite dimensional group is \( \mathbb{R}^{(N)} \) (the set of real-valued sequences with only finitely many nonzero entries) with the inductive limit topology w.r.t. the natural inclusions \( \mathbb{R}^n \subset \mathbb{R}^{(N)} \). This group is well-studied in stochastic analysis, and will be the main object of study also in this paper. Our aim here is to construct explicitly \( C^* \)-algebras which have useful host algebra properties for \( \mathbb{R}^{(N)} \). Recall that for the group \( C^* \)-algebras we have:

\[
C^*(\mathbb{R}^n) \otimes C^*(\mathbb{R}^m) \cong C^*(\mathbb{R}^{n+m})
\]

and this suggests that for a host algebra of \( \mathbb{R}^{(N)} \) we should try an infinite tensor product of \( C^*(\mathbb{R}) \). This is difficult to do, for two reasons:

- \( C^*(\mathbb{R}) \cong C_0(\mathbb{R}) \) is nonunital, and the standard infinite tensor products of \( C^* \)-algebras require unital algebras.

- There is a definition for an infinite tensor product of nonunital algebras developed by Blackadar cf. [Bl77], but this requires the algebras to have nonzero projections, and the construction depends on the choice of projections. (We used this construction in [GrN09] to construct an infinite tensor product to produce a host algebra.) However, \( C^*(\mathbb{R}) \cong C_0(\mathbb{R}) \) has no nonzero projections, so this method will not work.

In the light of these difficulties, we will develop here an infinite tensor product of \( C_0(\mathbb{R}) \) relative to a choice of approximate identity in each entry, to replace the choice of projections in Blackadar’s approach. As expected, the construction will depend on the choice of approximate identities, though it still produces for each choice an algebra with strong host algebra properties.

The construction of ("semi-")host algebras for \( \mathbb{R}^{(N)} \) will aid our understanding of the Bochner–Minlos theorem. We first recall:

**1.1 Theorem (Bochner–Minlos Theorem for \( \mathbb{R}^{(N)} \))** There is a bijection between continuous normalized positive definite functions (states) \( \omega \) of \( \mathbb{R}^{(N)} \) and regular Borel probability measures \( \mu \) on \( \mathbb{R}^{N} \) (with product topology) given by the Fourier transform:

\[
\omega(x) = \int_{\mathbb{R}^N} e^{i \langle x, y \rangle} d\mu(y), \quad x \in \mathbb{R}^{(N)}
\]

where \( \langle x, y \rangle := \sum_{n=1}^{\infty} x_n y_n \), \( x \in \mathbb{R}^{(N)} \), \( y \in \mathbb{R}^{N} \).

If we replace both \( \mathbb{R}^{(N)} \) and \( \mathbb{R}^{N} \) by \( \mathbb{R}^{n} \), this is the classical Bochner theorem, which we can obtain immediately from the state space integral decomposition of any state of \( C^*(\mathbb{R}^{n}) \cong C_0(\mathbb{R}^{n}) \) in terms of
pure states. This suggests that if we have a host algebra of $\mathbb{R}^N$, we can obtain the Bochner–Minlos theorem from state space decompositions of states on the host algebra in terms of pure states. We will see below that we can already obtain the Bochner–Minlos theorem from the weaker "semi–host" algebras which we will construct.

The structure of this paper is as follows. In Section 2 we collect the basic definitions and notation for host algebras, in Section 3 we give a detailed treatment of the aspects of infinite tensor products which we will need for this paper. In Section 4 we start in a concrete setting on $L^2(\mathbb{R}^N, \mu)$, where $\mu$ is a product measure of probability measures, each absolutely continuous w.r.t. the Lebesgue measure, and we construct an infinite tensor product of $C_0(\mathbb{R})$ w.r.t. a choice (compatible with $\mu$) of approximate identity in each entry. This concrete C*-algebra can already produce Bochner–Minlos decompositions for the limited class of positive definite functions on $\mathbb{R}^N$ associated with it. In Section 5 we develop abstractly the infinite tensor product of $C_0(\mathbb{R})$ w.r.t. an arbitrary choice of elements of a fixed approximate identity, we analyze its representation theory and through the unitary embedding of $\mathbb{R}^N$ in its multiplier algebra, we consider the relation of its representation theory to that of $\mathbb{R}^N$. We find that it can adequately model a subset of the representation theory of $\mathbb{R}^N$, but there is a small additional part. We show that the Bochner–Minlos decompositions for any continuous positive definite function on $\mathbb{R}^N$ can be obtained from the pure state space decomposition of these algebras. Finally, in Section 6 we collect these algebras together in one large C*-algebra, which we show, can model the full continuous representation theory of $\mathbb{R}^N$. However, the representation theory of this algebra also has an additional part which essentially consists of the representation theory of a multiplicative semigroup $\mathcal{Q}$ which depends on the initial fixed choice of approximate identity.

2 Definitions and notation

We will need the following notation and concepts for our main results.

- In the following, we write $M(A)$ for the multiplier algebra of a C*-algebra $A$ and, if $A$ has a unit, $U(A)$ for its unitary group. We have an injective morphism of C*-algebras $\iota_A: A \to M(A)$ and will just denote $A$ for its image in $M(A)$. Then $A$ is dense in $M(A)$ with respect to the strict topology, which is the locally convex topology defined by the seminorms

$$p_a(m) := \|m \cdot a\| + \|a \cdot m\|, \quad a \in A, \ m \in M(A)$$

(cf. [Wo95]).

- For a complex Hilbert space $\mathcal{H}$, we write $\text{Rep}(A, \mathcal{H})$ for the set of non-degenerate representations of $A$ on $\mathcal{H}$. Note that the collection $\text{Rep} A$ of all non-degenerate representations of $A$ is not a set, but a (proper) class in the sense of von Neumann–Bernays–Gödel set theory, cf. [Tak75], and in this framework we can consistently manipulate the object $\text{Rep} A$. However, to avoid set–theoretical subtleties, we will express our results below concretely, i.e., in terms of $\text{Rep}(A, \mathcal{H})$ for given Hilbert spaces $\mathcal{H}$. We have an injection

$$\text{Rep}(A, \mathcal{H}) \rightarrow \text{Rep}(M(A), \mathcal{H}), \quad \pi \mapsto \overline{\pi} \quad \text{with} \quad \overline{\pi} \circ \iota_A = \pi,$$

which identifies the non-degenerate representation $\pi$ of $A$ with that representation $\overline{\pi}$ of its multiplier algebra which extends $\pi$ and is continuous with respect to the strict topology on
\( M(A) \) and the topology of pointwise convergence on \( B(H) \). We will refer to \( \pi \) as the strict extension of \( \pi \), and it is easily obtained by

\[
\tilde{\pi}(M) = \lim_{\lambda \to \infty} \pi(M \lambda) \quad \forall M \in M(A)
\]

where \( \{E_\lambda\}_{\lambda \in \Lambda} \subset A \) is any approximate identity of \( A \).

- For topological groups \( G \) and \( H \) we write \( \text{Hom}(G, H) \) for the set of continuous group homomorphisms \( G \to H \). We also write \( \text{Rep}(G, H) \) for the set of all (strong operator) continuous unitary representations of \( G \) on \( H \). Endowing \( U(H) \) with the strong operator topology turns it into a topological group, denoted \( U(H)_s \), so that \( \text{Rep}(G, H) = \text{Hom}(G, U(H)_s) \). The set of continuous normalized positive definite functions on \( G \) (also called states) and denoted by \( \mathcal{S}(G) \), is in bijection with the state space of the group C*-algebra \( C^*(G) \) when \( G \) is locally compact. If \( G \) is not locally compact, \( \mathcal{S}(G) \) is in bijection with a subset of the state space of \( C^*(G_d) \), where \( G_d \) denotes \( G \) with the discrete topology, and the question arises as to whether there is a C*-algebra which can play the role of \( C^*(G) \). We clarify first what is meant by this:

### 2.1 Definition

A host algebra for \( G \) is a pair \( (L, \eta) \) where \( L \) is a C*-algebra and \( \eta: G \to U(M(L)) \) is a homomorphism such that for each complex Hilbert space \( H \) the corresponding map

\[
\eta^*: \text{Rep}(L, H) \to \text{Rep}(G, H), \quad \pi \mapsto \tilde{\pi} \circ \eta
\]

is injective. We then write \( \text{Rep}(G, H)_\eta \subseteq \text{Rep}(G, H) \) for the range of \( \eta^* \). We say that \( (L, \eta) \) is a full host algebra of \( G \) if \( \eta^* \) is surjective for each Hilbert space \( H \). If the map \( \eta^* \) is not injective, we will call the pair \( (L, \eta) \) a semi-host algebra for \( G \).

Note that by the universal property of group algebras, the homomorphism \( \eta: G \to U(M(L)) \) extends uniquely to the discrete group C*-algebra \( C^*(G_d) \), i.e. we have a *-homomorphism \( \eta: C^*(G_d) \to U(M(L)) \) (still denoted by \( \eta \)).

A similar notion can also be defined for projective representations (cf. [GrN09]).

### 2.2 Remark

1. It is well known that for each locally compact group \( G \), the group C*-algebra \( C^*(G) \), and the natural map \( \eta_G: G \to M(C^*(G)) \) provide a full host algebra (Dix77 Sect. 13.9]). The map \( \eta_G: G \to M(C^*(G)) \) is continuous w.r.t. the strict topology of \( M(C^*(G)) \) (this is an easy consequence of the fact that \( \text{im}(\eta_G) \) is bounded and that the action on the corresponding \( L^1 \)-algebra is continuous).

2. Note that for a host algebra \( (L, \eta) \) the map \( \eta^* \) preserves direct sums, unitary conjugation, subrepresentations, and for full host algebras, irreducibility (cf. [Gr05]).

3. When \( (L, \eta) \) is merely a semi-host algebra for \( G \), then the map \( \eta^* \) still preserves direct sums, unitary conjugation, subrepresentations, but in general, not irreducibility. However, in the case that \( G \) is Abelian (as it will be in this paper), since irreducible representations are just characters, and the map \( \eta^* \) takes one-dimensional representations to one-dimensional ones, here it will preserve irreducibility. So for Abelian groups, semi-hosts are useful to carry representation structure (e.g. integral decompositions) from the representation theory of \( L \) to the representation theory of \( G \), and we will use that in this paper to analyze the Bochner–Minlos theorem.
3 Basic Theory of Infinite Tensor Products

Since we need to develop the concept of infinite tensor products of non-unital algebras, it is necessary to collect first some basic material on infinite tensor products, and to fix notation. We follow Bourbaki [Bou89] and Wegge-Olsen [WO93]. There are several different concepts of infinite tensor products of unital algebras. See Bourbaki [Bou89], Guichardet [Gu67], Araki [AW66], though infinite tensor products of algebras without identity are only done in Blackadar [Bl77].

3.1 Algebraic tensor products of arbitrary many factors.

3.1 Definition Let \((X_t)_{t \in T}\) be an indexed set of non-zero complex vector spaces, where \(T\) can have any cardinality. We write \(x = (x_t)_{t \in T}\) for the elements of the product space \(\prod_{t \in T} X_t\). A map \(f: \prod_{t \in T} X_t \to V\) to a vector space \(V\) is said to be multilinear if it is linear in each entry. That is, for each \(t_0 \in T\) and \(x \in \prod_{t \in T \setminus \{t_0\}} X_t\), the map

\[
X_{t_0} \to V, \quad y_{t_0} \to f(x \times y_{t_0})
\]

is linear, where \(x \times y_{t_0} := z \in \prod_{t \in T} X_t\) is that element for which \(z_t = x_t\) if \(t \neq t_0\) and \(z_{t_0} = y_{t_0}\).

A pair \((\iota, V)\) consisting of a vector space \(V\) and a multilinear map \(\iota: \prod_{t \in T} X_t \to V\) is called an (algebraic) tensor product of \((X_t)_{t \in T}\) if it has the following universal property

(UP) For each multilinear map \(\varphi: \prod_{t \in T} X_t \to W\), there exists a unique linear map \(\tilde{\varphi}: V \to W\) with

\[
\tilde{\varphi} \circ \iota = \varphi.
\]

The usual arguments (cf. Proposition T.2.1 [WO93]) show that the universal property determines a tensor product up to linear isomorphism (factoring through the maps \(\iota\)). We may thus denote \(V\) by \(\bigotimes_{t \in T} X_t\) and denote the elementary tensors by

\[
\otimes_{t \in T} x_t := \iota(x) \in \bigotimes_{t \in T} X_t, \quad \text{for} \quad x \in \prod_{t \in T} X_t.
\]

To simplify notation, we write \(X := \prod_{t \in T} X_t\) in the following. Observe that no order in \(T\) appears in this definition, so e.g. \(X_1 \otimes X_2\) and \(X_2 \otimes X_1\) (in the usual notation) will be identified.

3.2 Lemma For each indexed set \((X_t)_{t \in T}\) of complex vector spaces, a tensor product \((\iota, \bigotimes_{t \in T} X_t)\) exists.

Proof: (cf. [Bou89] Ch. II.§3.9 for a more general construction) We consider the free complex vector space

\[
\mathbb{C}^{(X)} := \{ f: X \to \mathbb{C} \mid \text{supp}(f) \text{ is finite} \} = \text{Span}\{ \delta_x \mid x \in X \}
\]

where \(\delta_x(y) = 1\) if \(x = y\) and zero otherwise. Note that \(\{\delta_x \mid x \in X\}\) is a basis for \(\mathbb{C}^{(X)}\). Define the sets

\[
\begin{align*}
N_a &:= \{ \delta_x + \delta_y - \delta_z \mid \exists r \in T \text{ such that } x_r + y_r = z_r, \text{ and } x_t = y_t = z_t \quad \forall t \neq r \} \\
N_m &:= \{ \delta_x - \mu \delta_y \mid \mu \in \mathbb{C}, \text{ and } \exists r \in T \text{ such that } x_r = \mu y_r, \text{ and } x_t = y_t \quad \forall t \neq r \} \\
\mathcal{N} &:= \text{Span}(N_a \cup N_m) \subset \mathbb{C}^{(X)}
\end{align*}
\]
We now consider the quotient space $V := \mathbb{C}(X)/\mathcal{N}$ and write $\iota: X \to V, x \mapsto \delta_x + \mathcal{N}$ for the induced map. The definition of $\mathcal{N}$ immediately implies that $\iota$ is multilinear and we only have to verify the universal property.

Let $\varphi: X \to M$ be a multilinear map. We extend $\varphi$ to a linear map $\varphi : \mathbb{C}(X) \to M$ by $\varphi(f) := \sum_{x \in X} f(x) \varphi(x)$. The multilinearity of $\varphi$ now implies that its linear extension annihilates the subspace $\mathcal{N}$, hence it factors through a linear map $\tilde{\varphi} : V \to M$ satisfying $\tilde{\varphi} \circ \iota = \varphi$. That $\tilde{\varphi}$ is uniquely determined by this property follows from the fact that $\text{im}(\iota)$ spans $V$.

### 3.3 Theorem (Associativity)

Let $\{ T_s \subset T \mid s \in S \}$ be a partition of $T$ such that $|S| < \infty$. Then the map

$$\psi : \prod_{t \in T} X_t \to \bigotimes_{s \in S} \left( \bigotimes_{t \in T_s} X_{t_s} \right), \quad \psi((x_t)_{t \in T}) := \bigotimes_{s \in S} \left( \bigotimes_{t \in T_s} x_{t_s} \right)$$

is multilinear and factors through a linear isomorphism $\tilde{\psi} : \bigotimes_{t \in T} X_t \to \bigotimes_{s \in S} \left( \bigotimes_{t \in T_s} X_{t_s} \right)$.

**Proof.** It is clear from the definition that $\psi$ is multilinear, so we obtain a unique linear map $\tilde{\psi} : \bigotimes_{t \in T} X_t \to \bigotimes_{s \in S} \left( \bigotimes_{t \in T_s} X_{t_s} \right)$ with $\tilde{\psi} \circ \iota = \psi$.

To see that $\tilde{\psi}$ is a linear isomorphism, it suffices to observe that the multilinear map $\psi$ has the universal property (UP). So let $\varphi: X \to V$ be a multilinear map. With $Y_s := \prod_{t \in T_s} X_t$, we have $X = \prod_{s \in S} Y_s$. Then for each $s_0 \in S$ and for each $y \in \prod_{s \in S \setminus s_0} Y_s$ we obtain a unique map

$$\varphi^{s_0}_y : Y_{s_0} = \prod_{t \in T_{s_0}} X_t \to V, \quad \varphi^{s_0}_y(y_{s_0}) := \varphi(y \times y_{s_0}),$$

which is clearly multilinear w.r.t. the factors $\prod_{t \in T_{s_0}} X_t = Y_{s_0}$ hence induces a linear map on $\bigotimes_{t \in T_{s_0}} X_t$. Since $y \mapsto \varphi^{s_0}_y(v)$ is multilinear in $y \in \prod_{s \in S \setminus s_0} Y_s$ for fixed $v \in \bigotimes_{t \in T_{s_0}} X_t$, we can apply the argument again to an $s_1 \neq s_0 \in S$ for this map, and then continue the process until we have exhausted $S$. This produces a multilinear map

$$\tilde{\varphi} : \prod_{s \in S} \left( \bigotimes_{t \in T_s} X_t \right) \to V$$

which factors through a linear map

$$\tilde{\varphi} : \bigotimes_{s \in S} \left( \bigotimes_{t \in T_s} X_t \right) \to V \quad \text{with} \quad \tilde{\varphi}\left( \bigotimes_{s \in S} \left( \bigotimes_{t \in T_s} x_{t_s} \right) \right) = \varphi((x_t)_{t \in T}),$$

i.e., $\tilde{\varphi} \circ \psi = \varphi$. Moreover, since $\bigotimes_{s \in S} \left( \bigotimes_{t \in T_s} X_t \right)$ is spanned by elements of the form $\bigotimes_{s \in S} \left( \bigotimes_{t \in T_s} x_{t_s} \right)$, it follows that $\tilde{\varphi}$ is uniquely determined by the last equation. Thus $\psi$ has the universal property (UP), hence $\tilde{\psi}$ is a linear isomorphism.

### 3.4 Remark

Associativity does not seem to hold for a partition of $T$ into infinitely many sets (i.e., for $|S| = \infty$). This is because $\bigotimes_{t \in T} X_t$ is spanned by elementary tensors, and $\bigotimes_{s \in S} \left( \bigotimes_{t = 1, r \in T_s} x^{(t)}_{r_s} \right)$...
cannot be written as a finite linear combination of elementary tensors if there are infinitely many \( s \in S \) with \( n_s > 1 \).

3.5 Definition (a) Assume that \((X_t)_{t \in T}\) is a family of complex algebras. We now construct an algebra structure on their tensor product. For each fixed \( x \in X = \prod_{t \in T} X_t \), define a map

\[
\mu_x : X \to \bigotimes_{t \in T} X_t \quad \text{by} \quad \mu_x(y) := \bigotimes_{t \in T} x_t y_t = \iota(x \cdot y)
\]

where \( x \cdot y \in X \) is given by \((x \cdot y)_t := x_t y_t \) for all \( t \in T \), and we will also let \( x^n \in X \) denote \((x^n)_t := (x_t)^n \) for all \( t \in T \) and \( n \in \mathbb{N} \). Since \( \mu_x \) is multilinear, it induces a linear map

\[
\mu_x : \bigotimes_{t \in T} X_t \to \bigotimes_{t \in T} X_t.
\]

This defines a multilinear map

\[
\mu : X \to \text{End}(\bigotimes_{t \in T} X_t) \quad \text{by} \quad \mu(x) := \mu_x
\]

and thus a linear map \( \mu : \bigotimes_{t \in T} X_t \to \text{End}(\bigotimes_{t \in T} X_t) \). Explicitly we have for \( a = \sum_i \iota(x_i) \) and \( b = \sum_j \iota(y_j) \in \bigotimes_{t \in T} X_t \) that

\[
\mu(a)(b) = \sum_i \mu_x \left( \sum_j \iota(y_j) \right) = \sum_i \sum_j \mu_x \left( \iota(y_j) \right) = \sum_i \sum_j \iota(x_i \cdot y_j)
\]

where the sums are finite. We denote the multiplication as usual by \( a \cdot b := \mu(a)(b) \) for \( a, b \in \bigotimes_{t \in T} X_t \).

Associativity for this multiplication follows from componentwise associativity, and hence \( \bigotimes_{t \in T} X_t \) is an algebra over \( \mathbb{C} \).

(b) Next, we assume, in addition, that each \( X_t \) is a \( \ast \)-algebra. We want to turn \( \bigotimes_{t \in T} X_t \) into a \( \ast \)-algebra. Given any vector space \( V \) over \( \mathbb{C} \), let \( V^c \) denote the conjugate vector space. Thus, for each \( t \in T \), the involution \( \ast : X_t \to X_t^c \) becomes a \( \mathbb{C} \)-linear map (instead of conjugate linear on \( X_t \)). Define a map

\[
\gamma : X \to \left( \bigotimes_{t \in T} X_t \right)^c \quad \text{by} \quad \gamma(x) := \bigotimes_{t \in T} x_t^* = \iota(x^*)
\]

where \( x^* \in X \) is given by \((x^*)_t := x_t^* \) for all \( t \in T \). Since \( \gamma \) is multilinear, it defines a linear map \( \gamma : \bigotimes_{t \in T} X_t \to \left( \bigotimes_{t \in T} X_t \right)^c \). Its intertwining properties with multiplication then follow from the componentwise properties. As usual, we write \( a^* := \gamma(a) \) for \( a \in \bigotimes_{t \in T} X_t \), and hence \( \bigotimes_{t \in T} X_t \) becomes a \( \ast \)-algebra over \( \mathbb{C} \).

