Unwrapping eigenfunctions to discover the geometry of almost-invariant sets in hyperbolic maps

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Abstract

Numerical approximation of Perron-Frobenius operators allows efficient determination of the physical invariant measure of chaotic dynamical systems as a fixed point of the operator. Eigenfunctions of the Perron-Frobenius operator corresponding to large subunit eigenvalues have been shown to describe “almost-invariant” dynamics in one-dimensional expanding maps. We extend these ideas to hyperbolic maps in higher dimensions. While the eigendistributions of the operator are relatively uninformative, applying a new procedure called “unwrapping” to regularised versions of the eigendistributions clearly reveals the geometric structures associated with almost-invariant dynamics. This unwrapping procedure is applied to a uniformly hyperbolic map of the unit square to discover this map’s dominant underlying dynamical structure, and to the standard map to pinpoint clusters of period 6 orbits.

Key words: almost-invariant set, decay of correlations, eigenfunction, eigendistribution, hyperbolic map, isolated spectrum, Perron-Frobenius operator, standard map, Ulam’s method.

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1 Introduction

Uniformly hyperbolic dynamical systems $T : M \rightarrow M$ experience an exponential decay of correlations. More precisely, let $M$ be a smooth compact manifold and $T : M \rightarrow M$ be a transitive $C^2$ Anosov or Axiom A diffeomorphism. Then $T$
possesses a unique SBR measure $\mu$ and satisfies the following property\textsuperscript{2} \[20]. For $\phi, \psi : M \to \mathbb{R}$ sufficiently regular (eg. $C^1$),

$$C_{\phi, \psi}(k) := \left| \int_M \phi \circ T^k \cdot \psi \, d\mu - \int_M \phi \, d\mu \cdot \int_M \psi \, d\mu \right| \leq K(\phi, \psi) r^k,$$ \hfill (1)

for some constant $K = K(\phi, \psi) > 0$ and $0 \leq r < 1$. The rate of decay of linear correlations for nonlinear systems describes how quickly observables become temporally decorrelated or how quickly the system “mixes”; see [24] for a description of broader classes of systems with this property.

Perron-Frobenius operator methods can often be used to estimate the rate of decay of linear correlations via their spectrum, with the leading subunit eigenvalue governing the rate of decay. Briefly this works as follows. Define $P : \mathcal{B}(M) \circlearrowleft$ to be the Perron-Frobenius operator of $T$:

$$Pf(x) = f(T^{-1}x) \cdot \det DT^{-1}(x).$$ \hfill (2)

If $P$ acts on a suitable\textsuperscript{3} Banach space $\mathcal{B}(M, \| \cdot \|)$ then $P$ is quasi-compact. Quasi-compactness implies that $\sigma(P) = \sigma_{\text{ess}}(P) \cup \{ \lambda_1, \ldots, \lambda_L \} \cup \{1\}$, where the essential spectrum $\sigma_{\text{ess}}(P) \subset \{ |z| \leq R \}$ for some $R < 1$, and there are a finite number (possibly zero) of eigenvalues $\lambda_1, \ldots, \lambda_L$ of finite multiplicity with magnitude strictly between $R$ and 1. We assume throughout that the eigenvalue 1 is simple and the corresponding eigenfunction\textsuperscript{4}, denoted by $h$, is positive, of bounded variation, and normalised so that $\int h \, dm = 1$, where $m$ denotes normalised Lebesgue measure on $M$. If $L \geq 1$, it is $\max_{1 \leq \ell \leq L} |\lambda_{\ell}|$ that governs the geometric rate of decay in (1).

Rigorous numerical methods to directly approximate the large subunit eigenvalues are described in [2,13,5]. Suppose now that we are able to find an estimate of an eigendistribution\textsuperscript{5} $g$ of $P$ for large isolated spectral values. What dynamical information can we extract from such an object?

If $M$ is one-dimensional and $T$ is uniformly expanding (and typically many-to-one), these eigenfunctions can directly reveal global properties of the system. For example, [8] describes a family of expanding interval maps for which the second (real and positive) eigenvalue may be as close as desired to 1. For

\textsuperscript{2} assuming $T$ is topologically mixing; if $T$ is not topologically mixing, then (1) applies to $T^p$ where $T$ cyclically permutes $p$ disjoint subsets of $M$.