This defines the basic objects which we will work with.

3.2 Stabilized spaces.

We will also need the following structures.

3.6 Definition We define an equivalence relation on \( X \) by \( x \sim y \) whenever the set \( \{ t \in T \mid x_t \neq y_t \} \) is finite. Denote the equivalence class of \( x \in X \) by \( [x]_\sim \) and define

\[
[x] := \text{Span} \{ \bigotimes_{t \in T} y_t \mid y \sim x \} \subset \bigotimes_{t \in T} X_t.
\]
3.7 Proposition  The following assertions hold:

(i) For any pair \((x, F)\) such that \(x \in X\) and \(F \subseteq T\) a finite subset with \(x_t \neq 0\) for \(t \notin F\), there exists a linear map

\[
\varphi_F : \bigotimes_{t \in T} X_t \to \bigotimes_{t \in F} X_t
\]

satisfying \(\llbracket y \rrbracket \subseteq \text{Ker} \varphi_F\) for \(y \not\sim x\) and

\[
\varphi_F \left( \left( \bigotimes_{t \in F} y_t \right) \otimes \left( \bigotimes_{t \notin F} x_t \right) \right) = \bigotimes_{t \in F} y_t \quad \text{for} \quad y_t \in X_t, t \in F.
\]

(ii) \(\llbracket x \rrbracket \neq \emptyset\) if and only if at most finitely many components of \(x\) vanish.

(iii) The subspace \(\llbracket x \rrbracket\) is isomorphic to the direct limit of the finite tensor products \(\bigotimes_{t \in J} X_t, J \subseteq T\) finite, with respect to the connecting maps

\[
\varphi_{K,J} : \bigotimes_{t \in J} X_t \to \bigotimes_{t \in K} X_t \quad \text{with} \quad \varphi_{K,J} \left( \bigotimes_{t \in J} y_t \right) := \left( \bigotimes_{t \in J} y_t \right) \otimes \left( \bigotimes_{s \in K \setminus J} x_s \right).
\]

(iv) \(\bigotimes_{t \in T} X_t\) is the direct sum of the subspaces \(\llbracket x \rrbracket\), \(x \in X\).

Proof: (i) For \(t \notin F\) we pick linear functionals \(\lambda_t \in X_t^*\) with \(\lambda_t(x_t) = 1\) and define a map

\[
\hat{\varphi}_F : X \to \bigotimes_{f \in F} X_f, \quad \hat{\varphi}_F(y) := \prod_{t \in T \setminus F} \lambda_t(y_t) \cdot \left( \bigotimes_{s \in F} y_s \right) \quad \text{for} \quad y \sim x
\]

\[
\begin{array}{ll}
0 & \text{for} \quad y \not\sim x.
\end{array}
\]

We claim that \(\hat{\varphi}_F\) is multilinear. To see that \(\hat{\varphi}_F\) is linear in the \(t\)-component, let \(y, y' \in X\) with \(y_s = y'_s\) for \(s \neq t\). Then either both are equivalent to \(x\) or none is. In either case, the definition of \(\hat{\varphi}_F\) implies the linearity of the map \(z_t \mapsto \hat{\varphi}_F(y \times z_t)\). Therefore \(\hat{\varphi}_F\) is multilinear, hence induces a linear map

\[
\varphi_F : \bigotimes_{t \in F} X_t \to \bigotimes_{t \in F} X_t
\]

satisfying all requirements.

(ii) If the set \(\{t \in T \mid x_t = 0\}\) is finite, then (i) implies that \(\llbracket x \rrbracket \neq \emptyset\) since none of the spaces \(X_t\) vanishes by our initial assumption. We also note that, if infinitely many \(x_t\) vanish, then \(\llbracket x \rrbracket\) is spanned by elements \(i(y)\), where \(y\) has at least one zero entry. Then \(i(y) = 0\), and consequently \(\llbracket x \rrbracket = \{0\}\).

(iii) Let \(J \subset K \subset T\) such that \(|K| < \infty\). Then we obtain linear maps

\[
\varphi_{K,J} : \bigotimes_{t \in J} X_t \to \bigotimes_{t \in K} X_t \quad \text{with} \quad \varphi_{K,J} \left( \bigotimes_{t \in J} y_t \right) := \left( \bigotimes_{t \in J} y_t \right) \otimes \left( \bigotimes_{s \in K \setminus J} x_s \right).
\]

Since \(\varphi_{L,K} \circ \varphi_{K,J} = \varphi_{L,J}\) for \(J \subset K \subset L\), and \(|L| < \infty\), this is an inductive system. We write \(\lim_{\longrightarrow} \left( \bigotimes_{t \in J} X_t, \varphi_{K,J} \right)\) for its limit. We also have linear maps

\[
\varphi_J : \bigotimes_{t \in J} X_t \to \llbracket x \rrbracket \quad \text{by} \quad \varphi_J \left( \bigotimes_{t \in J} y_t \right) := \left( \bigotimes_{t \in J} y_t \right) \otimes \left( \bigotimes_{s \in T \setminus J} x_s \right) \in \llbracket x \rrbracket
\]

satisfying \(\varphi_K \circ \varphi_{K,J} = \varphi_J\), so that they induce a linear map \(\varphi : \lim_{\longrightarrow} \left( \bigotimes_{t \in J} X_t, \varphi_{K,J} \right) \to \llbracket x \rrbracket\). As every element of \(\llbracket x \rrbracket\) lies in the image of some map \(\varphi_J\), and by (i) this map is injective if \(J \supseteq \{t \in T \mid x_t = 0\}\), \(\varphi\) is a linear isomorphism.
(iv) Since \( \iota(x) \) is contained \([x]\), it suffices to show that the sum of the non-zero subspaces \([x]\) is direct. Suppose that the elements \(x_1, \ldots, x_n\) are pairwise non-equivalent with \([x_i] \neq \{0\}\), and that \(v_i \in [x_i]\) satisfy \(\sum_i v_i = 0\). From (i) we know that there exists for each \(i\) and each finite subset \(F \supseteq \{t \in T \mid x_{i,t} = 0\}\) a linear map

\[
\varphi^{(i)}_F : \bigotimes_{t \in T} X_t \to \bigotimes_{t \in F} X_t \quad \text{with} \quad \varphi^{(i)}_F \left( \left( \otimes_{t \in F} y_t \right) \otimes \left( \otimes_{t \in F} x_{i,t} \right) \right) = \otimes t \in F y_t
\]

and vanishing on \([x_j]\) for \(j \neq i\). We conclude that \(\varphi^{(i)}_F(v_i) = 0\) for each \(F\). Since \(F\) can be chosen arbitrarily large, the definition of \([x_i]\) now implies that \(v_i = 0\). \(\blacksquare\)

3.8 Remark If each \(X_t\) is an algebra and \(x_t^2 = x_t\) holds for all but finitely many \(t \in T\), then the linear space \([x]\) is a subalgebra. If each \(X_t\) is an \(\ast\)-algebra and \(x_t^2 = x_t\) for all but finitely many \(t \in T\), then \([x]\) is a \(\ast\)-subalgebra. In the literature (on topological tensor products), suitable closures of \([x]\) are often called stabilized infinite tensor products (stabilized by \(x\)).

3.9 Remark In particular, for \(x, y \in X\) with \([x] \neq \{0\} \neq [y]\), we have that \([x] \cap [y] = \{0\}\) if and only if \(x \not\sim y\). So, if \(y_t = \lambda_t x_t\) where \(\lambda_t \neq 1\) for infinitely many \(t \in T\), then \(x \not\sim y\) and hence \(\otimes_{t \in T} \lambda_t x_t\) is not a multiple of \(\otimes_{t \in T} x_t\). This is different in Guichardet’s version \([Gu67]\) of continuous tensor products.

When the \(X_t\) are algebras, we have the following algebraic relations for the spaces \([x]\) in the algebra \(\bigotimes_{t \in T} X_t\).

3.10 Theorem If each \(X_t\) is a complex algebra, then

(i) \([x] \cdot [y] \subseteq [x \cdot y]\) for all \(x, y \in X\). If \(X_t \cdot X_t = X_t\) for all \(t\), then we have the equality: Span\((\{x \cdot y\}) = [x \cdot y]\).

(ii) \([x]^* = [x^*]\) for all \(x \in X\) if all \(X_t\) are \(\ast\)-algebras.

(iii) If \(\emptyset \neq G_t \subset X_t \setminus \{0\}\) is a nonzero multiplicative semigroup for each \(t \in T\), then

\[
\mathcal{M} := \sum_{a \in \prod_{t \in T} G_t} [a] \quad \text{(finite sums)}
\]

is a subalgebra of \(\bigotimes_{t \in T} X_t\). If in addition, each \(X_t\) is a \(\ast\)-algebra and each \(G_t\) is \(\ast\)-invariant, then \(\mathcal{M}\) is a \(\ast\)-subalgebra.

Proof. (i) Since \([x]\) is spanned by elements of the form \(\iota(a), a \sim x\) and \([y]\) likewise by elements \(\iota(b)\) with \(b \sim y\), and we have \(a \cdot b \sim x \cdot y\), the first assertion follows from \(\iota(a)\iota(b) = \iota(a \cdot b) \in [x \cdot y]\).

To show that we have equality when \(X_t \cdot X_t = X_t\) for all \(t\), note that \([x \cdot y]\) is spanned by elements of the form \(\iota(a) = \left( \otimes_{s \in S} a_s \right) \otimes \left( \otimes_{t \in T \setminus S} x_{t,y_t} \right)\), where \(S\) is finite. Since each \(a_s \in X_{s,x_s}\) by assumption, it follows that \(\iota(a) \in [x][y]\), which proves the required equality.

(ii) Since \(\ast\) is involutive, it suffices to show that \([x]^* \subseteq [x^*]\). As \([x]^*\) is spanned by elements of the form \(\iota(a)^*\), \(a \sim x\), the assertion follows from \(\iota(a)^* = \iota(a^*)\) with \(a^* \sim x^*\).
(iii) Since the set \( \{ x \in X \mid x_t \in G_t \ \forall t \in T \} \) is a semigroup w.r.t. the componentwise multiplication, the first statement regarding \( \mathcal{M} \) follows from (i). The second statement likewise follows from (ii).

3.11 Remark (a) Regarding the condition \( X_t \cdot X_t = X_t \) in part (i), this is easily fulfilled, since by Theorem 5.2.2 in [Pa94], we know that if \( \mathcal{A} \) is a Banach algebra with a bounded left approximate identity and \( T : \mathcal{A} \to \mathcal{B}(X) \) is a continuous representation of \( \mathcal{A} \) on the Banach space \( X \), then for each \( y \in \text{Span}(T(\mathcal{A})X) \) there are elements \( a \in \mathcal{A} \) and \( x \in X \) with \( y = T(a)x \). Thus, if \( X = \mathcal{A} \) and \( T : \mathcal{A} \to \mathcal{B}(X) \) is defined by \( T(A)B := AB \),\( \) then since \( \mathcal{A} \) has an approximate identity, we have \( \mathcal{A} = \text{Span}(T(\mathcal{A})X) \) and hence \( \mathcal{A} \cdot \mathcal{A} = \mathcal{A} \). In particular, \( \mathcal{A} \cdot \mathcal{A} = \mathcal{A} \) for any \( \mathcal{C}^*\)-algebra \( \mathcal{A} \).

(b) In regard to the choice of semigroup \( G_t \) in (iii) above, when one has unital algebras, the conventional choice is to set all \( G_t = \{ 1 \} \). If the \( \mathcal{C}^*\)-algebras \( X_t \) are nonunital but have projections, then one can take each \( G_t \) to be a projection (cf. Blackadar [Bl77]) though the final tensor product algebra depends on this choice of projections. If the \( \mathcal{C}^*\)-algebras \( X_t \) have no nonzero projections, e.g. \( C_0(\mathbb{R}) \) below, then we will choose each \( G_t \) to be a small \( \mathcal{C}^*\)-closed semigroup generated by one element (which will be positive, of norm 1).

3.3 Tensor products of representations.

Below, we will need to complete some \( \mathcal{C}^*\)-subalgebras of the algebraic tensor product in the operator norm of a suitable representation, hence need to make explicit the structures involved with infinite tensor products of Hilbert space representations.

Let \( (\mathcal{H}_t)_{t \in T} \) be a family of Hilbert spaces. We want to equip selected subspaces of \( \bigotimes_{t \in T} \mathcal{H}_t \) with the inner product \( (\iota(x), \iota(y)) := \left( \prod_{t \in T} (x_t, y_t)_t \right) \) whenever the right hand side makes sense. There are many possibilities, but here we recall the tensor product constructions of von Neumann [vN61]. Let

\[
\mathcal{L} := \left\{ x \in \prod_{t \in T} \mathcal{H}_t \mid \sum_{t \in T} \| x_t \|_t - 1 < \infty \right\}
\]

where we interpret the convergence of a sum (resp. product) over an uncountable set \( T \) as convergence of the net of finite partial sums, resp., products. For sums such as \( S := \sum_{t \in T} \alpha_t, \alpha_t \in \mathbb{C} \), this implies that only countably many summands \( \{ \alpha_{t_n} \mid n \in \mathbb{N} \} \) are non-zero and that \( S = \sum_{n=1}^{\infty} \alpha_{t_n} \), and it converges absolutely (cf. Lemmas 2.3.2 and 2.3.3 in [vN61]). Moreover, we have that \( P = \prod_{t \in T} |\alpha_t| < \infty \) if and only if either \( \alpha_t = 0 \) for some \( t \) (in which case \( P = 0 \)), or else \( \sum_{t \in T} |1 - |\alpha_t| | < \infty \) (cf. [vN61, Lemma 2.4.1]). We will not need to use general products \( P = \prod_{t \in T} \alpha_t, \alpha_t \in \mathbb{C} \), for which the convergence is a more difficult notion (cf. Lemma 2.4.2 and Definition 2.5.1 in [vN61]).

Thus \( x \in \mathcal{L} \) implies that \( \| x_t \|_t = 1 \) for all \( t \in T \setminus R \) where \( R \) is at most countable, and that the product \( \prod_{t \in T} \| x_t \|_t \) converges. Obviously, any \( x \) such that \( \| x_t \|_t = 1 \) for all \( t \in T \) is in \( \mathcal{L} \). Note that if \( x \in \mathcal{L} \) then \( [x]_\approx \subset \mathcal{L} \) also. For \( x, y \in \mathcal{L} \), we define

\[
x \approx y \quad \text{if} \quad \sum_{t \in T} |(x_t, y_t)_t - 1| < \infty.
\]
Then \( \approx \) is an equivalence relation by Lemma 3.3.3 in \cite{vN61}, and we denote its equivalence classes by \([x]_{\approx}\). Observe that if \( x \in \mathcal{L} \) then \([x]_{\approx} \subset [x]_{\approx} \) and moreover, each \( \approx \)-equivalence class contains an \( a \in \mathcal{L} \) such that \( \|a_{t}\|_{t} = 1 \) for all \( t \in T \) (cf. Lemma 3.3.7 in \cite{vN61}).

**3.12 Definition** Given such an \( a \in [x]_{\approx} \subset \mathcal{L} \), we can define an inner product on \([a]_{\approx}\) by sesquilinear extension of
\[
(\iota(x), \iota(y)) := \prod_{t \in T} (x_{t}, y_{t})_{t} \quad \text{for} \quad x \sim a \sim y .
\]
(Note that the infinite products occurring here have only finitely many entries different from 1 hence are unproblematic). Denote the closure of \([a]_{\approx}\) w.r.t. this Hilbert norm by \( \bigotimes_{t \in T} [a]_{\approx} \). Then this is von Neumann’s “incomplete direct product,” and it contains \( \operatorname{Span}(\iota([a]_{\approx})) \) as a dense subspace (cf. Lemma 4.1.2 in \cite{vN61}). The direct sum of the spaces \( \bigotimes_{t \in T} [a]_{\approx} \) where we take one representative \( a \) from each \( \approx \)-equivalence class, is von Neumann’s “complete direct product” (cf. Lemma 4.1.1 in \cite{vN61}). An analogous associativity theorem to Theorem 3.3 holds for this complete direct product (cf. Theorem VII in \cite{vN61}).

Next, consider the case where \((\mathcal{A}_{t})_{t \in T}\) is a family of \(*\)-algebras, each equipped with a bounded Hilbert space \(*\)-representation \( \pi_{t} : \mathcal{A}_{t} \to \mathcal{B}(\mathcal{H}_{t}) \). For any \( A \in \prod_{t \in T} \mathcal{A}_{t} \) we can define a linear map \( \pi(\iota_{\mathcal{A}}(A)) \) on \( \bigotimes_{t \in T} \mathcal{H}_{t} \) by
\[
\pi(\iota_{\mathcal{A}}(A)) \iota(x) = \bigotimes_{t \in T} \pi_{t}(A_{t})x_{t} = \iota(A) x \quad \text{for all} \quad x \in \prod_{t \in T} \mathcal{H}_{t}
\]
where \( (\pi(A)x)_{t} := \pi_{t}(A_{t})x_{t} \) for all \( t \in T \). Then \( \pi \) is a representation, because it is one for each entry. To obtain Hilbert space \(*\)-representations from \( \pi \), we need to restrict it to suitable pre-Hilbert subspaces of \( \bigotimes_{t \in T} \mathcal{H}_{t} \) hence need to restrict to those \( A \) such that \( \pi(\iota_{\mathcal{A}}(A)) \) preserves the Hilbert space involved (and produces a bounded operator).

**3.13 Definition** Consider the Hilbert space completion \( \bigotimes_{t \in T} [a]_{\approx} \) of \([a]_{\approx}\), as above. When the algebras \( \mathcal{A}_{t} \) are all unital, then \( [1] \subset \bigotimes_{t \in T} \mathcal{A}_{t} \) is a \(*\)-subalgebra, where \( (1)_{t} = 1_{t} \in \mathcal{A}_{t} \) for all \( t \in T \). Then \( \pi(A)x \in [a]_{\approx} \) for all \( x \in [a]_{\approx} \subset \prod_{t \in T} \mathcal{H}_{t} \) and \( A \sim 1 \). In particular, \( \pi(\iota_{\mathcal{A}}(A)) \) preserves \([a]_{\approx}\) and it is bounded, since it is a tensor product of a finite tensor product (of bounded operators) with the identity operator. Thus it extends to a bounded operator on \( \bigotimes_{t \in T} [a]_{\approx} \). This defines a \(*\)-representation of the \(*\)-algebra \([1]\) on the (stabilized) tensor product \( \bigotimes_{t \in T} [a]_{\approx} \), and it is the most commonly used definition of a tensor representation.

When the \(*\)-algebras \( \mathcal{A}_{t} \) are not unital, consider the case where they contain nontrivial hermitian projections \( P_{t} \in \mathcal{A}_{t} \). Then, for any choice of such projections \( P \in \prod_{t \in T} \mathcal{A}_{t} \), the subspace \([P] \subset \bigotimes_{t \in T} \mathcal{A}_{t}\) is a \(*\)-subalgebra. For any \( a \in \prod_{t \in T} \mathcal{H}_{t} \) with \( \pi_{t}(P_{t})a_{t} = a_{t} \) for all \( t \in T \), we can now define a tensor product representation of \([P] \) on \( \bigotimes_{t \in T} [a]_{\approx} \). Below we will consider more general tensor product representations.
4 Semi–host algebras for Gaussians

In this section, $\mu$ will be a fixed Gaussian product measure on $\mathbb{R}^N$ and $\mu_n$ denotes its projection on the $n$th component. For $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^n$, we write $\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$ for the standard pairing. Recall that from $\mu$ one constructs a unitary representation

$$\pi_\mu : \mathbb{R}^n \to \mathcal{U}(L^2(\mathbb{R}^n, \mu)) \quad \text{by} \quad (\pi_\mu(x)f)(y) := \exp(i\langle x, y \rangle) f(y), \quad x \in \mathbb{R}^n, \ y \in \mathbb{R}^N.$$  

Then there is a unitary map $U : \bigotimes_{n=1}^{\infty} \mathcal{H}_n \to L^2(\mathbb{R}^N, \mu)$, where $\mathcal{H}_n := L^2(\mathbb{R}, \mu_n)$. The sequence $e = (e_1, e_2, \ldots)$ of stabilizing vectors $e_n \in \mathcal{H}_n$ is given by the constant functions $e_n(x) = 1$ for all $x \in \mathbb{R}$. Explicitly, $U$ is given by

$$U(f_1 \otimes f_2 \otimes \cdots \otimes f_k \otimes e_{k+1} \otimes e_{k+2} \otimes \cdots)(x_1, x_2, \ldots) = f_1(x_1) \cdot f_2(x_2) \cdots f_k(x_k)$$

which is clearly a cylinder function on $\mathbb{R}^N$. Then $\pi_\mu = U \left( \bigotimes_{n=1}^{\infty} \pi_{\mu_n} \right) U^{-1}$, where each

$$\pi_{\mu_n} : \mathbb{R} \to \mathcal{U}(L^2(\mathbb{R}, \mu_n)) \quad \text{is} \quad (\pi_{\mu_n}(x)f)(y) := e^{ixy} f(y) \quad \text{for} \quad x, y \in \mathbb{R}.$$  

The stabilizing sequence defines a cyclic vector $\Omega := \bigotimes_{n=1}^{\infty} e_n$. Immediate calculation establishes that the corresponding positive definite function satisfies:

$$\omega_\mu(t) := \langle \Omega, \pi_\mu(t)\Omega \rangle = \int_{\mathbb{R}^N} \exp(i\langle t, y \rangle) \, d\mu(y) \quad \text{for} \quad t \in \mathbb{R}^n,$$

which is part of the Bochner–Minlos Theorem (cf. [GV64]). We will show that it expresses the decomposition of a state into the pure states of a (semi-) host algebra for $\mathbb{R}^n$, and that there is a similar expression for other states (which is also part of the Bochner–Minlos theorem).