\textsuperscript{3} The authors [6,15,3] describe suitable Banach spaces for Anosov $T$.

\textsuperscript{4} For transitive $C^2$ Anosov $T$, a necessary and sufficient condition for $\mu$ to be absolutely continuous is that $| \det DT^k(x) | = 1$ whenever $T^k x = x$; [19,22]. The density $h$ of $\mu$ satisfies $P h = h$. Simplicity of the eigenvalue $\lambda = 1$ follows from ergodicity of $\mu$.

\textsuperscript{5} For Anosov $T$, eigenprojections of $P$ corresponding to isolated eigenvalues inside the unit circle map to distributions, not functions; see [6,15].
In the case of uniformly hyperbolic maps, however, it is far from clear what information is being carried by the eigendistributions of large subunit eigenvalues. Theory \cite{6,3,15} and early rigorous numerics \cite{14} suggest these eigendistributions are striated along unstable directions. Numerical calculations for the standard map, Anosov’s cat map, and a uniformly hyperbolic map of the square are discussed in \cite{14}.

The main aim of this paper is to begin developing theory and numerics in the uniformly hyperbolic setting to aid the dynamical interpretation of eigendistri-
tributions corresponding to large eigenvalues. Our focus is not on the rigorous approximation of eigendistributions, which is still a question of intense research, but the manipulation of numerical eigendistribution approximations to provide useful dynamical information.

2 A motivational example

Froyland [14] constructed an area-preserving piecewise linear hyperbolic bijection $T$ of the unit square with the property that $\mathcal{P} : B([0,1]^2) \circ \mathcal{P}$ is quasi-compact with $\sigma_{\text{ess}}(\mathcal{P}) \subset \{ z \in \mathbb{C} : |z| \leq 1/4 + \delta, \delta > 0 \}$ and an eigenvalue $\lambda = 1/2$. Moreover, $\sigma(\mathcal{P}) = \{ \sigma_{\text{ess}}(\mathcal{P}), 1/2, 1 \}$. This is borne out by a numerical calculation of the largest 100 eigenvalues of a finite approximation of $\mathcal{P}$; see Figure 2. The map $T$ stretches uniformly by a factor of 4 in the vertical direction and compresses by a factor of 4 horizontally; see Section 5 and [14] for a detailed definition of $T$. These facts account for the bound $r_{\text{ess}}(\mathcal{P}) \leq 1/4$. We shall see that the isolated eigenvalue 1/2 corresponds to the slow dynamics between the left and right halves of the unit square. These two halves are the almost-invariant sets of $T$.

Fig. 2. The 100 largest (in magnitude) spectral values obtained from a numerical approximation of $\mathcal{P}$ based upon a fine grid of 16384 squares and 64 test points per square. The circle of radius 1/4 denotes the bound for the essential spectral radius. A numerical eigenvalue 0.5 approximates the isolated eigenvalue 1/2. The remaining numerical eigenvalues are 1 and a cluster of 98 eigenvalues near 0. See Section 5 for more details.
We do not go into detail here about the precise definition of $\mathcal{B}([0,1]^2)$, apart from stating that the elements of $\mathcal{B}([0,1]^2)$ are Hölder in unstable directions and distribution-like in stable directions; for further details see [14,6]. A numerical estimate of the eigendistribution $\tilde{g}$ for the eigenvalue $\lambda = 1/2$ is shown in Figure 3. The numerical estimates of this eigendistribution and the eigenvalue $\lambda = 1/2$ converge to the theoretical objects [14]. There is some dynamical sense to this eigendistribution; if one stained phase space with a blob of black ink and iterated, a similar picture would appear. However, the complexity of the eigendistribution masks any underlying dynamical structure.

A related observable, which is arguably more revealing of the dynamics is the following: Define

$$g_0 = \begin{cases} 1, & x \in [0, 1/2) \times [0, 1]; \\ -1, & x \in [1/2, 1] \times [0, 1]. \end{cases}$$ (3)

We now discuss the importance of this function in relation to analysing the map $T$.

(1) **Decay Rate**: The function $g_0$ has the property that

$$C_{g_0/h, g_0/h}(k) = \int_M g_0 \circ T^k \cdot g_0 \, d\mu = 1/2^k,$$
where $\mu = m$ is $T$-invariant. The fact that this decay rate is slow compared to the local rate of expansion (a factor of 4 at each iteration of $T$) is used in the proof of Theorem 1 [14] to demonstrate that $P$ has an isolated eigenvalue $\lambda = 1/2$.

(2) **Eigendistribution Generation:** The function $g_0$ can be considered as a generator of $\tilde{g}$ as it is shown in [14]\(^6\) that $\tilde{g} = \lim_{n \to -\infty} 2^n P^n g_0$. In fact a similar result is true if $g_0$ is replaced by any $\tilde{g}_0$ with zero $m$-integral that does not lie in a co-dimension 2 subspace associated with the essential spectrum.

(3) **Almost-Invariance:** The function $g_0$ describes the dominant underlying geometric structure of the dynamics of $T$. We recall the heuristic argument from [9] for converting an eigenfunction\(^7\) of $P$ into two almost-invariant sets $A^+$ and $A^-$. Since any eigenfunction $g$ of $P$ with eigenvalue $\lambda$ such that $|\lambda| < 1$ must have $\int_M g \, dm = 0$, define

$$A^+ = \{g > 0\} \quad \text{and} \quad A^- = \{g < 0\}.$$  

The sets $A^+$ and $A^-$ are called almost-invariant. This heuristic is precisely the one used in Figure 1. In our example, this heuristic would yield sets $A^+ = [0, 1/2) \times [0, 1]$ and $A^- = [1/2, 1) \times [0, 1]$. These sets are key geometric structures for this map, as the dynamics between the left and right halves of the square is slower than suggested by the local expansion rate. More formally, the proof of Theorem 1 [14] demonstrates that the action of $T$ on the sets $A^+, A^-$ is effectively a 2-state Markov chain with dominant subunit eigenvalue $3/4$. In terms of how invariant these sets are, the conditional probability

$$\rho_\mu(A) := \frac{\mu(A \cap T^{-1} A)}{\mu(A)}$$

quantifies the likelihood that a point in $A$ will remain in $A$ after one iteration. In this example $\rho_\mu(A^+) = \rho_\mu(A^-) = 3/4$. Almost-invariant sets may be viewed as a set-wise version of the slow geometric decay of correlations of observables with $\rho_\mu(A^+)$ playing the role of $C_{g_0/h,g_0/h}(1)$.

In this example, it is reasonably obvious from the structure of the map $T$ how to construct the function $g_0$ with the properties outlined above. However, for an arbitrary hyperbolic map it is not at all obvious, and it would be extremely useful to have a numerical method that could approximate a function $g_0$ with the above properties. In the sequel, we describe a method by which a function $g_0$ with the properties 1–3 above can be recovered from a numerical estimate.

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\(^6\) The notation $h_n$ and $\tilde{h}$ used in [14] correspond to $g_n$ and $\tilde{g}$.

\(^7\) Small smooth random perturbations are added to $P$ in the setting of [9] so that the resulting Perron-Frobenius operator is compact and its eigenprojections are eigenfunctions, not eigendistributions.
of a raw eigendistribution $\tilde{g}$.

3 An introduction to unwrapping

In this section we define the action of the unwrapping operator $\hat{P}$ on integrable functions and state results analogous to the three properties outlined in our motivational example. Denote by $U : L^{\infty}(M, \mu) \odot$ the Koopman operator of $T$, with action $U\psi = \psi \circ T$, $\psi \in L^{\infty}(M, \mu)$. Let $\| \cdot \|_1$ be the standard norm in $L^{1}(M, m)$.

**Definition 1.** Let $g \in L^{1}(M, m)$. We call

$$\hat{P}^q g := h \cdot U^q \left( \frac{g}{h} \right)$$

the $q$-th unwrapping of $g$.

**Remark 1.** For invertible maps, $\hat{P} = P^{-1}$; see Proposition 2 below. We have deliberately represented $\hat{P}$ in terms of $U$ rather than $P^{-1}$, as we shall see in Section 5 that we may obtain a numerical approximation to $U$ from an approximation of $P$ “for free”. Numerically approximating $P^{-1}$ would require a new set of numerical calculations. We have borrowed the notation $^\wedge$ from Markov chain theory, where $^\wedge$ denotes reverse time dynamics.