Specialize the notation of the last section by setting: $T = \mathbb{N}$ and $X_t = C_0(\mathbb{R}) \cong C^*(\mathbb{R})$ for all $t$. We first try to define an appropriate infinite tensor product $C^*$-algebra of all the $C_0(\mathbb{R})$’s, which seems to be a problem because $C_0(\mathbb{R})$ is nonunital, and has no nontrivial projection. By the last section we always have the algebraic tensor product $\bigotimes_{k=1}^{\infty} C_0(\mathbb{R})$, but this is too large. We want to look at its $*$-subalgebra of the type defined in Theorem 3.10(iii), and will consider the following multiplicative semigroups in $C_0(\mathbb{R})$. For each $n \in \mathbb{N}$, define

$$V_n := \{ f \in C_0(\mathbb{R}) \mid f(\mathbb{R}) \subseteq [0,1], \ f \upharpoonright [-n,n] = 1, \ \text{supp}(f) \subseteq [-n-1, n+1] \}$$

and observe that it is a semigroup, that $\|f\| = 1$ for all $f \in V_n$ and that any sequence $\{u_n \in V_n \mid n \in \mathbb{N} \}$ is an approximate identity for $C_0(\mathbb{R})$. Moreover $V_n \cdot V_m = V_n$ if $m > n$ and hence $\bigcup_{n=1}^{\infty} V_n$ is a semigroup. For each $f \in V_n$ we have the subsemigroup

$$V_n(f) := \{ f^k \mid k \in \mathbb{N} \} \subset V_n,$$

and for these we also have that $V_n(f) \cdot V_m(g) = V_n(f)$ if $m > n$.

For any sequence $f = (f_1, f_2, \ldots) \in C_0(\mathbb{R})^\infty$ with $f_n \in V_{k_n}$ for all $n$, we consider the $*$-algebra generated in $\bigotimes_{k=1}^{\infty} C_0(\mathbb{R})$ by $[f]$, and note that

$$\text{*-alg([f])} = \text{Span}\{ [f^k] \mid k \in \mathbb{N} \} \subset \bigotimes_{i=1}^{\infty} C_0(\mathbb{R}), \quad \text{where} \quad (f^k)_n := f^k_n \forall n$$  

(3)
and for the equality we needed the fact that $C_0(\mathbb{R}) \cdot C_0(\mathbb{R}) = C_0(\mathbb{R})$ (Remark \ref{rem:linharmprod1}, and Theorem \ref{thm:linharmprod}).

Next, we want to define a convenient representation of $*\text{-alg}(\left[\ell^1\right])$ to provide us with a $C^*$-norm to close it in. We will show that there are $f$ for which we can define a representation of $*\text{-alg}(\left[\ell^1\right])$ on $\bigotimes_{n=1}^\infty e_n H_n$ in a natural way.

4.1 Proposition We now have:

(i) Let $P_k$ denote multiplication of functions on $\mathbb{R}$ by $\chi_{[-k,k]}$. Then there exists a sequence $(k_i)_{i \in \mathbb{N}}$ such that $\sum_{n=1}^\infty |(P_{k_n} e_n, e_n) - 1| < \infty$.

(ii) Fix a sequence $(k_i)_{i \in \mathbb{N}}$ as in (i) as well as $f \in \bigotimes_{j=1}^\infty V_{k_j}$. Then there is a $*\text{-representation}$

$$\pi_e : *\text{-alg}(\left[f\right]) \to B\left(\bigotimes_{n=1}^\infty e_n H_n\right)$$

such that

$$\pi_e\left(\bigotimes_{n=1}^\infty g_n \otimes \cdots \otimes \cdots \otimes \cdots \otimes e_n \right) = \bigotimes_{n=1}^\infty g_n e_n \in \bigotimes_{n=1}^\infty e_n H_n$$

for all $g \sim f^\ell$, $c \sim e$ and $\ell \in \mathbb{N}$, and where $g_n e_n$ is the usual pointwise product of functions on $\mathbb{R}$.

Proof. (i) For any $\varepsilon > 0$, there is a $k \in \mathbb{N}$ such that $|(P_k e_n, e_n) - 1| < \varepsilon$ by the Monotone Convergence Theorem. Thus there is a sequence $(k_i)_{i \in \mathbb{N}}$ such that $\sum_{n=1}^\infty |(P_{k_n} e_n, e_n) - 1| < \infty$.

(ii) Recall from Definition \ref{def:linharmprod2} that $\text{Span}(\left(e_i\right))$ is dense in the closure $\bigotimes_{n=1}^\infty e_n H_n$ of $\left[e\right]$, where

$$\left[e\right] = \left\{ v \in \prod_{n=1}^\infty H_n \mid \sum_{n=1}^\infty \|v_n\| - 1 < \infty \text{ and } \sum_{n=1}^\infty (e_n, v_n) - 1 < \infty \right\}.$$

With the given choice of $(k_i)_{i \in \mathbb{N}}$ and $f$ we have

$$(P_{k_n} e_n, e_n) = \mu_n([-k_n, k_n]) \leq \int_{-k_n}^{k_n+1} f_n(x) \, d\mu_n(x) = (f_n e_n, e_n) \leq 1$$

so that $|(f_n e_n, e_n) - 1| \leq |(P_{k_n} e_n, e_n)|$, and hence $\sum_{n=1}^\infty |(f_n e_n, e_n) - 1| < \infty$. As $(f_j)_{j=1}^\ell \in V_{k_j}$ for all $\ell \in \mathbb{N}$, we have in fact that $\sum_{n=1}^\infty |(f_n^\ell e_n, e_n) - 1| < \infty$ for all $\ell \in \mathbb{N}$. This implies that $\sum_{n=1}^\infty \|f_n^\ell e_n\| - 1 < \infty$ which implies via Lemma 3.3.2 in \cite{VN61} that $\sum_{n=1}^\infty \|f_n^\ell e_n\| < \infty$. Hence $f^\ell e \in \left[e\right]$ and so

$$\left(\bigotimes_{n=1}^\infty f_n^\ell \right)\left(\bigotimes_{k=1}^\infty e_k\right) = \bigotimes_{n=1}^\infty f_n^\ell e_n \in \bigotimes_{n=1}^\infty e_n H_n.$$

Since any $c \sim e$ only differs from $e$ in finitely many entries, the convergence arguments above will still hold if we replace $e$ by $c$. Likewise, we can replace $f^\ell$ by any $g^\ell$, i.e., we have shown that $\bigotimes_{n=1}^\infty g_n e_n \in \bigotimes_{n=1}^\infty e_n H_n$ for all $g \sim f^\ell$ and $c \sim e$. Since the multiplication map

$$\left(\bigcup_{\ell \in \mathbb{N}} \left[f^\ell\right]\right) \times [e] \to \bigotimes_{n=1}^\infty H_n, \quad (g, c) \mapsto \bigotimes_{n=1}^\infty g_n e_n$$
is multilinear, it defines a bilinear map on $\operatorname{Span} \left( \bigcup_{k \in \mathbb{N}} \left[ f^k \right] \right) \times \left[ e \right]$, denoted by $(a, b) \mapsto \pi_e(a)b$, thus obtaining the formula for $\pi_e$ in the theorem. That $\pi_e$ is a representation of $\ast$-$\operatorname{alg}(\left[ f \right])$ follows from the explicit formula, and the $\ast$-property is also clear. It remains to show that each $\pi_e(a)$ is bounded (hence extends as a bounded operator to $\bigotimes_{n=1}^\infty \mathcal{H}_n$). It suffices to check this for the generating elements of $\ast$-$\operatorname{alg}(\left[ f \right])$. Let $a \in \left[ f \right]$ with $a \sim f$

$$a = (a_1 \otimes \cdots \otimes a_p) \otimes f_{p+1} \otimes f_{p+2} \otimes \cdots$$

for some $p < \infty$. Moreover any $b \in \left[ e \right]$ can also be written in the form:

$$b = b_p \otimes e_{p+1} \otimes e_{p+2} \otimes \cdots \quad \text{with} \quad b_p \in \bigotimes_{j=1}^p \mathcal{H}_j,$$

where we may take the same $p$ as in the preceding expression (e.g. by adjusting the initial part). Then

$$\|\pi_e(a)b\| = \|A_pb\| \cdot \prod_{k=p+1}^\infty \|f_k e_k\|, \quad \text{where} \quad A_pv = (a_1 \otimes \cdots \otimes a_p)v.$$ 

Since $A_p$ is bounded on the completion $\bigotimes_{j=1}^\infty \mathcal{H}_j$ of $\bigotimes_{j=1}^p \mathcal{H}_j$, we have $\|A_pb\| \leq \|A_p\| \cdot \|b_p\|$, and as $\|f_k e_k\| \leq \|e_k\| = 1$, we see that

$$\|\pi_e(a)b\|^2 = \|A_p\|^2 \|b_p\|^2 \cdot \prod_{k=p+1}^\infty \|e_k\|^2 = \|A_p\|^2 \cdot \|b\|^2$$

and hence $\pi_e(a)$ is a bounded operator on $\left[ e \right]$ so extends to a bounded operator on $\bigotimes_{n=1}^\infty \mathcal{H}_n$. ■

4.2 Definition Thus for any $f \in \prod_{j=1}^\infty V_{k_j}$, we can define

$$\mathcal{L}_\mu[f] := C^\ast \left( \pi_e \left( \ast \operatorname{alg}(\left[ f \right]) \right) \right) \subset B \left( \bigotimes_{n=1}^\infty \mathcal{H}_n \right).$$

4.3 Remark Recall that we also have the unitaries $\pi_\mu(\mathbb{R}^N) \subset \mathcal{U} \left( \bigotimes_{n=1}^\infty \mathcal{H}_n \right)$, where

$$\pi_\mu(x) \bigotimes_{k=1}^\infty c_k = \bigotimes_{n=1}^\infty \left( \exp_{xn} \cdot c_n \right) \in \left[ e \right], \quad x \in \mathbb{R}^N, \quad c \sim e, \quad \exp_{xn}(t) := e^{ixn t}.$$ 

Then

$$\pi_\mu(x) \cdot \pi_e(\ell(g)) = \pi_e(\ell(g)) \cdot \pi_\mu(x) = \pi_e \left( \bigotimes_{n=1}^\infty \left( \exp_{xn} \cdot g_n \right) \right) \in \mathcal{L}_\mu[f],$$

for all $x \in \mathbb{R}^N$, $g \sim f^\ell$ and $\ell \in \mathbb{N}$. The inclusion needed the fact that $x$ has only finitely many nonzero entries, and that $\exp_{xn}C_0(\mathbb{R}) \subset C_0(\mathbb{R})$. Thus $\pi_\mu(\mathbb{R}^N) \cdot \mathcal{L}_\mu[f] \subset \mathcal{L}_\mu[f]$. Since for each $x \in \mathbb{R}^N$ we can find a sequence $\left( \ell(g_n) \right)_{n \in \mathbb{N}} \subset \left[ f \right]$ such that $\pi_e(\ell(g_n)) \cdot \pi_\mu(x)$ converges in norm to $\pi_\mu(x)$, we have a faithful embedding of $\mathbb{R}^N$ as unitaries into the multiplier algebra $\mathcal{M}(\mathcal{L}_\mu[f])$ denoted $\eta: \mathbb{R}^N \to \mathcal{M}(\mathcal{L}_\mu[f])$. In the next section we will investigate to what extent $\mathcal{L}_\mu[f]$ is a host algebra of $\mathbb{R}^N$. 

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4.4 Lemma With $f$ as in Proposition 4.1(ii), we have

(i) The $C^*$-algebra $L_\mu[f]$ is separable.

(ii) Let $\omega$ be a pure state on $L_\mu[f]$, and let $\tilde{\omega}$ be its strict extension to the unitaries $\eta(\mathbb{R}^N) \subset M(L_\mu[f])$. Then $\tilde{\omega} \circ \eta$ is a character and there exists an element $a \in \mathbb{R}^N$ with $\tilde{\omega}(\eta(x)) = \exp(i \langle x, a \rangle)$ for all $x \in \mathbb{R}^N$.

Proof. (i) Since $\pi_\mu\left(*-\text{alg}(\mathbb{F})\right)$ is dense in $L_\mu[f]$, where

$$*\text{-alg}(\mathbb{F}) = \text{Span}\{ \mathbb{F}^k \mid k \in \mathbb{N} \}$$

and $\mathbb{F}^k = \bigcup_{m=1}^{\infty} \left\{ \bigotimes_{\ell=1}^{m} C_0(\mathbb{R}) \otimes f_{m+1}^k \otimes f_{m+2}^k \otimes \cdots \right\}$,

(i) follows immediately from the separability of $C_0(\mathbb{R})$.

(ii) As $L_\mu[f]$ is commutative, any pure state $\omega$ of it is a point evaluation, hence a $*-\text{homomorphism}$. Thus the strict extension $\tilde{\omega}$ to $\eta(\mathbb{R}^N) \subset M(L_\mu[f])$ is also a $*-\text{homomorphism}$, hence $\tilde{\omega} \circ \eta$ is a character. The restriction of $\tilde{\omega} \circ \eta$ to the subgroup $\mathbb{R}^n \subset \mathbb{R}^N$ is still a character, and it is continuous (since it is determined by the factor $\otimes_{j=1}^{n} C_0(\mathbb{R})$ in $L_\mu[f]$ which is the group algebra of $\mathbb{R}^n$) hence of the form $\tilde{\omega} \circ \eta(x) = \exp(i \langle x \cdot a^{(n)} \rangle)$ for some $a^{(n)} \in \mathbb{R}^n$. Since $\tilde{\omega} \circ \eta$ is a character on all of $\mathbb{R}^N$, the family $\{ a^{(n)} \in \mathbb{R}^n \mid n \in \mathbb{N} \}$ is a consistent family, i.e., if $n < m$ then $a^{(n)}$ is the first $n$ entries of $a^{(m)}$. Thus there is an $a \in \mathbb{R}^N$ such that $a^{(n)}$ is the first $n$ entries of $a$ for any $n \in \mathbb{N}$. Then $\tilde{\omega} \circ \eta(x) = \exp(i \langle x, a \rangle)$ since for any $x \in \mathbb{R}^n \subset \mathbb{R}^N$ this restricts to the previous formula for $\tilde{\omega} \circ \eta$.

Since $L_\mu[f]$ is separable and commutative, it follows from Theorem II.2.2 in [Da96] that all its cyclic representations are multiplicity free, and hence by Theorem 4.9.4 in [Ped89], for any state $\omega$ on $L_\mu[f]$, there is a regular Borel probability measure $\nu$ on the states $\mathcal{G}(L_\mu[f])$ concentrated on the pure states $\mathcal{E}_\mu(L_\mu[f])$ such that

$$\omega(A) = \int_{\mathcal{E}_\mu(L_\mu[f])} \varphi(A) \, d\nu(\varphi) \quad \forall A \in L_\mu[f].$$

We will show that this decomposition produces similar decompositions to the one in [2] for other continuous positive definite functions than $\omega_\mu$.

Since $L_\mu[f]$ is separable, it has a countable approximate identity $\{ E_n \}_{n \in \mathbb{N}} \subset L_\mu[f]$ (cf. Remark 3.1.1 [Mu90]). For a state $\omega$ on $L_\mu[f]$, let $\tilde{\omega}$ be its strict extension to the unitaries $\eta(\mathbb{R}^N) \subset M(L_\mu[f])$, then we have for any countable approximate identity $\{ E_n \}_{n \in \mathbb{N}} \subset L_\mu[f]$ that

$$\tilde{\omega} \circ \eta(x) = \lim_{n \to \infty} \omega(\eta(x) E_n) = \lim_{n \to \infty} \int_{\mathcal{E}_\mu(L_\mu[f])} \varphi(\eta(x) E_n) \, d\nu(\varphi)$$

$$= \int_{\mathcal{E}_\mu(L_\mu[f])} \lim_{n \to \infty} \varphi(\eta(x) E_n) \, d\nu(\varphi) = \int_{\mathcal{E}_\mu(L_\mu[f])} \tilde{\varphi} \circ \eta(x) \, d\nu(\varphi)$$

where we used the Lebesgue dominated convergence theorem in the second line, since $|\varphi(\eta(x) E_n)| \leq 1$ and the constant function 1 is integrable.

By Lemma 4.4(ii) we can define a map

$$\xi : \mathcal{G}_\mu(L_\mu[f]) \to \mathbb{R}^N \quad \text{by} \quad \tilde{\varphi} \circ \eta(x) = \exp\left(i \langle x, \xi(\varphi) \rangle \right) \quad \text{for} \quad x \in \mathbb{R}^N,$$
so using $\xi$ we define a probability measure $\bar{\nu}$ on $\mathbb{R}^N$ by $\bar{\nu} := \xi \nu$, and so:

$$\bar{\omega} \circ \eta(x) = \int_{\mathbb{R}^N} \exp \left( i\langle x, y \rangle \right) d\bar{\nu}(y) \quad \text{for} \quad x \in \mathbb{R}^N,$$

which generalises the integral representation (2) to those positive definite functions $\bar{\omega}$ which are strict extensions of states of $L_\mu[\mathcal{F}]$ (and this includes $\omega_\mu$). We will obtain the full Bochner–Minlos theorem for $\mathbb{R}^N$ in a $C^*$-algebraic context, if we can show that every continuous normalized positive definite function is of this type for some $\mu$ and some $f$. This is what we will do in the next section.

5 Semi-host algebras for $\mathbb{R}^N$

Inspired by the good properties which we found for $L_\mu[\mathcal{F}]$ above, we now examine more general versions of these algebras. The semi-host algebras which we obtain will be the building blocks for the algebra hosting the full representation theory of $\mathbb{R}^N$, which will be constructed in the next section.

For the rest of this section we fix a sequence $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $f \in \prod_{n=1}^\infty V_{k_n}$ such that $\llbracket f \rrbracket \neq 0$. Then we have that

$$\ast\text{-alg}(\llbracket f \rrbracket) = \text{Span}\{ \llbracket f^k \rrbracket \mid k \in \mathbb{N} \} = \lim_{\longrightarrow \infty} A_m[\mathcal{F}], \quad \text{where}$$

$$A_m[\mathcal{F}] := \text{Span}\{ A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^1 \otimes f_{m+2}^2 \otimes \cdots \mid A_i \in C_0(\mathbb{R}) \ \forall i \in \mathbb{N}, \ k \in \mathbb{N} \}$$

and the inductive limit is w.r.t. set inclusion of the $\ast$-algebras $A_m[\mathcal{F}] \subset A_\ell[\mathcal{F}]$ if $m < \ell$. By the Associativity Theorem we can write

$$A_m[\mathcal{F}] = \left( \bigotimes_{k=1}^m C_0(\mathbb{R}) \right) \otimes \left( \ast\text{-alg} \left( \bigotimes_{j=m+1}^\infty f_j \right) \right).$$

The natural $C^*$-norm on the first factor is clear, but not on the second factor. So we next investigate possible bounded $\ast$-representations to provide $\ast\text{-alg}(\llbracket f \rrbracket)$ with a $C^*$-norm. Since $\ast\text{-alg} \left( \bigotimes_{j=m+1}^\infty f_j \right)$ is generated by the single element $E := \bigotimes_{j=m+1}^\infty f_j$, any representation $\pi$ of this $\ast$-algebra is given by specifying the single operator $\pi(E)$. Since $E$ is positive, we require $\pi(E) \geq 0$, and as we want a tensor norm on the larger $\ast\text{-alg}(\llbracket f \rrbracket)$, we need that $\|\pi(E)\| \leq \prod_{j=m+1}^\infty \|f_j\| = 1$.