We note that $\hat{P}$ preserves the $L^{1}(m)$ norm.

**Lemma 1.** If $g \in L^{1}(M, m)$ then $\hat{P}^q g \in L^{1}(M, m)$ and $\| \hat{P}^q g \|_1 = \| g \|_1$.

**Proof.** See proofs section.

We now demonstrate three key relationships between $\hat{P}^q g/h$ and $g/h$ in analogy to the three relationships between $g_0/h$ and $\tilde{g}/h$ listed in Section 2.

Firstly, that the decay of linear correlations experienced by $\hat{P}^q f/h$ and $\hat{P}^q g/h$ is identical to that experienced by $f/h$ and $g/h$.

**Proposition 1** (Decay Rate). Let $f \in L^{\infty}(M, m), g \in L^{1}(M, m)$. Then for $q \geq 0$,

$$C_{\hat{P}^q f/h, \hat{P}^q g/h}(k) = C_{f/h,g/h}(k) \quad \text{for all } k \geq 0.$$  \hspace{1cm} (6)

**Proof.** See proofs section.

Secondly, we show that we can exactly recover any function $g \in L^{1}(M, m)$ from $\hat{P}^q g$ by pushing forward $q$ steps with $P$.

**Proposition 2** (Invertibility). Suppose that $g \in L^{1}(M, m)$ and that $| \det DT(x) | \geq c > 0, x \in M$. Then
(1) $\hat{P}^q \hat{P}^q g = g,$
(2) $\hat{P}^q P^q g = g.$

Proof. See proofs section.

Thirdly, we characterise almost-invariant sets obtained from $\hat{P}^q g$, and demonstrate that the ratio $\rho_\mu$ is unchanged by the unwrapping procedure.

**Proposition 3** (Almost-Invariance). Let $g \in L^1(M, m)$ with $\int_M g \, dm = 0$. Define $\mathcal{A}^+$ as in (4) and set

$$\hat{A}^+ := \{\hat{P}^q g > 0\}.$$  

Then

(1) $\hat{A}^+ = T^{-q} A^+$
(2) $\rho_\mu(\hat{A}^+) = \rho_\mu(A^+).$

Proof. See proofs section.

An analogous result holds for $\hat{A}^- := \{\hat{P}^q g < 0\}$. Thus by Proposition 3 (2), the set $\hat{A}^+$ inherits the almost-invariance properties of $A^+$, and we see that by Proposition 3 (1), the operator $\hat{P}$ pushes around the sets $A^+$ and $A^-$ via $T^{-1}$.

3.1 A Geometrical Interpretation of Unwrapping

Each application of $\hat{P}$ to a function $g$ amounts to a stretching of $\{g > 0\}$ in the stable direction and a compression of $\{g > 0\}$ in the unstable direction. We have already discussed the fact that eigendistributions of $P$ are “striated” along unstable directions. Thus the application of $\hat{P}$ to regularised versions of these eigendistributions will serve to increase their regularity. We expect that if $g$ is a regularised version of an eigendistribution of $P$ then the set $\{\hat{P}^q g > 0\}$ will be more “regular” than the set $\{g > 0\}$, while both sets share identical almost-invariance properties. In fact, we may think of the sets $\{\{\hat{P}^q g > 0\}\}_{q \in \mathbb{Z}} = \{\hat{A}^+\}_{q \in \mathbb{Z}}$ as belonging to a single equivalence class of Borel measurable sets, where $A \sim B$ if $\exists q \in \mathbb{Z}$ so that $A = T^{-q} B$. By Proposition 3 (2) all elements of this equivalence class are equally almost-invariant, but there will be a particular $q$ that maximises regularity of $\hat{P}^q g$. As greater regularity is accompanied by a lack of striation in either stable or unstable directions, we argue that $P^q g$ should more clearly reveal dynamical structures associated with almost-invariance than $g$, even though both functions effectively carry the same information. We demonstrate this in Section 5 via two examples.
We now describe in more detail the formal assumptions we will operate under. **Assumptions 1.** Assume that we have selected a Banach space of distributions \((B(M), \| \cdot \|)\) so that \(P : B(M) \ominus \) is quasi-compact. We assume that the nontrivial isolated spectrum is non-empty; that is, there is an eigenvalue \(\lambda\) with \(\text{ess}(P) < |\lambda| < 1\). Denote by \(\tilde{g}\) the eigendistribution corresponding to \(\lambda\), so that \(P\tilde{g} = \lambda \tilde{g}\) where \(\tilde{g} \in B(M)\).

We have assumed that we are in a setting where the Perron-Frobenius operator is quasi-compact for two reasons. Firstly, quasi-compactness implies exponential decay of correlations with respect to suitably regular observables and the rate of decay is intimately connected with the spectrum. Secondly, the numerics we employ in Section 5 are finite in nature and we can reasonably expect to only approximate spectral values that are isolated and finite in multiplicity; in other words, that part of the spectrum that lives outside the essential spectrum and governs the rate of decay. Even more succinctly, our numerical approximations are compact operators and can therefore only approximate the compact part of \(P\).

Classical Banach spaces on which \(P\) is quasi-compact and \(B(M)\) is a function space are:

- \(M = [0, 1], T : [0, 1] \ominus\) is piecewise \(C^2\) expanding,
  \[B(M) = \{\phi : [0, 1] \to \mathbb{R}, \phi \text{ has bounded variation}\},\]
  \[\| \cdot \| = \| \cdot \|_{L^1} + \text{var}(\cdot)\] (see eg. Theorem 1 [16]).
- \(M \subset \mathbb{R}^d, T : M \ominus\) is \(C^{1+\alpha}\) expanding, \(B(M) = \{\phi : M \to \mathbb{R}, \phi \text{ is } \alpha\text{-Hölder}\}\) and \(\| \cdot \|\) is the standard \(\alpha\text{-Hölder norm}\) (see eg. Theorem 3.2 [21]).

As mentioned in the introduction, in the one-dimensional expanding setting the almost-invariant set analysis of eigenfunctions corresponding to large eigenvalues is relatively well understood and the techniques we describe here are not necessary. We are concerned with the hyperbolic setting, for which for which the dynamical interpretation of eigendistributions is not at all clear. An example of a Banach space on which \(P\) is quasi-compact and \(B(M)\) is a space of distributions is is:

- \(M = \mathbb{T}^2, T : \mathbb{T}^2 \ominus\) is \(C^3\) Anosov,
  \[B(M) = \{\text{distributions that are “smooth in unstable directions”}\}\]
(see Theorem 1 [6] or Theorem 2.3 [15]; see also [3]).

Because of the contracting dynamics in stable directions, from a spectral point
of view it is not useful to consider \( \mathcal{P} \) to act on smooth function spaces with standard strong norms. By considering Banach spaces of distributions as above with a norm that is strong in unstable directions and weak in stable directions, a desirable spectral picture can be obtained. However, the weakness of the norm in the stable directions allows for eigendistributions of infinite irregularity in stable directions.

In Section 3 we defined \( \hat{\mathcal{P}} \) on integrable functions; moreover the heuristic described in (4) and the results of Propositions 1–3 only make sense when applied to integrable functions. In fact, the geometrical arguments at the conclusion of Section 3 rely on the fact that the function \( g \) has finite irregularity. Clearly, we cannot apply the unwrapping formalism to eigendistributions of \( \mathcal{P} \). In analogy with the geometric arguments given at the end of Section 3 we instead propose to apply the unwrapping operator \( \hat{\mathcal{P}} \) to a function \( g \) of finite irregularity that is close to an eigendistribution of \( \mathcal{P} \) in terms of the norms of eg. [6,15]. Such a function \( g \) may also be arrived at by applying a suitable regularisation directly to the eigendistribution \( \tilde{g} \). In Section 6 we will argue informally that eigenfunctions of numerical approximations of \( \mathcal{P} \) provide good regularisations of \( \tilde{g} \). For concreteness, we briefly outline here an alternative approach to regularising \( \tilde{g} \), however, we emphasise that this is not the practical approach we take in Section 6.