5.1 Lemma Let $f \in \prod_{n \in \mathbb{N}} V_{k_n}$ and let $\{ \pi_k : C_0(\mathbb{R}) \rightarrow B(\mathcal{H}) \mid k \in \mathbb{N} \}$ be a set of $\ast$-representations on the same space with commuting ranges. Then

(i) The strong limit $F^\ell_k := \lim_{n \rightarrow \infty} \pi_k(f^\ell_k) \cdots \pi_n(f^\ell_n) \in B(\mathcal{H})$ exists, and $0 \leq F^\ell_k \leq 1$ for $k, \ell \in \mathbb{N}$.

(ii) $P[\mathcal{F}] := \lim_{k \rightarrow \infty} F^\ell_k$ (an increasing limit) is a projection independent of $\ell \in \mathbb{N}$ satisfying $F^\ell_k P[\mathcal{F}] = P[\mathcal{F}] = F^\ell_k$.

(iii) Let $Q \in B(\mathcal{H})$ be such that $0 \leq Q \leq 1$, and such that it commutes with $\pi_k(C_0(\mathbb{R}))$ for each $k \in \mathbb{N}$. Let $A := A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^\ell \otimes f_{m+2}^\ell \otimes \cdots \in \ast\text{-alg}(\llbracket f \rrbracket)$ and define

$$\pi_Q(A) := \pi_1(A_1) \pi_2(A_2) \cdots \pi_m(A_m) F^\ell_{m+1} Q^\ell.$$

Then $\pi_Q$ defines a $\ast$-representation $\pi_Q : \ast\text{-alg}(\llbracket f \rrbracket) \rightarrow B(\mathcal{H})$. 

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(iv) The representation $\pi_Q$ is non-degenerate if and only if all $\pi_i$ are non-degenerate, $P[f] = 1$ and $\ker Q = \{0\}$. If $\pi_Q$ is degenerate, $\ker Q = 0$, and all $\pi_j$ are non-degenerate, then $P[f]$ is the projection onto the essential subspace of $\pi_Q$.

Proof. (i) Since the operators $\pi_k(f_k^\ell), \pi_j(f_j^\ell) \in \mathcal{B}(\mathcal{H})$ commute and are positive, it follows from joint spectral theory that their product $\pi_k(f_k^\ell) \cdot \pi_j(f_j^\ell)$ is also a positive operator. From $\pi_k(f_k^\ell) \leq 1$ for all $k, \ell \in \mathbb{N}$, we derive that $\pi_k(f_k^\ell) \cdot \pi_j(f_j^\ell) \leq \pi_k(f_k^\ell)$ and hence, for a fixed $k$, the operators $C_n := \pi_k(f_k^\ell) \cdots \pi_n(f_n^\ell)$ form a decreasing sequence of commuting positive operators. Thus, by [Mu90 Thm 4.1.1, p. 113], $C_n$ converges in the strong operator topology to some limit $F_k^{(\ell)}$. It is clear that $F_k^{(\ell)}$ is positive, and using

$$
\|T\| = \sup \{ |\langle \psi, T\psi \rangle | : \psi \in \mathcal{H}, \|\psi\| = 1 \},
$$

it follows from $\|C_n\| = \|\pi_k(f_k^\ell) \cdots \pi_n(f_n^\ell)\| \leq 1$ for all $n$ that $\|F_k^{(\ell)}\| \leq 1$ and hence that $0 \leq F_k^{(\ell)} \leq 1$.

(ii) By definition, $F_k^{(\ell)} = \pi_k(f_k^\ell) F_{k+1}^{(\ell)}$ and $0 \leq \pi_k(f_k^\ell) \leq 1$ and so the commuting sequence of operators $(F_k^{(\ell)})_{k \in \mathbb{N}}$ is increasing, and bounded above by 1. Thus it follows again from Theorem 4.1.1 in [Mu90] that the strong limit $P^{(\ell)}[f] := s-lim_{k \to \infty} F_k^{(\ell)}$ exists, is positive and bounded above by 1. Since the operator product is jointly strong operator continuous on bounded sets, we get

$$
F_k^{(\ell)} P^{(\ell)}[f] = s-lim_{n \to \infty} \pi_k(f_k^\ell) \cdots \pi_{n-1}(f_{n-1}^\ell) \cdot s-lim_{n \to \infty} F_n^{(\ell)} = s-lim_{n \to \infty} \pi_k(f_k^\ell) \cdots \pi_{n-1}(f_{n-1}^\ell) F_n^{(\ell)} = s-lim_{n \to \infty} F_k^{(\ell)} = F_k^{(\ell)}.
$$

Thus by $P^{(\ell)}[f] = s-lim_{k \to \infty} F_k^{(\ell)} = s-lim_{k \to \infty} P^{(\ell)}[f] = (P^{(\ell)}[f])^2$ and the fact that $P^{(\ell)}[f]$ is positive we conclude that it is a projection. To see that $P^{(\ell)}[f]$ is independent of $\ell$, note that for $k \leq m$ we have:

$$
F_k^{(\ell)} F_m^{(j)} = s-lim_{n \to \infty} \pi_k(f_k^\ell) \cdots \pi_n(f_n^\ell) \cdot s-lim_{p \to \infty} \pi_m(f_m^j) \cdots \pi_p(f_p^j) = s-lim_{n \to \infty} \pi_k(f_k^\ell) \cdots \pi_m(f_m^j) \pi_m(f_m^{j+1}) \cdots \pi_n(f_n^{j+1}) = \pi_k(f_k^\ell) \cdots \pi_m(f_m^{j+1}) F_m^{(\ell+j)}.
$$

This leads to

$$
P^{(\ell)}[f] \cdot P^{(j)}[f] = s-lim_{k \to \infty} F_k^{(\ell)} s-lim_{m \to \infty} F_m^{(j)} = s-lim_{n \to \infty} F_n^{(\ell)} F_n^{(j)} = s-lim_{n \to \infty} F_n^{(\ell+j)} = P^{(\ell+j)}[f].
$$

However, each $P^{(\ell)}[f]$ is idempotent, i.e., $P^{(\ell)}[f] = P^{(2\ell)}[f]$ for all $\ell \in \mathbb{N}$, hence $P^{(\ell)}[f]$ is independent of $\ell$.

(iii) Since $* \cdot \alpha \cdot \left( F_{m+1}^{(\ell)} \right) = \lim_{m \to \infty} \mathcal{A}_m[f] = \bigcup_{m \in \mathbb{N}} \mathcal{A}_m[f]$, it suffices to show that $\pi_Q$ defines a *-representation on each *-algebra $\mathcal{A}_m[f]$, and that $\pi_Q$ restricts to its correct values on any $\mathcal{A}_k[f] \subset \mathcal{A}_m[f]$ for $k \leq m$. Recall that

$$
\mathcal{A}_m[f] = \left( \bigotimes_{k=0}^m C_0(\mathbb{R}) \right) \otimes \left( * \cdot \alpha \cdot \left( \bigotimes_{j=m+1}^{\infty} f_j \right) \right).
$$

Now

$$
\pi^{(m)}_a : \bigotimes_{k=0}^m C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}), \quad \pi^{(m)}_a(A_1 \otimes \cdots \otimes A_m) := \pi_1(A_1) \cdots \pi_m(A_m)
$$

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is a well-defined \(*\)-representation obtained by the universal property of the tensor product. Moreover, since \(*\)-alg\(\left(\bigotimes_{j=m+1}^{\infty} f_j\right)\) is generated by a single element not satisfying any polynomial relation, the assignment \(\pi_b^{(m)}\left(\bigotimes_{j=m+1}^{\infty} f_j\right) := F_{m+1}^{(1)} Q \geq 0\) defines a \(*\)-representation \(\pi_b^{(m)} : \text{*-alg}\(\bigotimes_{j=m+1}^{\infty} f_j\) \rightarrow \mathcal{B}(\mathcal{H})\). Note from Equation (7) that \(F_{m+1}^{(k)} F_{m+1}^{(1)} = F_{m+1}^{(k+1)}\), which leads to the factorization

\[
\pi_Q(A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^{( k)} \otimes f_{m+2}^{( k)} \otimes \cdots) = \pi_a^{(m)}(A_1 \otimes \cdots \otimes A_m) \cdot \pi_b^{(m)}\left(\bigotimes_{j=m+1}^{\infty} f_j\right)
\]

Thus, since it is multilinear, we obtain a linear map \(\pi_Q\) on \(A_m[\ell]\), and as the ranges of the \(*\)-representations \(\pi_a\) and \(\pi_b\) commute, \(\pi_Q\) is a \(*\)-representation on \(A_m[\ell]\). For \(k < m\) we have from the definition that

\[
\pi_b^{(k)}\left(\bigotimes_{j=k+1}^{\infty} f_j\right) = F_{k+1}^{(1)} Q = \pi_{k+1}(f_{k+1}) \cdots \pi_m(f_m) F_{m+1}^{(1)} Q
\]

and hence

\[
\pi_a^{(m)}(A_1 \otimes \cdots \otimes A_k \otimes f_{k+1} \cdots \otimes f_m) \cdot \pi_b^{(m)}\left(\bigotimes_{j=m+1}^{\infty} f_j\right) = \pi_a^{(k)}(A_1 \otimes \cdots \otimes A_k) \cdot \pi_b^{(k)}(\bigotimes_{j=k+1}^{\infty} f_j)
\]

so it is clear that the value of \(\pi_Q\) on \(A_m[\ell] \subset A_m[\ell]\) is the same as the restriction of the map \(\pi_Q\) defined on \(A_m[\ell]\). Hence \(\pi_Q\) is consistently defined as a \(*\)-representation of \(*\text{-alg}(\ell)]\).

(iv) Note that by \(F_{k}^{(1)} P[\ell] = F_{k}^{(1)}\), we have \(\pi_Q(A) P[\ell] = \pi_Q(A)\) for all \(A \in \text{-alg}(\ell)]\), hence, if \(P[\ell] \neq 1\), then \(\pi_Q(\text{-alg}(\ell)]\) has null spaces, i.e., \(\pi_Q\) is degenerate. Likewise, if \(\text{Ker} Q \neq \{0\}\) then \(\pi_Q\) is degenerate. Moreover, if any \(\pi_i\) is degenerate, then since by commutativity:

\[
\pi_Q(A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^{( k)} \otimes f_{m+2}^{( k)} \otimes \cdots) = \pi_1(A_1) \cdots \pi_i(A_i) \cdots \pi_m(A_m) F_{m+1}^{(k)} Q^\ell \pi_i(A_i),
\]

where the hat means omission, it follows that \(\pi_Q\) is also degenerate.

Conversely, let \(\pi_Q\) be degenerate, i.e., there is a nonzero \(\psi \in \mathcal{H}\) such that \(\pi_Q(A) \psi = 0\) for all \(A\), hence

\[
\pi_Q(A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^{( k)} \otimes f_{m+2}^{( k)} \otimes \cdots) \psi = \pi_1(A_1) \cdots \pi_m(A_m) F_{m+1}^{(k)} Q^\ell \psi = 0
\]

for all \(A_i \in C_0(\mathbb{R})\) and \(m, \ell \in \mathbb{N}\). If all \(\pi_j\) are non-degenerate, then it follows inductively that \(F_{m}^{(k)} Q^\ell \psi = 0\) for all \(m\) and \(\ell\). If \(\text{Ker} Q = 0\), then \(F_{m}^{(k)} \psi = 0\) for all \(m\), hence \(P[\ell] \psi = 0\), i.e., \(P[\ell] \neq 1\).

By the last step we also see that when \(\pi_Q\) is degenerate, \(\text{Ker} Q = 0\), and all \(\pi_j\) are non-degenerate, then \(P[\ell] = 0\) is zero on the null space of \(\pi_Q\). Since \(F_{k}^{(1)} P[\ell] = F_{k}^{(1)}\) by (ii) it follows from the definition of \(\pi_Q\) that \(\pi_Q(A) P[\ell] = \pi_Q(A)\) for all \(A \in \text{-alg}(\ell)]\). Thus \(P[\ell]\) is the identity on the essential subspace of \(\pi_Q\), i.e. it is the projection onto this essential subspace. ■

### 5.2 Definition
Using this lemma, we can now investigate natural representations of \(*\text{-alg}(\ell)]\). Start with the universal representation of \(\mathbb{R}^{(N)}\) denoted \(\pi_u : \mathbb{R}^{(N)} \rightarrow \mathcal{U}((\mathcal{H}_u))\) which we recall, is the direct sum of the cyclic strong–operator continuous unitary representations of \(\mathbb{R}^{(N)}\), one from each unitary equivalence class. For the \(k^{th}\) component we have an inclusion \(\mathbb{R} \subset \mathbb{R}^{(N)}\) by \(x \mapsto\)
(0,\ldots,0,x,0,\ldots) \ (k^{th} \text{ entry}), \ \pi_u \ \text{restricts to a representation on the } k^{th} \text{ component, denoted by } \pi_k^u : \mathbb{R} \to \mathcal{U}(\mathcal{H}_u). \ \text{By the host algebra property of } C^*(\mathbb{R}) \cong C_0(\mathbb{R}), \ \text{this produces a unique representation } \pi_k^u : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_u), \ \text{which is non-degenerate. Since the set of representations } \{\pi_k^u : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_u) \mid k \in \mathbb{N}\} \ \text{have commuting ranges, we can apply Lemma 5.1 with } Q = 1, \ \text{to define a representation } \pi_u : \ast-\text{alg}(\mathcal{L}[f]) \to \mathcal{B}(\mathcal{H}_u) \ \text{by an abuse of notation. Below we will use the notation } F_k^{(\ell)} \ \text{for the operator } F_k^{(\ell)} \ \text{of } \pi_u.

5.3 Definition The C*-algebra \mathcal{L}[f] \ is the C*-completion of \pi_u(\ast-\text{alg}(\mathcal{L}[f])) \ in \mathcal{B}(\mathcal{H}_u).

5.4 Remark (1) We see directly from (1) and the separability of \( C_0(\mathbb{R}) \) that \mathcal{L}[f] \ is separable.

(2) Observe that the representation \pi_u \ of \ast-\text{alg}(\mathcal{L}[f]) \ may be degenerate. Although all \pi_k^u \ are non-degenerate, it is possible that \( P[f] \neq 1 \). By Lemma 5.1(iv) it then follows that \( P[f] \) \ is the projection onto the essential subspace of \( \pi_u \).

(3) Since \mathcal{L}[f] \subset \mathcal{B}(\mathcal{H}_u) \ is given as a concrete C*-algebra, this selects the class of those representations of \mathcal{L}[f] \ which are normal maps w.r.t. the \sigma-strong topology of \mathcal{B}(\mathcal{H}_u) \ on \mathcal{L}[f]. We will say that such a representation \pi \ is normal w.r.t. the defining representation \pi_u. This will be the case if the vector states of \( \pi(\mathcal{L}[f]) \) \ are normal states for \( \pi_u(\mathcal{L}[f]) \) \ (cf. Proposition 7.1.15 [KR86]).

(4) From Fell’s Theorem [Fe60 Thm. 1.2] we know that any state of \( \mathcal{L}[f] \) \ is in the weak-*closure of the convex hull of the vector states of \pi_u.

We will need the following proposition.

5.5 Proposition If \( S \subset \mathbb{N} \) \ is a finite subset, then

(i) there is a C*-algebra \( B_S[f] \subset \mathcal{B}(\mathcal{H}_u) \) \ and a copy of the C*-complete tensor product \( \mathcal{L}^S := \bigotimes_{s \in S} C_0(\mathbb{R}) \) \ in \( \mathcal{B}(\mathcal{H}_u) \) \ such that

\[
\mathcal{L}[f] = C^*(\mathcal{L}^S \cdot B_S[f]) \cong \mathcal{L}^S \overline{\otimes} B_S[f].
\]

(ii) the natural embeddings \( \zeta_S : M(\mathcal{L}^S) \rightarrow M(\mathcal{L}[f]) = M(\mathcal{L}^S \overline{\otimes} B_S[f]) \) \ by \( \zeta_S((M \cdot A) \otimes B) \ := (M \cdot A) \otimes B \) \ for all \( A \in \mathcal{L}^S \) \ and \( B \in B_S[f] \) \ are topological embeddings w.r.t. the strict topology on each bounded subset of \( M(\mathcal{L}^S) \). Moreover, \( \mathcal{L}^S \) \ is dense in \( M(\mathcal{L}^S) \) \ w.r.t. the relative strict topology of \( M(\mathcal{L}[f]) \).

(iii) The group homomorphism \( \eta : \mathbb{R}^{(N)} \rightarrow M(\mathcal{L}[f]) \) \ is strictly continuous.

Proof. (i) By associativity (Theorem 5.2.3): \( \bigotimes_{k=1}^{\infty} C_0(\mathbb{R}) = \bigotimes_{s \in S} C_0(\mathbb{R}) \otimes \bigotimes_{\ell \in \mathbb{N} \setminus S} C_0(\mathbb{R}) \), and so, applying this to \ast-\text{alg}(\mathcal{L}[f]), \ and using the fact that it is the span of elementary tensors of the type \( \ldots \otimes A_m \otimes f_{m+1}^\ell \otimes f_{m+2}^\ell \otimes \ldots \) \ with \( A_i \in C_0(\mathbb{R}) \) \ and \( m, \ell \in \mathbb{N}, \) \ we get

\[
\ast-\text{alg}(\mathcal{L}[f]) = \bigotimes_{s \in S} C_0(\mathbb{R}) \otimes \left( \ast-\text{alg}(\mathcal{L}^{S \setminus \{s\}}) \right).
\]
where \((f_{i\cap S})_t = f_t\) for \(t \in \mathbb{N}\setminus S\) and \(*\text{-alg}(\{f_{i\cap S}\})\) denotes the *-algebra generated in \(\bigotimes_{t \in \mathbb{N}\setminus S} C_0(\mathbb{R})\) by

\[
\left\{ \otimes_{t \in \mathbb{N}\setminus S} g_t \mid g \in \prod_{t \in \mathbb{N}\setminus S} C_0(\mathbb{R}), \; g \sim f_{i\cap S} \right\}.
\]

Below, we need unital algebras, so adjoin identities, and define

\[
C_0 := \left( \mathbb{C}1 + \bigotimes_{s \in S} C_0(\mathbb{R}) \right) \otimes \left( \mathbb{C}1 + *\text{-alg}(\{f_{i\cap S}\}) \right) \subset \bigotimes_{k=1}^{\infty} (\mathbb{C}1 + C_0(\mathbb{R}))
\]

which contains \(*\text{-alg}(\{f\})\) as a *-ideal. Since \(\pi_u(*\text{-alg}(\{f\}))\) acts non-degenerately on its essential space \(H_{\text{ess}} \subset H_u\), it determines a unique extension of \(\pi_u\) to a representation \(\pi_u : C_0 \to \mathcal{B}(H_u)\), if we let the null space of \(\pi_u\) be \(H^+_{\text{ess}}\). Define \(C := C^*(\pi_u(C_0)) = C^*(\mathcal{A} \cdot \mathcal{B})\) where

\[
\mathcal{A} := C^*\left(\pi_u\left(\left( \bigotimes_{s \in S} C_0(\mathbb{R}) \right) \otimes 1 \right)\right) \quad \text{and} \quad \mathcal{B} := C^*\left(\pi_u\left( 1 \otimes \left( \mathbb{C}1 + *\text{-alg}(\{f_{i\cap S}\}) \right) \right)\right).
\]

Thus the unital \(C^*\)-algebra \(C\) is generated by the two commuting unital \(C^*\)-algebras \(\mathcal{A}\) and \(\mathcal{B}\). Moreover, since \(\pi_u\) contains tensor representations (w.r.t. the two factors of \(*\text{-alg}(\{f\})\) above), it follows that if \(AB = 0\) for an \(A \in \mathcal{A}\) and a \(B \in \mathcal{B}\), then either \(A = 0\) or \(B = 0\). Thus by [Tak79, Ex. 2, p. 220], it follows that \(C \cong \mathcal{A} \otimes \mathcal{B}\), where the tensor \(C^*\)-norm is unique, since both \(\mathcal{A}\) and \(\mathcal{B}\) are commutative, hence nuclear. We conclude that the original \(C^*\)-norm defined on \(C\) is in fact a cross-norm. Since its restriction to

\[
*\text{-alg}(\{f\}) = \left( \bigotimes_{s \in S} C_0(\mathbb{R}) \right) \otimes \left( *\text{-alg}(\{f_{i\cap S}\}) \right) \subset C_0
\]

is still a cross-norm, and the latter is unique by commutativity of the algebras (given the norms on the factors), it follows from \(C^*[\pi_u(\bigotimes_{s \in S} C_0(\mathbb{R}))] = \bigotimes_{s \in S} C_0(\mathbb{R})\) that

\[
L[f] = \left( \bigotimes_{s \in S} C_0(\mathbb{R}) \right) \otimes C^*[\pi_u\left( 1 \otimes (*\text{-alg}(\{f_{i\cap S}\})) \right)] = L^S \otimes \mathcal{B}_S[f].
\]

where \(\mathcal{B}_S[f] := C^*[\pi_u\left( 1 \otimes (*\text{-alg}(\{f_{i\cap S}\})) \right)]\).

(ii) This follows from (i) and Lemma A.2 in [GrN09].