**Definition 2 ([6]).** Define a smooth averaging operator \( Q_\epsilon \) by

\[
Q_\epsilon f(y) := \int_M f(x)q_\epsilon(x,y) \, dm(x),
\]

(7)

where \( q_\epsilon(x,y) = \epsilon^{-d} \bar{q}(x-y)/\epsilon), \epsilon \in (0,1] \) and \( \bar{q} \) is nonnegative, smooth, with support contained in the unit ball in \( \mathbb{R}^d \) centred at the origin, \( \int_M \bar{q} \, dm = 1 \) and \( \inf \{ \bar{q}(x) : |x| \leq 1/2 \} > 0 \).

Form a smoothed Perron-Frobenius operator \( \mathcal{P}_\epsilon := Q_\epsilon \mathcal{P} \). Theorem 2 [6] states that the isolated spectrum outside the bound for \( r_{\text{ess}}(\mathcal{P}) \) given in Theorem 1 [6] is stable under the smooth perturbations of Definition 2, as are the eigenprojections corresponding to these spectral values with respect to a suitable norm. Thus, by adding a very small random perturbation we may approximate eigendistributions \( \tilde{g} \) of \( \mathcal{P} \) by smooth eigenfunctions \( g_\epsilon \) of \( \mathcal{P}_\epsilon \). While in principle, this is an elegant method of creating regularised versions of \( \tilde{g} \), we do not pursue this approach in Section 6 as it is not amenable to numerical computations.
5 Numerically unwrapping a regularised version of an eigendistribution

We explore the possibility of *numerically* unwrapping regularisations $g$ of eigendistributions $\tilde{g}$ of $P$. Our approach is to use Ulam’s method [23] as a basis for our new unwrapping method, as Ulam’s method and its various extensions have proven to be efficient and effective means of numerically studying the ergodic properties of piecewise smooth transformations. We begin by providing relevant background information on Ulam’s method.

5.1 Ulam approximation

Let $T$ satisfy Assumptions 1. Let $\mathfrak{P}_n = \{A_1, \ldots, A_n\}$ denote a measurable partition of $M$ with $m(A_i) > 0$ and $A_i$ connected for $i = 1, \ldots, n$. The standard Ulam approximation of $P$ is $\pi_n P$ where $\pi_n : L^1(M, m) \to \text{sp}\{\chi_{A_1}, \ldots, \chi_{A_n}\}$ is the canonical projection onto the span of characteristic functions on elements of $\mathfrak{P}_n$ defined by

$$\pi_n g = \sum_{k=1}^{n} \frac{\int_{A_k} g \, dm}{m(A_k)} \chi_{A_k}. \quad (8)$$

When restricted to $\text{sp}\{\chi_{A_1}, \ldots, \chi_{A_n}\}$, the action of $\pi_n P$ is described by the matrix\(^8 [18] \)

$$P_{n,ij} := \frac{m(A_i \cap T^{-1}A_j)}{m(A_j)}. \quad (9)$$

The left eigenvector $p_n$ satisfying $p_n P_n = \lambda_n P_n$ with $|\lambda_n| < 1$ is the method we used to construct the numerical eigenfunction shown in Figure 3 where $\mathfrak{P}_n$ was a regular $128 \times 128$ partition of $[0, 1]^2$ into squares.

\(^8\) The practical numerical calculation of (9) may be carried out efficiently using the algorithms described in [7] and the software GAIO.
After numerically determining the entries in the matrix $P_{128^2}$ we computed the eigenvector $v_{128^2}$ corresponding to the eigenvalue $\lambda_{128^2} \approx 1/2$. This eigenvector $v_{128^2}$ defined a piecewise constant function $g_{128^2} = \sum_{i=1}^{128^2} v_i \chi_{A_i}$ which was shown [14] to converge to the eigendistribution $\tilde{g}$ satisfying $P \tilde{g} = \tilde{g}/2$.

This convergence result relied upon the partition elements being aligned with the stable and unstable directions of $T$ and the ability to separate the action of $T$ into two one-dimensional dynamics; one expanding and one contracting. The approximation of eigendistributions corresponding to subunit eigenvalues is less well-studied than the approximation of the leading fixed eigenfunction of $P$. For expanding maps of the interval without periodic turning points, Blank and Keller [5] showed that any eigenvalue of $P$ lying outside the essential spectral radius may be approximated by eigenvalues of the Ulam matrix; this extended a corresponding result for convolution-type perturbations [4].