(iii) Since \(\eta(\mathbb{R}^n)\) consists of unitary multipliers, it suffices to verify that the set of all elements \(A \in L[f]\) for which the map

\[
\eta^A : \mathbb{R}^n \to L[f], \quad x \mapsto \eta(x)A
\]

is continuous span a dense subalgebra. To establish this, let \(A = \iota(y)\) for some \(y \sim f^k\) for some \(k \in \mathbb{N}\). Now \(\mathbb{R}^n\) is a topological direct limit, so that it suffices to verify continuity on the finite dimensional subgroups \(\mathbb{R}^n\). For these, it follows from the strict continuity of the action of the group \(\mathbb{R}^n\) on its \(C^*\)-algebra \(C^*(\mathbb{R}^n) \cong C_0(\mathbb{R}^n)\) and the fact that by part (i) we have

\[
L[f] \cong C_0(\mathbb{R}^n) \hat{\otimes} \mathcal{A},
\]

for a \(C^*\)-algebra \(\mathcal{A}\), where \(\mathbb{R}^n\) acts by unitary multipliers on the first tensor factor and the identity on the second factor.

\[
\text{■}
\]
Note that for $S = \{1, 2, \ldots, n\}$, the map $\zeta_S$ identifies $\mathbb{R}^n \subset UM(L^S)$ with the unitaries $\mathbb{R}^n \subset \mathbb{R}^{(n)} \subset UM(L[f])$. Below we will abbreviate the notation to $L^{(n)} := L^{(1, 2, \ldots, n)} = \bigotimes_{k=1}^{n} C_0(\mathbb{R})$. For ease of notation, we sometimes also omit explicit indication of the embeddings $\zeta_S$, using inclusions instead.

Next, let $\pi : L[f] \to B(H_\pi)$ be a given fixed non-degenerate *-representation. Let $\tilde{\pi}$ denote the strict extension of $\pi$ to $M(L[f])$, so that $\pi_k := \tilde{\pi} | L^{(k)}$ and $\tilde{\pi}^{(n)} := \tilde{\pi} | L^{(n)}$ are the strict extensions of $\pi$ to $L^{(k)} \subset M(L^{(k)}) \xrightarrow{\zeta^{(k)}} M(L[f])$ and $L^{(n)} \subset M(L^{(n)}) \xrightarrow{\zeta^{(1, \ldots, n)}} M(L[f])$ respectively. Then $\{\pi_k \mid k \in \mathbb{N}\}$ is a set of non-degenerate representations with commuting ranges as in Lemma 5.1 hence we specialize its notation to:

$$F_{\pi,k}^{(\ell)} := \lim_{n \to \infty} \pi_k(f_k^1 \cdots f_n^1) \in B(H_\pi)$$

Since the commuting sequence of operators $(F_{\pi,k}^{(\ell)})_{k=1}^\infty$ is increasing, $P_{\pi}[f] \neq 1$ implies that there is a nonzero $\psi \in H_\pi$ such that $F_{\pi,k}^{(\ell)} \psi = 0$ for all $k$ and $\ell$.

We will show in the next proposition that, for a certain choice of $Q$, there is a representation $\pi_Q$ constructed as in Lemma 5.1 from the set $\{\pi_k \mid k \in \mathbb{N}\}$ which coincides with $\pi$.

**5.6 Proposition** Fix a non-degenerate *-representation $\pi : L[f] \to B(H_\pi)$ with $H_\pi \neq \{0\}$.

(i) Let $B_n := \tilde{\pi}(1 \otimes \cdots \otimes 1 \otimes f_n \otimes f_{n+1} \otimes \cdots)$. Then the strong limit $Q := \lim_{n \to \infty} B_n$ exists and satisfies $0 < Q \leq 1$.

(ii) If $A := A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^1 \otimes f_{m+2}^1 \otimes \cdots \in \* \text{-alg}(L[f])$, then

$$\pi(A) = \pi_1(A_1) \pi_2(A_2) \cdots \pi_m(A_m) F_{\pi,m+1}^{(\ell)} Q^\ell = \pi_Q(A),$$

i.e., $\pi_Q = \pi | \* \text{-alg}(L[f])$. Moreover $P_{\pi}[f] = 1$ and Ker $Q = \{0\}$.

(iii) Let $\pi^{(n)} : L^{(n)} \to B(H_\pi)$ denote the strict extension of $\pi$ to $L^{(n)} \subset M(L[f])$. Then

$$\pi(L_1 \otimes L_2 \otimes \cdots) = \lim_{n \to \infty} \pi^{(n)}(L_1 \otimes L_2 \otimes \cdots \otimes L_n)^\ell$$

for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in [f^\ell] \subset \* \text{-alg}(L[f])$.

**Proof.** (i) We need to prove this claim in greater generality than stated above, for use in the subsequent part. By definition, we have for $A := A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^1 \otimes f_{m+2}^1 \otimes \cdots \in \* \text{-alg}(L[f])$, that

$$\pi(A) = \pi_1(A_1) \pi_2^2(A_2) \cdots \pi_m^m(A_m) F_{\pi,m+1}^{(\ell)} \in L[f],$$

where $F_{\pi,k}^{(\ell)} := \lim_{n \to \infty} \pi_k(f_k^1 \cdots f_n^1) = \tilde{\pi}(1 \otimes \cdots \otimes 1 \otimes f_n^1 \otimes f_{n+1}^1 \otimes \cdots) \in B(H_\pi)$. Hence we have that $F_{\pi,n}^{(\ell)} \in M(L[f])$. Thus the operator

$$B_n^{(\ell)} := \tilde{\pi}(1 \otimes \cdots \otimes 1 \otimes f_n \otimes f_{n+1} \otimes \cdots) = \tilde{\pi}(F_{\pi,n}^{(\ell)})$$

satisfies $0 \leq B_n^{(\ell)} \leq 1$ since $0 \leq F_{\pi,n}^{(\ell)} \leq 1$. As $B_n^{(\ell)} = \pi_n(f_n^1) B_n^{(\ell,n)}$ and $\pi_n(f_n^1) \leq 1$ is a positive operator commuting with $B_n^{(\ell,n)}$, we see that $B_n^{(\ell)} \leq B_n^{(\ell,n)}$. Thus the strong limit $Q^{(\ell)} := \lim_{n \to \infty} B_n^{(\ell,n)}$
exists by Theorem 4.1.1 in [Mu90], and satisfies $0 < Q^{(l)} \leq 1$ (note that $Q^{(l)} \neq 0$ since $\pi$ is non-degenerate and $\mathcal{H}_\pi \neq \{0\}$). Since the operator product is jointly strongly continuous on bounded sets we have:

\[
Q^{(l)}Q^{(m)} = \lim_{n \to \infty} \pi(1 \otimes \cdots \otimes 1 \otimes f_{n}^{l} \otimes f_{n+1}^{l} \otimes \cdots) \lim_{k \to \infty} \pi(1 \otimes \cdots \otimes 1 \otimes f_{k}^{m} \otimes f_{k+1}^{m} \otimes \cdots)
\]

\[
= \lim_{n \to \infty} \pi(1 \otimes \cdots \otimes 1 \otimes f_{n}^{l} \otimes f_{n+1}^{l} \otimes \cdots) \pi(1 \otimes \cdots \otimes 1 \otimes f_{n}^{m} \otimes f_{n+1}^{m} \otimes \cdots)
\]

\[
= \lim_{n \to \infty} \pi(1 \otimes \cdots \otimes 1 \otimes f_{n}^{l+m} \otimes f_{n+1}^{l+m} \otimes \cdots) = Q^{(l+m)}.
\]

Thus $Q^{(l)} = Q^{l}$ where $Q := Q^{(1)}$.

(ii) Now $B^{(l)}_n = \pi\left(\prod_{k=1}^{n-1} f_{k}^{l} \otimes f_{n}^{l} \otimes f_{n+1}^{l} \otimes \cdots\right)$

\[
= \pi(f_{n}^{1}) \pi(1 \otimes \cdots \otimes 1 \otimes f_{n}^{l} \otimes f_{n+1}^{l} \otimes \cdots)
\]

\[
= \pi_{n}(f_{n}^{1}) \pi(1 \otimes \cdots \otimes 1 \otimes f_{n+1}^{l} \otimes f_{n+2}^{l} \otimes \cdots)
\]

\[
= \lim_{k \to \infty} \pi_{n}(f_{n}^{1}) \cdots \lim_{k \to \infty} \pi_{k}(f_{k}^{l}) \pi(1 \otimes \cdots \otimes 1 \otimes f_{k+1}^{l} \otimes f_{k+2}^{l} \otimes \cdots)
\]

\[
= \lim_{m \to \infty} \pi_{n}(f_{n}^{1}) \cdots \lim_{m \to \infty} \pi_{m}(f_{m}^{l}) \pi(1 \otimes \cdots \otimes 1 \otimes f_{m+1}^{l} \otimes f_{m+2}^{l} \otimes \cdots)
\]

\[
= F^{(l)}_{\pi,n}Q^{(l)} = F^{(l)}_{\pi,n}Q^{l}
\]

where we used again the joint strong operator continuity of the product on bounded sets. Let $A := A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^{l} \otimes f_{m+2}^{l} \otimes \cdots \in \ast\text{-alg}([F])$. Then

\[
\pi(A) = \pi_1(A_1) \cdot \pi(1 \otimes A_2 \otimes A_3 \otimes \cdots \otimes A_m \otimes f_{m+1}^{l} \otimes f_{m+2}^{l} \otimes \cdots) = \cdots
\]

\[
= \pi_1(A_1) \pi_2(A_2) \cdots \pi_m(A_m) \cdot \pi(1 \otimes \cdots \otimes 1 \otimes f_{m+1}^{l} \otimes f_{m+2}^{l} \otimes \cdots)
\]

\[
= \pi_1(A_1) \pi_2(A_2) \cdots \pi_m(A_m) \cdot F^{(l)}_{\pi,m+1}Q^{l} = \pi_Q(A)
\]

making use of (8) above. Since $\pi$ is non-degenerate, it follows from Lemma 5.4(iii) that $P_{\pi}[F] = 1$ and $\text{Ker} Q = \{0\}$.

(iii) Note first that from Proposition 5.5(ii) above and [Tak79, Lemma 4.1 on p.203] that $\pi_1(A_1) \pi_2(A_2) \cdots \pi_n(A_n) = \pi^{(n)}(A_1 \otimes \cdots \otimes A_n)$ for all $A_i \in C_0(\mathbb{R})$. Thus, if we continue equation (9) above

\[
\pi(A) = \pi_1(A_1) \pi_2(A_2) \cdots \pi_m(A_m) \cdot F^{(l)}_{\pi,m+1}Q^{l}
\]

\[
= \pi_1(A_1) \pi_2(A_2) \cdots \pi_m(A_m) \lim_{n \to \infty} \pi_{m+1}(f_{m+1}^{l}) \cdots \pi_n(f_{n}^{l}) Q^{l}
\]

\[
= \lim_{n \to \infty} \pi^{(n)}(A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^{l} \otimes \cdots \otimes f_{n}^{l}) Q^{l}
\]

which establishes the claim.

\[\square\]

5.7 Definition Given a representation $\pi$ of $\mathcal{L}[F]$, we will call its associated operator $Q$ its excess.

This proposition creates a difficulty for the host algebra project, because by part (iii) we can see that to construct its representations, we need more information than what is contained in the representations.
of $\mathbb{R}^{(N)}$, i.e., we need the excess operators $Q$. It is therefore very important to establish whether there are representations $\pi_Q$ with $Q \neq 1$ (below we will see such $\pi_Q$ will not be normal w.r.t. $\pi_u$).

5.8 Proposition Let $f$ be as before and let $\{\pi_k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}) \mid k \in \mathbb{N}\}$ be a set of $\ast$-representations on the same space with commuting ranges. Then for any positive operator $Q \in \mathcal{B}(\mathcal{H})$ with $Q \leq 1$ which commutes with the ranges of all $\pi_k$, we have that $\pi_Q : \ast$-alg$([f]) \to \mathcal{B}(\mathcal{H})$ extends to a $\ast$-representation of $\mathcal{L}[f]$.

Proof. We show first that $\sigma(F_{u,k}) = [0,1]$. Let $\omega$ be a character of $\mathbb{R}^{(N)}$. Then since it is a one-dimensional subrepresentation of $\pi_u$ there is a vector $\psi_\omega \in \mathcal{H}_u$ such that $(\psi_\omega, \pi_u(x)\psi_\omega) = \omega(x)$ for all $x \in \mathbb{R}^{(N)}$. Then $\omega_k(h) = (\psi_\omega, \pi_k(h)\psi_\omega)$ for all $h \in \mathcal{L}^{(k)} = C_0(\mathbb{R})$ is also a character, hence a point evaluation at a point $x_k^\omega \in \mathbb{R}$, and in fact we obtain all point evaluations of $\mathcal{L}^{(k)} = C_0(\mathbb{R})$ this way. Thus

$$F_{\omega,k} := \lim_{n \to \infty} \omega_k(f_k) \cdots \omega_n(f_n) = \prod_{n=k}^{k-1 \text{ factors}} f_k(x_k^\omega) \cdots f_n(x_n^\omega) = \prod_{n=k}^\infty f_n(x_n^\omega) \in [0,1],$$

and as we can choose our $\omega$, hence points $x_k^\omega \in \mathbb{R}$ arbitrarily, it is clear that we can find $\omega$ to set $F_{\omega,k}$ equal to any value in $[0,1]$. Since

$$F_{\omega,k} := \lim_{n \to \infty} \omega_k(f_k) \cdots \omega_n(f_n) = (\psi_\omega, F_{u,k} \psi_\omega)$$

defines a character on $C^*(F_{u,k})$ we see that $\sigma(F_{u,k}) = [0,1]$. Since for $\{\pi_k \mid k \in \mathbb{N}\}$ and $Q$ as in the initial hypotheses we always have that $0 \leq F_{\pi,k}Q \leq 1$, it follows that $\sigma(F_{\pi,k}Q) \subseteq [0,1] = \sigma(F_{u,k})$ for all $k$.

Next, note that in a diagonalization of $F_{u,k} \geq 0$ we can write it as $F_{u,k}(x) = x$ for $x \in \sigma(F_{u,k})$, and hence $\|p(F_{u,k})\| = \sup \{\|p(x)\| \mid x \in \sigma(F_{u,k})\}$. From this it is immediate that $\sigma(F_{\pi,k}Q) \subseteq \sigma(F_{u,k})$ implies $\|p(F_{\pi,k}Q)\| \leq \|p(F_{u,k})\|$ for all polynomials $p$.

Finally, recall that $\ast$-alg$([f]) = \lim_m \mathcal{A}_m[f]$ where

$$\mathcal{A}_m[f] := \text{Span} \{A_1 \otimes \cdots \otimes A_m \otimes f^k_{m+1} \otimes f^k_{m+2} \otimes \cdots \mid A_i \in C_0(\mathbb{R}) \forall i \in \mathbb{N}, k \in \mathbb{N}\}$$

and the inductive limit is w.r.t. to the inclusion $\mathcal{A}_m[f] \subseteq \mathcal{A}_l[f]$. Thus $\mathcal{L}[f]$ is the inductive limit of the $C^*$-closures $\mathcal{L}_m$ of $\pi_u(\mathcal{A}_m[f])$ w.r.t. set inclusion. Since

$$\mathcal{A}_m[f] = \bigotimes_{k=0}^m C_0(\mathbb{R}) \otimes (\ast\text{-}\text{alg}(\bigotimes_{j=m+1}^\infty f_j)),$$

and the norm of $\mathcal{L}[f]$ is a product norm by Proposition 5.3(i), we have that $\mathcal{L}_m \cong \mathcal{L}^{(m)} \otimes C^*(F_{u,m+1})$. Next we define (as in the proof of Lemma 5.4(iii)) two $\ast$-representations $\pi^{(m)}_a : \bigotimes_{k=0}^m C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ and $\pi^{(m)}_b : \ast\text{-}\text{alg}(\bigotimes_{j=m+1}^\infty f_j) \to \mathcal{B}(\mathcal{H})$ as follows. First, we have that

$$\pi^{(m)}_a : \bigotimes_{k=0}^m C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}), \quad \pi^{(m)}_a(A_1 \otimes \cdots \otimes A_m) := \pi_1(A_1) \cdots \pi_m(A_m)$$

defines a well-defined $\ast$-representation by the universal property of the tensor product. Moreover, since $\ast\text{-}\text{alg}(\bigotimes_{j=m+1}^\infty f_j)$ is generated by a single element not satisfying any polynomial relation, the
assignment
\[ \pi_b^{(m)} \left( \bigotimes_{j=m+1}^{\infty} f_j \right) := F_{m+1}^{(1)} Q \geq 0 \text{ defines a } \ast \text{-representation } \pi_b^{(m)} : \ast \text{-alg} \left( \bigotimes_{j=m+1}^{\infty} f_j \right) \rightarrow B(H). \]

Note from Equation (7) that
\[ F_{m+1}^{(k)} \cdot F_{m+1}^{(\ell)} = F_{m+1}^{(k+\ell)}, \]
which leads to the factorization
\[ \pi_Q(A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^\ell \otimes f_{m+2}^\ell \otimes \cdots) = \pi_a^{(m)}(A_1 \otimes \cdots \otimes A_m) \cdot \pi_b^{(m)} \left( \bigotimes_{j=m+1}^{\infty} f_j \right)^\ell. \]

Now \( \pi_a^{(m)} \) has a unique extension to \( \mathcal{L}^{(m)} \), and as \( \pi_b^{(m)} \) is defined on the dense \( \ast \)-algebra \( \ast \text{-alg} \left( \bigotimes_{j=m+1}^{\infty} f_j \right) \) on which it is continuous by the fact proven above, that
\[ \| \pi_b^{(m)} (p(F_{u,k})) \| = \| p(F_{u,k}Q) \| \leq \| p(F_{u,k}) \|. \]
Thus it extends uniquely to \( C^*(F_{u,m+1}) \), hence \( \pi_Q \) has a unique continuous extension to \( \mathcal{L}^m \). Since \( \pi_Q \) respects the inductive limit structure (since it does so on the dense subalgebra \( \ast \text{-alg}([\mathbb{I}]) \) and is continuous on all \( \mathcal{A}_m \) ) it follows that \( \pi_Q \) extends uniquely to a continuous \( \ast \)-representation of \( \mathcal{L}^m \).

We conclude that there is an abundance of representations \( \pi \) of \( \mathcal{L}^m \) with \( Q \neq 1 \).

Having investigated the representations of \( \mathcal{L}^m \), we next consider its host algebra properties. First label the unitary embedding \( \eta : \mathbb{R}^{(n)} \rightarrow M(\mathcal{L}^m) \) where
\[
\eta(x_1, \ldots, x_n, 0, 0, \ldots)(L_1 \otimes L_2 \otimes \cdots) = \eta_1(x_1)L_1 \otimes \cdots \otimes \eta_n(x_n)L_n \otimes L_{n+1} \otimes L_{n+2} \otimes \cdots
\]
and any approximate identity \( \{E_\alpha\}_{\alpha \in \Lambda} \) in \( \mathcal{L}^m \). Since \( \mathcal{L}^m \) and \( \mathbb{R}^{(n)} \) commute, their irreducible representations are all one-dimensional, hence \( \eta^* \) takes irreducible representations to irreducible representations.

### 5.9 Theorem
Given the preceding notation, we have that
(i) \( \eta : \mathbb{R}^{(n)} \rightarrow M(\mathcal{L}^m) \) is continuous w.r.t. the strict topology of \( M(\mathcal{L}^m) \).

(ii) Let \( \text{Rep}_0(\mathcal{L}^m, H) \) denote those non-degenerate \( \ast \)-representations of \( \mathcal{L}^m \) with excess operators \( Q = 1 \) (cf. Proposition 5.8). Then \( \eta^* \) is injective on \( \text{Rep}_0(\mathcal{L}^m, H) \).

(iii) The range \( \eta^* (\text{Rep}(\mathcal{L}^m, H)) \) is the same as \( \eta^* (\text{Rep}_0(\mathcal{L}^m, H)) \) and consists of those \( \pi \in \text{Rep}(\mathbb{R}^{(n)}, H) \) such that \( 1 = s\text{-lim}_{k \rightarrow \infty} F_k \) where \( F_k := s\text{-lim}_{n \rightarrow \infty} \pi_k(f_k) \cdots \pi_n(f_n) \) with \( \pi_k \) the unique representation in \( \text{Rep}(\mathcal{L}^k, H) \) such that \( \eta_k^*(\pi_k) = \pi | \mathbb{R}e_k \), where \( e_k \in \mathbb{R}^{(n)} \) is the \( k \)th basis vector.

(iv) For a state \( \omega \in \mathcal{G}(\mathcal{L}^m) \), its GNS-representation \( \pi^\omega \) is in \( \text{Rep}_0(\mathcal{L}^m, H_\omega) \) if and only if
\[
\omega \in \mathcal{G}_0(\mathcal{L}^m) := \{ \varphi \in \mathcal{G}(\mathcal{L}^m) \mid \lim_{n \rightarrow \infty} \varphi\left( 1 \otimes \cdots \otimes 1 \otimes f_n \otimes f_{n+1} \otimes \cdots \right) = 1 \}. 
\]
Moreover, the restriction $\eta^* : \mathcal{G}_0(\mathcal{L}[f]) \to \mathcal{G}(\mathbb{R}^{(N)}) \equiv \text{states of } \mathbb{R}^{(N)}$, is injective, with range consisting of

$$\omega \in \mathcal{G}(\mathbb{R}^{(N)}) \text{ such that } \lim_{k \to \infty} \lim_{n \to \infty} (\Omega_\omega, \pi^\omega_k(f_k) \cdots \pi^\omega_n(f_n)\Omega_\omega) = 1$$

with $\pi^\omega_j$ as in (iii), and $\Omega_\omega$ is the cyclic GNS-vector.

(v) $\pi$ is normal w.r.t. the defining representation $\pi_u$ of $\mathcal{L}[f]$ if and only if $Q = 1$.

Proof. (i) This is proven already in Proposition 5.5 (iii).

(ii) Let $\pi \in \text{Rep}_0(\mathcal{L}[f], \mathcal{H})$ and let $\tilde{\pi}$ be its strict extension to $M(\mathcal{L}[f])$. As $\tilde{\pi}$ is strictly continuous, (i) implies that the unitary representation $\eta^*(\pi) = \tilde{\pi} \circ \eta : \mathbb{R}^{(N)} \to \mathcal{U}(\mathcal{H})$ is strongly operator continuous. We need to show that $\eta^*$ is injective on $\text{Rep}_0(\mathcal{L}[f], \mathcal{H})$. If $\eta^*(\pi) = \eta^*(\pi')$ for two representations $\pi, \pi' \in \text{Rep}_0(\mathcal{L}[f], \mathcal{H})$, then $\eta^*_{(1,\ldots,n)}(\pi) = \eta^*_{(1,\ldots,n)}(\pi')$ on $\mathbb{R}^n \subset \mathbb{R}^{(N)}$ for all $n \in \mathbb{N}$. But $\text{Span}(\eta(n)(\mathbb{R}^n)) \subset M(\mathcal{L}^{(n)})$ is strictly dense, and by Proposition 5.5 (ii) this is still true for the strict topology of $M(\mathcal{L}[f]) \supset \zeta_{(1,\ldots,n)}(M(\mathcal{L}^{(n)}))$. Thus $\tilde{\pi} \upharpoonright \zeta_{(1,\ldots,n)}(\mathcal{L}^{(n)}) = \pi^{(n)} = \tilde{\pi}' \upharpoonright \zeta_{(1,\ldots,n)}(\mathcal{L}^{(n)})$, i.e., $\pi$ and $\pi'$ produce the same representation $\pi^{(n)} : \mathcal{L}^{(n)} \to \mathcal{B}(\mathcal{H})$. Thus by Proposition 5.6 (iii) (using $Q = 1$) we find

$$\pi(L_1 \otimes L_2 \otimes \cdots) = \text{s-lim}_{n \to \infty} \pi^{(n)}(L_1 \otimes L_2 \otimes \cdots \otimes L_n) = \pi'(L_1 \otimes L_2 \otimes \cdots \otimes L_n)$$

for the elementary tensors in $\ast$-alg$([\mathcal{L}])$, i.e., $\pi = \pi'$. Thus $\eta^*$ is injective on $\text{Rep}_0(\mathcal{L}[f], \mathcal{H})$.

(iii) To see that $\eta^*(\text{Rep}(\mathcal{L}[f], \mathcal{H})) = \eta^*(\text{Rep}_0(\mathcal{L}[f], \mathcal{H}))$, note that for $\pi_Q$ as in Lemma 5.1

$$\pi_Q(\eta(x_1,\ldots,x_n,0,0,\ldots)(L_1 \otimes L_2 \otimes \cdots))$$

$$= \pi_Q(\eta(x_1)L_1 \otimes \cdots \otimes \eta(x_n)L_n \otimes L_{n+1} \otimes L_{n+2} \otimes \cdots)$$

$$= \text{s-lim}_{k \to \infty} \pi^{(k)}(\eta(x_1)L_1 \otimes \cdots \otimes \eta(x_n)L_n \otimes L_{n+1} \otimes L_{n+2} \otimes \cdots \otimes L_k)\mathcal{Q}^\ell$$

by Prop. 5.6 (iii)

$$= \text{s-lim}_{k \to \infty} \pi_1(\eta(x_1)L_1) \cdots \pi_n(\eta(x_n)L_n)\pi_{n+1}(L_{n+1})\pi_{n+2}(L_{n+2}) \cdots \pi_k(L_k)\mathcal{Q}^\ell$$

$$= \eta^*_1(\pi_1(x_1)) \cdots \eta^*_n(\pi_n(x_n))\text{s-lim}_{k \to \infty} \pi_1(L_1) \cdots \pi_k(L_k)\mathcal{Q}^\ell$$

$$= \eta^*_1(\pi_1(x_1)) \cdots \eta^*_n(\pi_n(x_n))\pi_Q(L_1 \otimes L_2 \otimes \cdots)$$

for $L_1 \otimes L_2 \otimes \cdots \in [\mathcal{F}] \subset \ast$-alg$([\mathcal{L}])$, which shows that

$$\eta^*(\pi_Q)(x_1,\ldots,x_n,0,0,\ldots) = \eta^*(\pi_1)(x_1,\ldots,x_n,0,0,\ldots),$$

and establishes the claim.

To characterize the range of $\eta^*$, let $\pi \in \text{Rep}_0(\mathcal{L}[f], \mathcal{H})$ and note that as it is non-degenerate, we have from Lemma 5.1 that

$$1 = \text{P}_\pi[f] := \text{s-lim}_{k \to \infty} \text{F}_{\pi,k}^{(l)} \quad \text{where} \quad \text{F}_{\pi,k}^{(l)} := \text{s-lim}_{n \to \infty} \pi_k(f_k^1) \cdots \pi_n(f_n^l) \in \mathcal{B}(\mathcal{H}_\pi),$$

and $\pi_k = \tilde{\pi} \upharpoonright L^{(k)}$. From the uniqueness of the strict extension $\tilde{\pi}$ on $M(\mathcal{L}[f])$ and the fact that the strict topology of $M(L^{(k)}) \subset M(\mathcal{L}[f])$ coincides with that of $M(\mathcal{L}[f])$ on bounded subsets, we see that $\eta^*_k(\pi_k) = \eta^* \upharpoonright \mathbb{R} e_k$ and hence $\tilde{F}_k = F_k^{(1)}$. Thus $1 = \text{s-lim}_{k \to \infty} \tilde{F}_k$.

Conversely, let $\pi \in \text{Rep}(\mathbb{R}^{(N)}, \mathcal{H}_\pi)$ be such that $1 = \text{s-lim}_{k \to \infty} \tilde{F}_k$. We want to define $\pi_\mathcal{L} \in \text{Rep}_0(\mathcal{L}[f], \mathcal{H}_\pi)$ such that $\eta^*(\pi_\mathcal{L}) = \pi$. Consider first the case that $\pi$ is cyclic. Recall that $\mathcal{L}[f]$
is the norm closure of $\pi_u(\ast\text{-alg}(\|f\|))$. By definition of $\pi_u$, $\mathcal{H}_\pi$ is a direct summand of $\mathcal{H}_u$ and there is a projection $P_\pi \in \pi_u(\mathbb{R}^{(n)})'$ such that $\pi(x) = P_\pi \pi_u(x) \mid \mathcal{H}_\pi$ for all $x \in \mathbb{R}^{(n)}$. Then $\pi_k(A) = P_\pi \pi_u(A) \mid \mathcal{H}_\pi$ for all $A \in \mathcal{L}^{(k)}$, and hence $\tilde{F}_k = P_\pi F_{u,k}^{(1)} \mid \mathcal{H}_\pi$. We define $\pi_\mathcal{L} : \mathcal{L}[f] \to \mathcal{B}(\mathcal{H}_\pi)$ by $\pi_\mathcal{L}(A) := P_\pi \pi_u(A) \mid \mathcal{H}_\pi$ which is obviously a $\ast$-representation, satisfying $F_{\pi_\mathcal{L},k} = \tilde{F}_k$, with excess 1 (as it is normal w.r.t. $\pi_u$), and as

$$P_{\pi_\mathcal{L}}[f] = s\text{-lim}_{k \to \infty} F_{\pi_\mathcal{L},k} = s\text{-lim}_{k \to \infty} \tilde{F}_k = 1$$

by hypothesis, $\pi_\mathcal{L}$ is non-degenerate. Next, relax the requirement that $\pi$ be cyclic. Then $\pi$ is a direct sum of cyclic representations. Let $(\pi^c, \mathcal{H}_c)$ be a cyclic subrepresentation of $\pi$, and denote the projection onto $\mathcal{H}_c$ by $P_c$. Since $\pi \mid \mathbb{R} e_k$ also preserves $\mathcal{H}_c$, it follows that $\pi_k^c(A) = P_c \pi_k^c(A) \mid \mathcal{H}_c$ for all $A \in \mathcal{L}^{(k)}$. Now, recalling that $1 = s\text{-lim}_{k \to \infty} \tilde{F}_k$ where $\tilde{F}_k := s\text{-lim}_{n \to \infty} \pi_k(f_1) \cdots \pi_n(f_n)$, we have that

$$1_{\mathcal{H}_c} = P_c \mid \mathcal{H}_c = P_c \ast\text{-lim}_{k \to \infty} \tilde{F}_k \mid \mathcal{H}_c = s\text{-lim}_{k \to \infty} s\text{-lim}_{n \to \infty} P_c \pi_k(f_1) \cdots \pi_n(f_n) \mid \mathcal{H}_c$$

$$= s\text{-lim}_{k \to \infty} s\text{-lim}_{n \to \infty} \pi_k^c(f_1) \cdots \pi_n^c(f_n) = s\text{-lim}_{k \to \infty} \bar{F}_k$$

where $\bar{F}_k := s\text{-lim}_{n \to \infty} \pi_k^c(f_1) \cdots \pi_n^c(f_n)$. Thus, by the previous part we can construct a nondegenerate representation $\pi_\mathcal{L}^c : \mathcal{L}[f] \to \mathcal{B}(\mathcal{H}_c)$ by $\pi_\mathcal{L}^c(A) := P_c \pi_u(A) \mid \mathcal{H}_\pi$, which is normal w.r.t. $\pi_u$. Define $\pi_\mathcal{L} : \mathcal{L}[f] \to \mathcal{B}(\mathcal{H}_\pi)$ as the direct sum of all the $\pi_\mathcal{L}^c$. Since this is normal w.r.t. $\pi_u$ and nondegenerate, we have that $\pi_\mathcal{L} \in \text{Rep}_0(\mathcal{L}[f], \mathcal{H}_\pi)$.

Since the strict extension of $\pi_\mathcal{L}$ produces the same representations $\pi_k$ on $\mathcal{L}^{(k)}$ than obtained from $\pi \mid \mathbb{R} e_k$, the strict extension of $\pi_\mathcal{L}$ must coincide on $\mathbb{R}^{(n)}$ with $\pi$, i.e. $\eta^*(\pi_\mathcal{L}) = \pi$.

(iv) It is immediate from the definitions that if $\pi^c \in \text{Rep}_0(\mathcal{L}[f], \mathcal{H}_\omega)$, then $\omega \in \mathcal{G}_0(\mathcal{L}[f])$. Conversely, let $\omega \in \mathcal{G}_0(\mathcal{L}[f])$. Then, as $\mathcal{L}[f]$ is commutative, we know $\mathcal{L}[f] \cong C_0(X)$, with $X$ its spectrum. Then there is a probability measure $\mu$ on $X$ and a unitary $U : \mathcal{H}_\omega \to L^2(X, \mu)$ such that $(U \pi^c(h)\psi)(x) = h(x)U\psi)(x)$ for all $h \in C_0(X)$, $\psi \in \mathcal{H}_\omega$, $x \in X$, and moreover $U\Omega_\omega = 1$. Then

$$\lim_{n \to \infty} \omega\left(1 \otimes \cdots \otimes 1 \otimes f_n \otimes f_{n+1} \otimes \cdots\right) = (\Omega_\omega, Q\Omega_\omega)$$

$$= \int_X (UQU^{-1})(x) \, d\mu(x) \quad \text{and as } 0 < Q \leq 1 \text{ we have:}$$

$$0 = \int_X \left|1 - (UQU^{-1})(x)\right| \, d\mu(x).$$

Hence $(UQU^{-1})(x) = 1$ $\mu$-a.e., i.e., $Q = 1$ and thus $\pi^c \in \text{Rep}_0(\mathcal{L}[f], \mathcal{H}_\omega)$.

The last part of the claim now follows from this, (iii), and the observation that $\eta^*\omega(g) = (\Omega_\omega, \eta^*\pi^c(g)\Omega_\omega)$ for all $g \in \mathbb{R}^{(n)}$. Note that the state condition on the range of $\eta^*$ implies the operator condition in (iii) by a similar argument than the one above for $Q$.

(v) Let $\pi$ be normal w.r.t. $\pi_u(\mathcal{L}[f])$. Then it is continuous on bounded sets w.r.t. the strong operator topologies of both sides, hence $Q = s\text{-lim}_{n \to \infty} B_{u,n}^{(1)} = s\text{-lim}_{n \to \infty} \bar{\pi}(F_{u,n}^{(1)}) = \bar{\pi}(s\text{-lim}_{n \to \infty} F_{u,n}^{(1)})$. However, by Lemma 5.1(iv) we have that $P_u[f] = s\text{-lim}_{n \to \infty} F_{u,n}^{(1)}$ is the projection onto the essential subspace of $\pi_u(\mathcal{L}[f])$. Thus, since $\mathcal{L}[f]$ is in fact defined in $\pi_u$, it follows that $P_u[f]$ is the identity for $\pi_u(\mathcal{L}[f])$, hence $Q = \bar{\pi}(s\text{-lim}_{n \to \infty} F_{u,n}^{(1)}) = 1$.
Conversely, let $Q = 1$, then by part (iii) $\eta^* \pi$ is a continuous representation of $\mathbb{R}^{(N)}$, and by Proposition 5.10(iii) (with $Q = 1$) we have that
\[
\pi(L_1 \otimes L_2 \otimes \cdots) = \text{s-lim}_{n \to \infty}(\pi^{(n)}(L_1 \otimes L_2 \otimes \cdots \otimes L_n)) = \text{s-lim}_{n \to \infty} \pi_1(L_1)\pi_2(L_2) \cdots \pi_n(L_n)
\]
for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in [\mathbb{F}] \subset \ast\text{-alg}([\mathbb{F}])$. This is precisely the formula in which Lemma 5.1 defined representations on $\ast\text{-alg}(\mathbb{F})$ which we used to define $\pi_u$. Now $\pi_u(\mathbb{R}^{(N)})'' = \{\pi_u^{(n)}(\mathcal{L}(n)) \mid n \in \mathbb{N}\}'' = \pi_u(\mathcal{L}[f])''$ and a similar equation holds for $\pi$. Since the cyclic components of $\pi$ are contained in the direct summands of $\pi_u$, there is a normal map $\varphi : \pi_u(\mathcal{L}[f])'' \to \mathcal{B}(\mathcal{H}_\pi)$ such that $\varphi \circ \pi_u = \pi$. Thus $\pi$ is normal to $\pi_u$.  

Thus, though $\mathcal{L}[f]$ is not actually a host algebra for $\mathbb{R}^{(N)}$, it does have good properties, e.g., $\eta^*$ is bijective between two large sets of representations, and it takes irreducible representations to irreducibles. In fact, using the algebras $\mathcal{L}[f]$, we can now give a full $C^*$-algebraic interpretation of the Bochner–Minlos Theorem. Our aim is not to re–prove the Bochner–Minlos Theorem in the $C^*$-context, but just to identify the measures and decompositions of it with the appropriate measures and decompositions arising from the current $C^*$-context. First, we transcribe Lemma 4.4 for the current context:

**5.10 Lemma** As before, let $\mathbf{f} \in \prod_{n=1}^{\infty} V_{k_n}$ such that $[\mathbb{F}] \neq 0$. Let $\omega$ be a pure state on $\mathcal{L}[f]$, and let $\tilde{\omega}$ be its strict extension to the unitaries $\eta(\mathbb{R}^{(N)}) \subset M(\mathcal{L}[f])$. Then $\tilde{\omega} \circ \eta$ is a character and there exists an element $\mathbf{a} \in \mathbb{R}^N$ with $\tilde{\omega}(\eta(\mathbf{x})) = \exp(i(\mathbf{x}, \mathbf{a}))$ for all $\mathbf{x} \in \mathbb{R}^N$.

**Proof.** As $\mathcal{L}[f]$ is commutative, any pure state $\omega$ of it is a $\ast$-homomorphism. Thus the strict extension $\tilde{\omega}$ to $\eta(\mathbb{R}^{(N)}) \subset M(\mathcal{L}[f])$ is also a $\ast$-homomorphism, hence $\tilde{\omega} \circ \eta$ is a character. The restriction of $\tilde{\omega} \circ \eta$ to the subgroup $\mathbb{R}^n \subset \mathbb{R}^{(N)}$ is still a character, and it is continuous (since it is determined by the factor $\otimes_{j=1}^n C_0(\mathbb{R})$ in $\mathcal{L}[f]$ which is the group algebra of $\mathbb{R}^n$) hence of the form $\tilde{\omega} \circ \eta(\mathbf{x}) = \exp(i(\mathbf{x}, \mathbf{a}^{(n)}))$ for some $\mathbf{a}^{(n)} \in \mathbb{R}^n$. Since $\tilde{\omega} \circ \eta$ is a character on all of $\mathbb{R}^n$, the family $\{\mathbf{a}^{(n)} \in \mathbb{R}^n \mid n \in \mathbb{N}\}$ is a consistent family, i.e., if $n < m$ then $\mathbf{a}^{(n)}$ is the first $n$ entries of $\mathbf{a}^{(m)}$. Thus there is an $\mathbf{a} \in \mathbb{R}^N$ such that $\mathbf{a}^{(n)}$ is the first $n$ entries of $\mathbf{a}$ for any $n \in \mathbb{N}$. Then $\tilde{\omega} \circ \eta(\mathbf{x}) = \exp(i(\mathbf{x}, \mathbf{a}))$ since for any $\mathbf{x} \in \mathbb{R}^n \subset \mathbb{R}^{(N)}$ this restricts to the previous formula for $\tilde{\omega} \circ \eta$.

Thus there is a map from the pure states $\mathcal{S}_\rho(\mathcal{L}[f])$ to $\mathbb{R}^N$ denoted by
\[
\xi : \mathcal{S}_\rho(\mathcal{L}[f]) \to \mathbb{R}^N \text{ satisfying } \tilde{\varphi}(\eta(\mathbf{x})) = \exp(i(\mathbf{x}, \xi(\varphi))) \forall \mathbf{x} \in \mathbb{R}^N, \varphi \in \mathcal{S}_\rho(\mathcal{L}[f]) .
\]

**5.11 Theorem** For each state $\omega$ of $\mathbb{R}^{(N)}$ there is an $\mathbf{f} \in \prod_{n=1}^{\infty} V_{k_n}$ where $k_n \in \mathbb{N}$ and a unique state $\omega_0 \in \mathcal{S}_\rho(\mathcal{L}[f])$ such that $\eta^* (\omega_0) = \omega$. Then

(i) there is a regular Borel probability measure $\nu$ on $\mathcal{S}(\mathcal{L}[f])$ concentrated on the pure states $\mathcal{S}_\rho(\mathcal{L}[f])$ such that
\[
\omega_0(A) = \int_{\mathcal{S}_\rho(\mathcal{L}[f])} \varphi(A) d\nu(\varphi) \quad \forall A \in \mathcal{L}[f] .
\]
(ii) The probability measure $\tilde{\nu}$ on $\mathbb{R}^N$ given by $\tilde{\nu} := \xi_*\nu$ is (up to sets of measure zero) the Bochner–Minlos measure for $\omega$, i.e.,

$$\omega(x) = \int_{\mathbb{R}^n} \exp\left(i\langle x, y \rangle\right) d\tilde{\nu}(y) \quad \forall x \in \mathbb{R}^N.$$ 

Proof. Fix an $\omega \in \mathcal{S}(\mathbb{R}^N)$. Then by Theorem 4.9.4 in [Ped89] it suffices to show that there is a $f \in \prod_{n=1}^\infty V_{k_n}$ such that $\lim_{k \to \infty} \lim_{n \to \infty} \langle \Omega, \pi_k^\omega(f) \cdots \pi_n^\omega(f_n) \Omega \rangle = 1$. However, since there is an approximate identity \{\(E_n\)\}_{n \in \mathbb{N}} of $C_0(\mathbb{R})$ in $\bigcup_{n=1}^\infty V_n$, it is possible to choose an $f$ satisfying this limit condition, and we do this as follows. Since $\lim_{n \to \infty} \pi_k^\omega(E_n)\Omega = \Omega$, choose for each $n \in \mathbb{N}$ an $f_n := E_{k_n}$ such that $\|\pi_k^\omega(E_{k_n})\Omega - \Omega\| \leq 1/n^2$. Then for $1 < k < n$ we have:

$$\pi_k^\omega(f_k) \cdots \pi_n^\omega(f_n)\Omega - \Omega = \pi_k^\omega(f_k) \cdots \pi_{n-1}^\omega(f_{n-1})(\pi_n^\omega(f_n) - 1)\Omega$$

$$+ \pi_k^\omega(f_k) \cdots \pi_{n-2}^\omega(f_{n-2})(\pi_{n-1}^\omega(f_{n-1}) - 1)\Omega + \cdots + (\pi_k^\omega(f_k) - 1)\Omega.$$ 

Hence:

$$\|\pi_k^\omega(f_k) \cdots \pi_n^\omega(f_n)\Omega - \Omega\| \leq \frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{k^2}$$

$$< \int_{k-1}^{n+1} \frac{1}{x^2} \, dx = \frac{1}{k-1} - \frac{1}{n+1}$$

from which we see that $\lim_{k \to \infty} \lim_{n \to \infty} \|\pi_k^\omega(f_k) \cdots \pi_n^\omega(f_n)\Omega - \Omega\| = 0$, and this implies the required limit condition.