In the Anosov setting Froyland [13] proved an equivalent result when the Ulam partition is a Markov partition, while Blank et al. [6] demonstrate convergence of a highly smoothed Ulam approximation, where the size of the smooth perturbation is $\epsilon$ and the partition set diameters approach zero as $\epsilon^3$. Counterexamples to the stability of the spectrum for a pure Ulam method (that is, using precisely equation (9)) are described in [6,5], however, in all cases, there are no isolated eigenvalues for Ulam’s method to approximate. Froyland [14] presents some positive numerical experiments for a pure Ulam’s method in the presence of an isolated eigenvalue. The Ulam approximation of subunit spectral values and eigendistributions in the Anosov setting is a difficult open question that we do not consider here. Our focus is on attempting to extract further information from the results of Ulam’s method.

5.2 Approximating $\hat{\mathcal{P}}^q g$

By analogy with Ulam’s method, a natural approximation of $\hat{\mathcal{P}}^q g$ as defined in (5) is

$$\hat{\mathcal{P}}^q_n g = h_n \cdot \left( (\pi_n \mathcal{U})^q \left( \frac{\pi_n g}{h_n} \right) \right).$$  \hspace{1cm} (11)

For brevity, we write $\pi_n g = \sum_{i=1}^{n} \tilde{g}_{n,i} \chi_{A_i}$. The action of $\pi_n \mathcal{U}$ on $\mathrm{sp}\{\chi_{A_1}, \ldots, \chi_{A_n}\}$ is described by the matrix [14]

$$U_{ij} := \frac{m(A_j \cap T^{-1}A_i)}{m(A_j)}.$$
As we have vector and matrix representations of $h_n$, $\pi_n U$, and $\pi_n g$, we can rewrite our approximation to $\hat{P}^q g$ in matrix form as

$$\hat{P}^q_n g = \sum_{i,j=1}^{n} \left( \frac{p_{n,i}(U_n^q)_{ji} g_{n,j}}{p_{n,j}} \right) \chi_{A_i}. \quad (12)$$

**Definition 3.** Define the $n \times n$ stochastic matrix $\hat{P}_n$ by

$$\hat{P}_{n,ji} = \frac{p_{n,i}(U_n^q)_{ji}}{p_{n,j}}. \quad (13)$$

As $(\hat{P}^q_n)_{ji} = p_{n,i}(U_n^q)_{ji}/p_{n,j}$, we may write

$$\hat{P}^q_n g = \sum_{i,j=1}^{n} \left( g_{n,j} \hat{P}^q_{n,ji} \right) \chi_{A_i}. \quad (14)$$

### 5.3 Properties of the approximation $\hat{P}^q_n g$

We seek to demonstrate that $g$ and $\hat{P}^q_n g$ enjoy relationships similar to those of $g$ and $\hat{P}^q g$ described in Propositions 1–3. Because our numerical constructions are finite in nature, we will see that we recover exactly these corresponding properties only in the $n \to \infty$ limit.

A natural approximation of $C_{f/h,g/h}(1) := |\int_M (f/h) \cdot T \cdot g/h \, d\mu| = |\int_M U(f/h) \cdot g/h \, d\mu|$ is the computable quantity

$$C_{\pi_n f/h_n,\pi_n g/h_n,n}(1) := \left| \int_M \pi_n U \left( \frac{\pi_n f}{h_n} \right) \cdot \frac{\pi_n g}{h_n} \, d\mu_n \right|.$$ 

**Assumptions 2.** Assume that $|\det DT| \geq c > 0$. Let $f \in L^\infty(M,\mu)$, $g \in L^1(M,m)$, with $f, g \leq F < \infty$. Assume that $h \leq H < \infty$, and that $\|h_n - h\|_1 \to 0$ as $n \to \infty$.

The following proposition states the property corresponding to (6) for our numerical approximations.

**Proposition 4 (Decay Rate).** Under Assumptions 2,

1. $|C_{\hat{P}^q_n,f/h_n,\hat{P}^q_n,g/h_n,n,n}(1) - C_{\pi_n f/h_n,\pi_n g/h_n,n,n}(1)| \to 0$ as $n \to \infty$. \quad (15)
2. $C_{\pi_n f/h_n,\pi_n g/h_n,n,n}(1) \to C_{f/h,g/h}(1)$ as $n \to \infty$. \quad (16)

**Proof.** See Proofs section.
The following proposition states the analogous property to Proposition 2 (1), and formalises the sense in which the matrices $P$ and $\hat{P}$ are “inverses” of one another.