(i) Since $\mathcal{L}[f]$ is separable and commutative, it follows from Theorem II.2.2 in [Da96] that all its GNS–representations are multiplicity free, and hence by Theorem 4.9.4 in [Ped89], for any state $\omega_0$ on $\mathcal{L}[f]$ there is a regular Borel probability measure $\nu$ on $\mathcal{S}(\mathcal{L}[f])$ concentrated on the pure states $\mathcal{S}_\rho(\mathcal{L}[f])$ such that

$$\omega_0(A) = \int_{\mathcal{S}_\rho(\mathcal{L}[f])} \varphi(A) \, d\nu(\varphi) \quad \forall A \in \mathcal{L}[f].$$

(ii) For the state $\omega_0$ on $\mathcal{L}[f]$, let $\tilde{\omega}_0$ be its strict extension to the unitaries $\eta(\mathbb{R}^N) \subset M(\mathcal{L}[f])$, then we have for any countable approximate identity \{\(E_n\)\}_{n \in \mathbb{N}} $\subset \mathcal{L}_\rho[\mathcal{L}]$ that

$$\tilde{\omega}_0 \circ \eta(x) = \lim_{n \to \infty} \omega_0(\eta(x)E_n) = \lim_{n \to \infty} \int_{\mathcal{S}_\rho(\mathcal{L}[f])} \varphi(\eta(x)E_n) \, d\nu(\varphi)$$

$$= \int_{\mathcal{S}_\rho(\mathcal{L}[f])} \lim_{n \to \infty} \varphi(\eta(x)E_n) \, d\nu(\varphi) = \int_{\mathcal{S}_\rho(\mathcal{L}[f])} \tilde{\varphi} \circ \eta(x) \, d\nu(\varphi)$$

where we used the Lebesgue Dominated Convergence Theorem in the second line, since $|\varphi(\eta(x)E_n)| \leq 1$ and the constant function 1 is integrable. If we define a probability measure $\tilde{\nu}$ on $\mathbb{R}^N$ by $\tilde{\nu} := \xi_*\nu$, where the map $\xi : \mathcal{S}_\rho(\mathcal{L}[f]) \to \mathbb{R}^N$ given by $\tilde{\varphi} \circ \eta(x) = \exp\left(i\langle x, \xi(\varphi) \rangle\right)$ for $x \in \mathbb{R}^N$ was mentioned above, we obtain

$$\omega(x) = \tilde{\omega}_0 \circ \eta(x) = \int_{\mathcal{L}[f]} \exp\left(i\langle x, y \rangle\right) \, d\tilde{\nu}(y) \quad \forall x \in \mathbb{R}^N.$$ 

Hence $\tilde{\nu}$ coincides (up to sets of measure zero) with the usual Bochner–Minlos measure on $\mathbb{R}^N$ by uniqueness of the measure on $\mathbb{R}^N$ producing this decomposition (cf. Lemma 7.13.5 in [Bo07]).
Thus we can interpret the Bochner–Minlos Theorem as an expression of the pure state space decompositions of the $C^*$-algebras $\mathcal{L}[\mathfrak{f}]$. We will not consider the uniqueness of the measures in the decompositions of the Bochner–Minlos Theorem, as that is easy to prove.

To understand $\mathcal{L}[\mathfrak{f}]$ at a more concrete level, we consider its spectrum $X$. Since $\mathcal{L}[\mathfrak{f}]$ is commutative, we know $\mathcal{L}[\mathfrak{f}] \cong C_0(X)$, and as each $\omega \in X$ is a character, we obtain from Propositions 5.6 and 5.8 that

$$\omega(L_1 \otimes L_2 \otimes \cdots) = \lim_{n \to \infty} \omega_1(L_1)\omega_2(L_2) \cdots \omega_n(L_n)q^\ell$$

for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in [\mathfrak{f}]^\otimes$. Here, $q \in (0, 1]$ and each $\omega_i$ is a character of $\mathcal{L}^{(i)} = C_0(\mathbb{R})$ hence a point evaluation $\omega_i(f) = f(x_i)$. Since $\omega$ is uniquely determined by its values on $\mathcal{L}(\mathfrak{f})$, this defines (via Proposition 5.8) a surjective map

$$\gamma : \mathbb{R}^n \times (0, 1] \to X \cup \{0\} \quad \text{by} \quad \gamma(x, q)(L_1 \otimes L_2 \otimes \cdots) := \lim_{n \to \infty} L_1(x_1)L_2(x_2) \cdots L_n(x_n)q^\ell$$

for $L_1 \otimes L_2 \otimes \cdots \in [\mathfrak{f}]^\otimes$. To obtain a bijection with $X$ from $\gamma$, note that if $A := L_1 \otimes L_2 \otimes \cdots = A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^\ell \otimes f_{m+2}^\ell \otimes \cdots \in \mathcal{L}(\mathfrak{f})$, then

$$\prod_{k=1}^\infty \omega_k(L_k) = A_1(x_1)A_2(x_2) \cdots A_m(x_m) \prod_{k=m+1}^\infty f_k(x_k)^\ell = 0 \quad \forall A_i, m, \ell$$

if and only if $\lim_{m \to \infty} \prod_{k=m}^\infty f_k(x_k) = 0$. Thus we define

$$N_\mathfrak{f} := \left\{ x \in \mathbb{R}^n \mid \lim_{m \to \infty} \prod_{k=m}^\infty f_k(x_k) = 0 \right\}$$

and hence the restriction $\gamma : (\mathbb{R}^n \setminus N_\mathfrak{f}) \times (0, 1] \to X$ is a surjection. That $\gamma$ is bijective, is clear since each $\gamma(x, q)$ is nonzero (as $x \not\in N_\mathfrak{f}$), and in each factor in the product, a component of $\mathcal{L}[\mathfrak{f}]$ will separate the characters, and in the last entry, by definition all elementary tensors will separate different values of $q$. Thus we may identify (as sets) $X$ with $(\mathbb{R}^n \setminus N_\mathfrak{f}) \times (0, 1]$. Note that $N_\mathfrak{f}$ contains the set $\left\{ x \in \mathbb{R}^n \mid x_n \in f_n^{-1}(0) \text{ for infinitely many } n \right\}$, hence since the $f_n$ are of compact support, $\mathbb{R}^n \setminus N_\mathfrak{f}$ is contained in the union of sets $\bigcap_{n=1}^\infty S_n \subset \mathbb{R}^n$ where only finitely many of the $S_n$ are not relatively compact.

The w *-topology of $X$ w.r.t. $\mathcal{L}[\mathfrak{f}]$ is not clear. The most important subset in $X$ is $X_0 := X \cap \text{Rep}_0(\mathcal{L}[\mathfrak{f}], \mathbb{C})$ which corresponds to $(\mathbb{R}^n \setminus N_\mathfrak{f}) \times \{1\}$. We prove that it is a $G_\delta$-set. To see this, note that $\omega \in X_0$ if and only if $\lim_{n \to \infty} \prod_{k=n}^{\infty} \omega_k(f_k) = 1$. This is an increasing limit. By using approximate identities in each factor $\mathcal{L}^{(k)}$, we can find for each $n$ a net $\{ A_n^{(n)} \} \subset \mathcal{L}[\mathfrak{f}]$, $0 < A_n^{(n)} < 1$, such that $\omega_1(1 \otimes \cdots \otimes 1 \otimes f_n \otimes f_{n+1} \otimes \cdots) = \sup \omega(A_n^{(n)})$ for all $\omega \in X$. Define a function $q_\mathfrak{f} : X \to [0, 1]$ by $q_\mathfrak{f}(\omega) := \sup \omega(A_n^{(n)})$ then $X_0 = q_\mathfrak{f}^{-\alpha}\{1\}$. Since $q_\mathfrak{f}$ is the supremum of continuous functions on $X$ it is lower semicontinuous (cf. 6.3 in [Ko69]), i.e., $q_\mathfrak{f}^{-1}(t, \infty)$ is open for all $t \in \mathbb{R}$. Since $X_0 = q_\mathfrak{f}^{-1}\{1\} = \bigcap_{n \in \mathbb{N}} q_\mathfrak{f}^{-1}(\left(\frac{n+1}{n}, \infty\right))$, it follows that $X_0$ is a $G_\delta$-set.

To make a host algebra out of $\mathcal{L}[\mathfrak{f}]$, i.e., to make $\eta^*$ injective, we need to reduce its spectrum to $X_0$. However, since we do not know whether $X_0$ is a locally compact subset of $X$ this is not easy. From the fact that it is a $G_\delta$-set, we can identify $X_0$ as the common characters of the decreasing
sequence of C*-algebras $C_0\left((\mathbb{R}^n, \infty)\right)$ in $\mathcal{L}[f]$, where of course $\eta(\mathbb{R}^n)$ still acts on these as multipliers (i.e., as elements of $C_0(X)$, with pointwise multiplication).

6 Hosting the full representation theory of $\mathbb{R}^n$

We first want to extend the semi-host algebra $\mathcal{L}[f]$ above to an algebra $\mathcal{L}_V$, such that $\eta^*(\text{Rep}(\mathcal{L}_V, \mathcal{H})) = \text{Rep}(\mathbb{R}^n, \mathcal{H})$. Recall that for

$$V_n := \{ f \in C_0(\mathbb{R}) \mid f(\mathbb{R}) \subseteq [0, 1], \ f \upharpoonright [-n, n] = 1, \ \text{supp}(f) \subseteq [-n - 1, n + 1] \}.$$ 

we obtain a multiplicative subsemigroup $\mathcal{V} := \bigcup_{n=1}^{\infty} V_n$ in $C_0(\mathbb{R})$. Thus, by Theorem 3(iii), $\mathcal{V} = \mathcal{V}^*$, implies that

$$\mathcal{A}(\mathcal{V}) := \text{Span}\{ b \in [f] \mid f \in \mathcal{V}^n \} = \text{Span}\{ \bigcap_{n=1}^{\infty} g_n \mid g \sim f \in \mathcal{V}^n \}$$

is a $*$-subalgebra of $\bigotimes_{n=1}^{\infty} C_0(\mathbb{R})$.

6.1 Proposition There is a $*$-representation $\pi_u : \mathcal{A}(\mathcal{V}) \to \mathcal{B}(\mathcal{H}_u)$ such that

$$\pi_u(L_1 \otimes L_2 \otimes \cdots) = \text{s-lim}_{n \to \infty} \pi_u^1(L_1) \pi_u^2(L_2) \cdots \pi_u^n(L_n)$$

for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in \mathcal{A}(\mathcal{V})$, where $\pi_u^k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_u)$ are as before (cf. text above Definition 5.3).

Proof. By Proposition 5(iii), $\pi_u$ is already a $*$-representation on each $*$-alg([f]) for $f \in \mathcal{V}^n$, hence it is a linear map on each $[f]$ for $f \in \mathcal{V}^n$. However, by Proposition 5.7(iv) we know that for $f, g \in \mathcal{V}^n$ with $[f] \neq \{0\} \neq [g]$ we have $[f] \cap [g] = \{0\}$ if and only if $[f] \not\sim [g]$. Thus the set of spaces $\{[f] \mid f \in \mathcal{V}^n\}$ is labelled by the equivalence classes $[f] \subset \mathcal{V}^n$, and by Proposition 5.7(iv), the sum of the subspaces $[f]$ is direct. Thus, since $\pi_u$ is defined as a linear map on each $[f]$, it extends uniquely to a linear map $\pi_u$ on $\mathcal{A}(\mathcal{V}) = \text{Span}\{ b \in [f] \mid f \in \mathcal{V}^n \}$.

To show that $\pi_u$ is a $*$-homomorphism, it suffices to check this on the elementary tensors $\bigotimes_{n=1}^{\infty} g_n$ with $g \sim f \in \mathcal{V}^n$. For $f, g \in \mathcal{V}^n$, let

$$A = A_1 \otimes \cdots \otimes A_{k-1} \otimes f_k \otimes f_{k+1} \otimes \cdots \in [f] \quad \text{and} \quad B = B_1 \otimes \cdots \otimes B_{k-1} \otimes g_k \otimes g_{k+1} \otimes \cdots \in [g]$$

where we can choose the same $k$ for both. Then by Proposition 5.6(ii) we have

$$\pi_u(A) = \pi_u^1(A_1) \cdots \pi_u^{k-1}(A_{k-1}) F_{u,k}[f] \quad \text{and} \quad \pi_u(B) = \pi_u^1(B_1) \cdots \pi_u^{k-1}(B_{k-1}) F_{u,k}[g]$$

and

$$\pi_u(AB) = \pi_u^1(A_1 B_1) \cdots \pi_u^{k-1}(A_{k-1} B_{k-1}) F_{u,k}[f \cdot g].$$

where

$$F_{u,k}[f] := \text{s-lim}_{n \to \infty} \pi_u^k(f_k) \cdots \pi_u^n(f_n).$$

Since $\pi_u^j$ is a representation for all $j$, we only need to show that $F_{u,k}[f \cdot g] = F_{u,k}[f] F_{u,k}[g]$ to establish that $\pi_u(AB) = \pi_u(A) \pi_u(B)$ . We have

$$F_{u,k}[f \cdot g] = \text{s-lim}_{n \to \infty} \pi_u^k(f_k g_k) \cdots \pi_u^n(f_n g_n) = \text{s-lim}_{n \to \infty} \pi_u^k(f_k) \cdots \pi_u^n(f_n) \pi_u^k(g_k) \cdots \pi_u^n(g_n)$$

$$= \text{s-lim}_{n \to \infty} \pi_u^k(f_k) \cdots \pi_u^n(f_n) \cdot \text{s-lim}_{m \to \infty} \pi_u^k(g_k) \cdots \pi_u^m(g_m) = F_{u,k}[f] F_{u,k}[g].$$
since the operator product is jointly continuous in the strong operator topology on bounded subsets. Thus \( \pi_u \) is a homomorphism. To see that it is a \(*\)-homomorphism, note that
\[
\pi_u(A)^* = \pi_u^1(A_1^*) \cdots \pi_u^{k-1}(A_{k-1}^*) F_{u,k}[f] = \pi_u(A^*)
\]
since all \( \pi_u^j \) are \(*\)-homomorphisms with commuting ranges, and \([f]^* = [f^*] = [f] \). Thus \( \pi_u \) is a \(*\)-homomorphism of \( \mathcal{A}(V) \).

As in Section 5.3 we define

**6.2 Definition** The \( C^* \)-algebra \( \mathcal{L}_V \) is the \( C^* \)-completion of \( \pi_u(\mathcal{A}(V)) \) in \( \mathcal{B}(H_u) \).

Note that \( \mathcal{L}_V = C^* \{ \mathcal{L}[f] \mid f \in \mathcal{V}^N \} \subset \mathcal{B}(H_u) \).

We extend the unitary embeddings \( \eta : \mathbb{R}^N \to U M(\mathcal{L}[f]) \) from above to \( \mathcal{L}_V \) as follows. Define \( \eta : \mathbb{R}^N \to M(\mathcal{L}_V) \), where
\[
\eta(x_1, \ldots, x_n, 0, 0, \ldots) (L_1 \otimes L_2 \otimes \cdots) = \eta_1(x_1) L_1 \otimes \cdots \otimes \eta_n(x_n) L_n \otimes L_{n+1} \otimes L_{n+2} \otimes \cdots
\]
for all \( (x_1, \ldots, x_n) \in \mathbb{R}^n, \ n \in \mathbb{N}, \ L_i \in \mathcal{L}^{(1)} = C_0(\mathbb{R}) \), and where \( \eta_i : \mathbb{R} \to M(C^*(\mathbb{R})) \) is the usual unitary embedding. Clearly, \( \eta \) restricts to the previous definition of it on each \( \mathcal{L}[f] \subset \mathcal{L}_V \). Then the map \( \eta^* : \text{Rep}(\mathcal{L}_V, \mathcal{H}) \to \text{Rep}(\mathbb{R}^N, \mathcal{H}) \) consists of the strict extension of (non-degenerate) representations of \( \mathcal{L}_V \) to \( \eta(\mathbb{R}^N) \), i.e.,
\[
(\eta^* \pi)(x) := \lim_{\alpha \to \infty} \pi (\eta(x) E_{\alpha}) \quad \forall \ x \in \mathbb{R}^N \quad \text{and any approximate identity} \quad \{E_{\alpha}\}_{\alpha \in \Lambda} \subset \mathcal{L}_V,
\]
and \( \eta^* \) obviously takes irreducibles to irreducibles by commutativity.

**6.3 Definition** Let \( \text{Rep}_0(\mathcal{L}_V, \mathcal{H}) \) denote those non-degenerate \(*\)-representations \( \pi : \mathcal{L}_V \to \mathcal{B}(\mathcal{H}) \) for which \( \pi \upharpoonright \mathcal{L}[f] \in \text{Rep}_0(\mathcal{L}[f], \mathcal{H}_f) \) for all \( f \), where \( \mathcal{H}_f := \pi(\mathcal{L}[f]) \mathcal{H} \). That is, each restriction of \( \pi \) to \( \mathcal{L}[f] \) has excess operator \( Q_f = \mathbb{1} \) on its essential subspace \( \mathcal{H}_f \).

By Proposition 5.7 this means that all
\[
Q_f(\pi) := \lim_{n \to \infty} B_n[f] \quad \text{are projections, where} \quad B_n[f] := \pi \left( \overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{n-1 \text{ factors}} \otimes f_n \otimes f_{n+1} \otimes \cdots \right).
\]
In fact, the projections \( Q_f(\pi) \) must be the range projections \( P_{\pi,f} = \lim_{k \to \infty} F_{\pi,k}^{(1)} \) where \( F_{\pi,k}^{(1)} := \lim_{n \to \infty} \pi_k(f_k) \cdots \pi_n(f_n) \). Note that a direct sum of representations \( \pi_i \in \text{Rep}_0(\mathcal{L}_V, \mathcal{H}_i), \ i \in I \) (an index set) is again of the same type, i.e., \( \bigoplus_{i \in I} \pi_i \in \text{Rep}_0(\mathcal{L}_V, \bigoplus_{i \in I} \mathcal{H}_i) \).

**6.4 Theorem** Given the preceding notation, we have that
\begin{enumerate}
  \item \( \eta : \mathbb{R}^N \to M(\mathcal{L}_V) \) is continuous w.r.t. the strict topology of \( M(\mathcal{L}_V) \).
  \item The map \( \eta^* \) is injective on \( \text{Rep}_0(\mathcal{L}_V, \mathcal{H}) \).
  \item The range \( \eta^* (\text{Rep}(\mathcal{L}_V, \mathcal{H})) \) is the same as \( \eta^* (\text{Rep}_0(\mathcal{L}_V, \mathcal{H})) \) and is all of \( \text{Rep}(\mathbb{R}^N, \mathcal{H}) \).
  \item \( \pi \in \text{Rep}_0(\mathcal{L}_V, \mathcal{H}) \) if and only if \( \pi \) is normal w.r.t. \( \pi_u \).
\end{enumerate}
Proof. (i) Since \( \eta : \mathbb{R}^{(N)} \to M(\mathcal{L}_V) \) is bounded, it suffices to show that the space

\[
\{ L \in \mathcal{L}_V \mid \text{the map } \mathbb{R}^{(N)} \ni x \mapsto \eta(x)L \in \mathcal{L}_V \text{ is norm continuous} \}
\]

is dense in \( \mathcal{L}_V \). But this follows from the fact that by Theorem 5.9(i), this space contains all \([f] \subset \mathcal{L}[f]\), and these spaces span \( \mathcal{A}(\mathcal{V}) \) which is dense in \( \mathcal{L}_V \).

(ii) Consider \( \pi, \pi' \in \text{Rep}_0(\mathcal{L}_V, \mathcal{H}) \) such that \( \eta^* \pi = \eta^* \pi' \). Then for the restrictions to \( \mathbb{R}^n \subset \mathbb{R}^{(N)} \) we have \( \tilde{\pi}^{(n)} := \eta^* \pi \mid \mathbb{R}^n = \eta^* \pi' \mid \mathbb{R}^n := \tilde{\pi}'^{(n)} \). Moreover, \( \mathcal{L}^{(n)} \) embeds in \( M(\mathcal{L}_V) \) as \( \mathcal{L}^{(n)} \otimes 1 \) (acting on the elementary tensors), hence \( \pi \) also extends to it to define a non-degenerate \( \pi^{(n)} : \mathcal{L}^{(n)} \to \mathcal{B}(\mathcal{H}_x) \). Since \( \eta \) is defined via the natural actions, we have \( \eta(x)\mathcal{L}^{(n)} \subseteq \mathcal{L}^{(n)} \) for all \( x \in \mathbb{R}^n \). Since \( \eta \) produces a unique non-degenerate representation \( \pi \), analogous expressions hold for \( \pi \).

Now for each elementary tensor \( L_1 \otimes L_2 \otimes \cdots \in \ast \text{-alg}([f]) \subset \mathcal{L}[f] \) we know by Proposition 5.6(iii) that

\[
\pi(L_1 \otimes L_2 \otimes \cdots) = \varprojlim_{n \to \infty} \pi^{(n)}(L_1 \otimes L_2 \otimes \cdots \otimes L_n)\mathcal{Q}_\pi(f).
\]

Recall that by hypothesis \( \mathcal{Q}_\pi(f) = P_{\pi}[f] = \varprojlim_{k \to \infty} F_{\pi,k}^{(1)} \), where \( F_{\pi,k}^{(1)} := \varprojlim_{n \to \infty} \pi_k(f_k) \cdots \pi_n(f_n) \). Analogous expressions hold for \( \pi' \), thus since \( \pi_k = \pi_k' \) for all \( k \), it follows that \( \mathcal{Q}_\pi(f) = \mathcal{Q}_\pi(f') \) and hence from Equation (10) it follows from \( \pi^{(n)} = \pi'^{(n)} \) for all \( n \), that \( \pi \) and \( \pi' \) coincides on all \( \mathcal{L}[f] \) hence on all of \( \mathcal{L}_V \), which proves the claim.

(iii) Let \( \pi \in \text{Rep}_0(\mathcal{L}_V, \mathcal{H}) \) and let \( \tilde{\pi} \) be its strict extension to \( M(\mathcal{L}_V) \). As \( \tilde{\pi} \) is strictly continuous, (i) implies that the unitary representation \( \eta^* \pi = \tilde{\pi} \circ \eta : \mathbb{R}^{(N)} \to \mathcal{U}(\mathcal{H}) \) is strong operator continuous, i.e., \( \eta^* (\text{Rep}(\mathcal{L}_V, \mathcal{H})) \subseteq \text{Rep}(\mathbb{R}^{(N)}, \mathcal{H}) \). To prove the claim of this theorem, we need to show that for each \( \pi \in \text{Rep}(\mathbb{R}^{(N)}, \mathcal{H}_x) \), there is a \( \pi(0) \in \text{Rep}_0(\mathcal{L}_V, \mathcal{H}_x) \) such that \( \eta^* \pi(0) = \pi \). Since each \( \pi \in \text{Rep}(\mathbb{R}^{(N)}, \mathcal{H}_x) \) is a direct sum of cyclic representations, and \( \eta^* \) preserves direct sums, it suffices to show that for each cyclic \( \pi \in \text{Rep}(\mathbb{R}^{(N)}, \mathcal{H}_x) \), there is a \( \pi(0) \in \text{Rep}_0(\mathcal{L}_V, \mathcal{H}_x) \) such that \( \eta^* \pi(0) = \pi \). Fix a cyclic \( \pi \in \text{Rep}(\mathbb{R}^{(N)}, \mathcal{H}_x) \), then there is a projection \( P_\pi \in \pi_\ast(\mathbb{R}^{(N)}) \) such that \( \pi = (P_\pi \pi_u) \upharpoonright \mathcal{H}_x \) where \( \mathcal{H}_x = P_\pi \mathcal{H}_u \). Recall the inclusion \( \mathbb{R} \to \mathbb{R}^{(N)}, x \mapsto xe_k \), so let \( \pi_k : \mathbb{R} \to \mathcal{U}(\mathcal{H}_x) \) be \( \pi_k(x) := \pi(xe_k) \). By the host algebra property of \( C^*(\mathbb{R}) \cong C_0(\mathbb{R}) \), this produces a unique non-degenerate representation \( \pi_k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_u) \), which is characterized by \( \pi_k(x)\pi_k(L) = \pi_k(\eta_k(x)L) = \pi(\eta(0,0,0,0,0,0,0,0,0,0,0))uu(1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \cdots) \) (\( x \) and \( L \) in the \( k \)th entries) for all \( x \in \mathbb{R} \) and \( L \in C_0(\mathbb{R}) \). Since

\[
\pi(\eta(0,0,0,0,0,0,0,0,0,0,0))uu(1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \cdots)
\]

we get that \( \pi_k(L) = P_\pi \pi_k(L) \upharpoonright \mathcal{H}_x \) for all \( L \in C_0(\mathbb{R}) \). Since the set of representations \( \{ \pi_k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_x) \mid k \in \mathbb{N} \} \) have commuting ranges, we can apply Lemma 5.1 (with the choice
\( Q = 1 \) to define a representation \( \pi_{(0)} : \ast\text{-alg}[\mathcal{I}] \to \mathcal{B}(\mathcal{H}_\pi) \), for all \( \mathcal{I} \), and we need to show that \( \pi_{(0)} \) extends to a representation of \( \mathcal{L}_\mathcal{V} \). Now \( P_\pi \) commutes with the images of all \( \pi_u^k \) (since it commutes with \( \pi_u(\mathbb{R}^N) \)) hence all \( \pi_u^k(L) \) preserve \( \mathcal{H}_\pi \) and so by its definition \( \pi_u(\mathcal{L}_\mathcal{V}) \) preserves \( \mathcal{H}_\pi \). Thus the map \( A \in \mathcal{L}_\mathcal{V} \to P_\pi \pi_u(A) \upharpoonright \mathcal{H}_\pi \) is a \( \ast\)-representation of \( \mathcal{L}_\mathcal{V} \) and it coincides with \( \pi_{(0)} \) on each \( \ast\text{-alg}[\mathcal{I}] \) because

\[
P_\pi \pi_u(L_1 \otimes L_2 \otimes \cdots) \upharpoonright \mathcal{H}_\pi = \operatorname{s-lim}_{n \to \infty} P_\pi \pi_u^1(L_1) \pi_u^2(L_2) \cdots \pi_u^n(L_n) \upharpoonright \mathcal{H}_\pi
\]

for all elementary tensors \( L_1 \otimes L_2 \otimes \cdots \in \mathcal{A}(\mathcal{V}) \). This defines a \( \ast\)-representation \( \pi_{(0)} : \mathcal{L}_\mathcal{V} \to \mathcal{B}(\mathcal{H}_\pi) \) by \( \pi_{(0)}(A) = P_\pi \pi_u(A) \upharpoonright \mathcal{H}_\pi \) for all \( A \in \mathcal{L}_\mathcal{V} \). To see that it is non-degenerate, note that its restriction to any \( \mathcal{L}[\mathcal{I}] \subset \mathcal{L}_\mathcal{V} \) has essential projection \( P_\pi[\mathcal{I}] = \operatorname{s-lim}_{k \to \infty} \tilde{F}_k \)

where \( \tilde{F}_k := \operatorname{s-lim} \pi_{(0)}(f_n) \cdots \pi_{(0)}(f_n) \) by Theorem [5.9](iii) and Lemma [5.1](ii). It is suffices to show that for each nonzero \( \psi \in \mathcal{H}_\pi \) there is a sequence \( \mathcal{I} \in \mathcal{V}^\infty \) such that \( P_\pi[\mathcal{I}] \psi \neq 0 \). Fix a nonzero \( \psi \in \mathcal{H}_\pi \). Since there is an approximate identity of \( C_0(\mathbb{R}) \) in \( \mathcal{V} \), it is possible to choose for each \( n \in \mathbb{N} \) a \( f_n \in \mathcal{V} \) such that \( \|\psi - \pi_{(0)}(f_n)\| < 1/n^2 \), hence we may write \( \pi_{(0)}(f_n)\psi = \psi + \xi_n/n^2 \) where \( \|\xi_n\| \leq 1 \). Then

\[
\pi_{(0)}(f_n)\psi = \psi + \frac{1}{n^2} \pi_{(0)}(f_n) \cdots \pi_{(0)}(f_n)\xi_n
\]

and hence \( P_\pi[\mathcal{I}] \psi = \psi \) as the series converges. Thus \( \pi_{(0)} \) is non-degenerate.

Since \( \pi_{(0)} \) is obviously normal to \( \pi_u \), it follows that the excess operator is \( Q = 1 \) for the restriction of \( \pi_{(0)} \) to any \( \mathcal{L}[\mathcal{I}] \), and hence \( \pi_{(0)} \in \operatorname{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H}_\pi) \). To see that \( \eta^* \pi_{(0)} = \pi \), note that for \( \mathcal{I} \in \mathbb{R}^k \subset \mathcal{V}^\infty \) we have

\[
\eta^* \pi_{(0)}(\mathcal{I}) \pi_{(0)}(L_1 \otimes L_2 \otimes \cdots) = \pi_{(0)}(\eta(\mathcal{I})(L_1 \otimes L_2 \otimes \cdots))
\]

for all elementary tensors \( L_1 \otimes L_2 \otimes \cdots \in \mathcal{A}(\mathcal{V}) \), hence \( \eta^* \pi_{(0)}(\mathcal{I}) \pi_{(0)}(A) = \pi(\mathcal{I}) \pi_{(0)}(A) \) for all \( A \in \mathcal{L}_\mathcal{V} \). Since \( \pi_{(0)} \) is non-degenerate, it follows that \( \eta^* \pi_{(0)} = \pi \) as required.

(iv) By Theorem [5.9](v) we have that \( \pi \in \operatorname{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H}) \) if and only if \( \pi \upharpoonright \mathcal{L}[\mathcal{I}] \) (on its essential subspace \( \mathcal{H}_\pi \)) is normal w.r.t. \( \pi_u(\mathcal{L}[\mathcal{I}]) \) for all \( \mathcal{I} \in \mathcal{V}^\infty \). Let \( \pi \in \operatorname{Rep}(\mathcal{L}_\mathcal{V}, \mathcal{H}) \) be normal w.r.t.
\[ \pi_u(\mathcal{L}_V) \]. Then it is continuous on bounded sets w.r.t. the strong operator topologies of both sides, and it follows that this is true for its restrictions to each \( \pi_u(\mathcal{L}[f]) \), and hence that each restriction is normal w.r.t. \( \pi_u \). Thus \( \pi \in \text{Rep}_0(\mathcal{L}_V, \mathcal{H}) \).

Conversely, given \( \pi \in \text{Rep}_0(\mathcal{L}_V, \mathcal{H}) \) then by part (iii) \( \eta^* \pi \) is a continuous representation of \( \mathbb{R}^{|N|} \), and by Proposition 5.6(iii) (with \( Q \neq 1 \)) we have that on each \( \mathcal{H}_f \n
\pi(L_1 \otimes L_2 \otimes \cdots) = \lim_{n \to \infty} \pi^n(L_1 \otimes L_2 \otimes \cdots \otimes L_n) = \lim_{n \to \infty} \pi_1(L_1)\pi_2(L_2) \cdots \pi_n(L_n)
\]

for all elementary tensors \( L_1 \otimes L_2 \otimes \cdots \in [f^n] \subset *-\text{alg}([f]) \). Now \( \pi_u(\mathbb{R}^{|N|})'' = \{ \pi_u^n(\mathbb{C}^{|N|}) | n \in \mathbb{N} \}'' = \pi_u(\{ \mathcal{L}[f] | f \in \mathcal{V}^N \})'' = \pi_u(\mathcal{L}_V)'' \) and a similar equation holds for \( \pi \). Since the cyclic components of \( \pi \) are contained in the direct summands of \( \pi_u \), there is a normal map \( \varphi : \pi_u(\mathcal{L}_V)'' \to \mathcal{B}(\mathcal{H}) \) such that \( \varphi \circ \pi_u = \pi \). Thus \( \pi \) is normal to \( \pi_u \).

Thus \( \mathcal{L}_V \) is a semi-host for the full representation theory of \( \mathbb{R}^{|N|} \), i.e., \( \eta^* : \text{Rep}(\mathcal{L}_V, \mathcal{H}) \to \text{Rep}(\mathbb{R}^{|N|}, \mathcal{H}) \) is surjective, but not necessarily injective. We want to examine the remaining representations of \( \mathcal{L}_V \) outside of \( \text{Rep}_0(\mathcal{L}_V, \mathcal{H}) \). Denote the universal representation of \( \mathcal{L}_V \) by \( \pi_U : \mathcal{L}_V \to \mathcal{B}(\mathcal{H}_U) \) (not to be confused with the defining representation \( \pi_u \)). Let

\[ Q := \{ Qf_{(\pi_U)} | f \in \mathcal{V}^N \} \subset \mathcal{L}_V'' := \pi_U(\mathcal{L}_V)'' , \]

i.e., the set of all excess operators w.r.t. \( \pi_U \). Since \( Q \) is in the positive part of the unit ball of \( \mathcal{L}_V'' \), it has a natural partial order, and in a moment we will see that \( Q \) is a multiplicative semigroup. Let

\[ \text{Rep}(Q, \mathcal{H}) := \{ \gamma : Q \to \mathcal{B}(\mathcal{H}) | 0 \leq \gamma(Q_1) \leq 1, \gamma(Q_1Q_2) = \gamma(Q_1)\gamma(Q_2) \forall Q_i \in Q \} . \]

### 6.5 Proposition

With notation above, we have

(i) \( Q_{f_1}(\pi_U) \cdot Q_{f_2}(\pi_U) = Q_{f_1 \cdot f_2}(\pi_U) \) for all \( f_i \in \mathcal{V}^N \). Thus \( Q \) is a multiplicative semigroup, and the map \( [f]_\sim \to Q_{\pi}(f) \) is a surjective homomorphism \( \mathcal{V}_\infty \to Q \) of multiplicative semigroups where \( \mathcal{V}_\infty := \{ [f]_\sim | f \in \mathcal{V}^N \} \).

(ii) Fix a non-degenerate \(*\)-representation \( \pi : \mathcal{L}_V \to \mathcal{B}(\mathcal{H}_\pi) \). Then the map \( [f]_\sim \to Q_{\pi}(f) \) defines a representation of \( \mathcal{V}_\infty \) as well as of \( Q \). Thus every \( \pi \in \text{Rep}(\mathcal{L}_V, \mathcal{H}) \) is of the form:

\[ \pi(A) := \pi_0(A)\gamma(f) \quad \text{for} \quad A \in [f] \]

for some \( \pi_0 \in \text{Rep}_0(\mathcal{L}_V, \mathcal{H}) \) and \( \gamma \in \text{Rep}(Q, \mathcal{H}) \) with \( \gamma(Q) \subset \pi(\mathcal{L}_V)' \).

**Proof.** (i) Recall that \( Q_{\pi}(f) := \lim_{n \to \infty} B_n[f] \), where \( B_n[f] := \pi \left( 1 \otimes \cdots \otimes 1 \otimes f_{n+1} \otimes \cdots \right) \). Since the operator product is jointly continuous on bounded subsets we have:

\[ Q_{\pi_U}(\pi_U) \cdot Q_{\pi}(\pi_U) = \lim_{n \to \infty} \pi_U(1 \otimes \cdots \otimes 1 \otimes f_{n+1} \otimes \cdots) \lim_{k \to \infty} \pi_U(1 \otimes \cdots \otimes 1 \otimes g_{k+1} \otimes \cdots) \]

\[ = \lim_{n \to \infty} \pi_U(1 \otimes \cdots \otimes 1 \otimes f_{n+1} \otimes \cdots) \pi_U(1 \otimes \cdots \otimes 1 \otimes g_{n+1} \otimes \cdots) \]

\[ = \lim_{n \to \infty} \pi_U(1 \otimes \cdots \otimes 1 \otimes f_{n+1} \otimes \cdots) \pi_U(1 \otimes \cdots \otimes 1 \otimes g_{n+1} \otimes \cdots) \]

\[ = Q_{\pi_U}(\pi_U) . \]

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It will suffice for this part to show that the map \([f] \to Q_{\pi}(\pi_U)\) is well-defined, i.e., that \(Q_{\pi}(\pi_U)\) only depends on the equivalence class \([f]\) not on any particular representative which is chosen. However, this is immediate from the definition of \(Q_{\pi}(\pi_U)\).

(ii) By the universal property of \(\pi_U\) (cf. Theorem 10.1.12 in [KR86]) there is a central projection \(P_\pi \in Z(\pi_U(\mathcal{L}_\mathcal{V})'')\) and a \(*\)-isomorphism of Von Neumann algebras \(\alpha: P_\pi \pi_U(\mathcal{L}_\mathcal{V})'' \to \pi(\mathcal{L}_\mathcal{V})''\) such that \(\pi(A) = \alpha(P_\pi \pi_U(A))\) for all \(A \in \mathcal{L}_\mathcal{V}\). The map \(\alpha\) is normal in both directions (cf. Proposition 2.5.2 in [Ped89]). It is also true that \(\tilde{\pi}(A) = \alpha(P_\pi \tilde{\pi}_U(A))\) for all \(A \in M(\mathcal{L}_\mathcal{V})\). So it follows from

\[
    Q_{\pi}(\pi) = s\text{-lim}_{n \to \infty} \tilde{\pi}(1 \otimes \cdots \otimes 1 \otimes f_n \otimes f_{n+1} \otimes \cdots) = \alpha(P_{\pi} s\text{-lim}_{n \to \infty} \tilde{\pi}_U(1 \otimes \cdots \otimes 1 \otimes f_n \otimes f_{n+1} \otimes \cdots))
\]

and part (i) that \(Q_{\pi}(\pi) \cdot Q_{\mathcal{G}}(\pi) = Q_{\mathcal{G}}(\pi)\) hence the map \([f]_\sim \to Q_{\pi}(\pi)\) defines a representation of \(\mathcal{V}_\infty\) as well as of \(\mathcal{Q}\). The second claim is immediate.

Thus the the additional part of \(\text{Rep}(\mathcal{L}_\mathcal{V}, \mathcal{H})\) to \(\text{Rep}(\mathcal{R}^{(N)}, \mathcal{H})\) is in \(\text{Rep}(\mathcal{Q}, \mathcal{H})\).

By definition, each \(Q \in \mathcal{Q}\) is the strong operator limit of increasing positive elements in \(\pi_U(\mathcal{L}_\mathcal{V})\), so it is a lower semi-continuous function on the spectrum of \(\mathcal{L}_\mathcal{V}\). In fact, \(\mathcal{Q}\) is in the monotone closure \(\mathcal{L}_\mathcal{V}^m\) (cf. [Tak79] Thm. 6.8 and above, p. 182]). Let \(X\) be the spectrum of \(\mathcal{L}_\mathcal{V}\), and let \(X_0 := X \cap \text{Rep}_0(\mathcal{L}_\mathcal{V}, \mathbb{C})\). Then since \(\omega(Q)\) must be idempotent for \(\omega \in X_0\), \(Q \in \mathcal{Q}\), it has to be 0 or 1. Thus \(X_0 \subset Q^{-1}(\{0\}) \cup Q^{-1}(\{1\})\), and by the definition of \(\text{Rep}_0(\mathcal{L}_\mathcal{V}, \mathbb{C})\) we get that

\[
    X_0 = \bigcap_{Q \in \mathcal{Q}} (Q^{-1}(\{0\}) \cup Q^{-1}(\{1\}))
\]

This suggests that to obtain a full host algebra for \(\mathcal{R}^{(N)}\) we only need to apply the homomorphism which factors out by \(\bigcup_{Q \in \mathcal{Q}} Q^{-1}((0, 1))\), but this is not possible, because we do not know whether the last set is open, as the \(\mathcal{Q}\) are only lower semi-continuous.

### 7 Discussion

Here we constructed an infinite tensor product of the algebras \(C_0(\mathbb{R})\), denoted \(\mathcal{L}_\mathcal{V}\), and used it to find semi-hosts for the full continuous representation theory of \(\mathcal{R}^{(N)}\). Due to commutativity, these were as useful as host algebras, because \(\eta^*\) preserves irreducibility in this context. We also interpreted the Bochner–Minlos theorem in \(\mathcal{R}^{(N)}\) as the pure state space decomposition of the partial hosts which \(\mathcal{L}_\mathcal{V}\) comprises of. We analyzed the representation theory of \(\mathcal{L}_\mathcal{V}\), and showed that \(\eta^*\) is a bijection between \(\text{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H})\) and \(\text{Rep}(\mathcal{R}^{(N)}, \mathcal{H})\), but that there is an extra part which essentially consists of the representation theory of a multiplicative semigroup \(\mathcal{Q}\).

Much further analysis remains, e.g. the topological structure of the spectrum \(X\) of \(\mathcal{L}_\mathcal{V}\), especially the important question as to whether \(X_0\) is locally compact with the relative topology. Moreover, one can easily apply the methods developed here to construct infinite tensor products of other \(C^*\)-algebras without nontrivial projections. A very important issue, is to extend the \(C^*\)-algebraic interpretation of the Bochner–Minlos theorem developed here, to general nuclear spaces.
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