**Proposition 5 (Invertibility).** Under Assumptions 2,

(1) \[ \| g - (\pi_n P) \hat{P}_n g \|_1 \to 0 \text{ as } n \to \infty. \]

(2) Writing $\pi_n g = \sum_{i=1}^{n} g_{n,i} \chi_{A_i}$, then equivalently,

\[ \| \bar{g}_n (\text{Id}_n - \hat{P}_n P_n) \|_{1,n} \to 0 \text{ as } n \to \infty, \]

where $\text{Id}_n$ is the $n \times n$ identity matrix and $\| w \|_{1,n} = \sum_{i=1}^{n} |w_i| m(A_i)$.

**Proof.** See proofs section.

Proposition 6 states that applying the almost-invariant set heuristic described in Section 2 to $\hat{P}_n g$ results in a set that is contained in a small neighbourhood of the preimage of the set formed by the heuristic applied to $g$.

**Proposition 6 (Almost-Invariance).** Let $g \in L^1(M,m)$ with $\int_M g \, dm = 0$. Denote $\delta_n = \max_{1 \leq i \leq n} \text{diam}(A_{n,i})$ and let $B_{\delta_n}(A) = \{ y \in M : \inf_{x \in A} \| x - y \| \leq \delta_n \}$ for $A \subset M$. Then

\[ \hat{A}_{1+}^n := \{ \hat{P}_n g > 0 \} \subset B_{\delta_n}(T^{-1} B_{\delta_n}(\{g > 0\})). \]

**Proof.** See proofs section.

A similar result holds for $\hat{A}_{1-}^n := \{ \hat{P}_n g < 0 \}$.

**Remark 2.** In Propositions 4 to 6 we have stated and proven results for $q = 1$. Similar proofs may be given for any finite $q$.

### 6 Numerical Implementation and Results

The purpose of the unwrapping procedure is to transform a function $g$ that is a regularised version of an eigendistribution of $P$ into a related function $\hat{P}_n^q g$ that reveals important geometric information about the dynamics of the map via almost-invariant set considerations. In practice, we will construct the function $g$ as an eigenfunction $g_n$ of $\pi_n P$.

**Remark 3.** Strictly speaking, we should apply $\hat{P}_n^q$ to a $g$ that arises via a regularisation as outlined at the conclusion of Section 4. However, the accurate numerical modelling of such regularisations on top of the numerical discretisation would be prohibitively expensive. We point to the positive numerical and
theoretical results of [14] to support the approach of using eigenfunctions of \(\pi_n P\) as regularisations of eigendistributions of \(P\).

We regard the properties described in Propositions 1, 2, 4, 5 as relevant consistency checks on the relationship between \(g, P^q g,\) and \(\hat{P}^q g,\) ensuring that the unwrapped functions retain important dynamical properties. It is not our intention to explicitly compute \(C^\hat{P}^q_n f/h_n, \hat{P}^q_n g/h_n, (k)\) for some \(f\) and \(g\) as this tells us no more than an estimate of \(C_{f,g}(k)\) would, and the outer spectral values of \(\pi_n P\) may already (for an appropriate functional analytic setting) suggest an estimate of the rate of decay. Similarly, we do not intend in practice to numerically re-generate \(g\) from \(\hat{P}^q_n g;\) nevertheless in principle it is important to know that one could do so. Our primary practical objective is to apply the unwrapping procedure to discover almost-invariant and almost-cyclic structures. Propositions 3 and 6 describe how the almost-invariant sets defined by \(g\) are related to those defined by \(P^q g\) and \(\hat{P}^q_n g,\)

As we have chosen to use eigenfunctions \(g_n\) of \(\pi_n P\) as regularisations of eigendistributions, the regularity of \(g_n\) depends on the fineness of the partition \(\{A_1, \ldots, A_n\}\) used in the numerical calculations. The choice of \(q\) in the unwrapping procedure therefore also depends upon the fineness of this partition; a finer partition generally means a larger value of \(q\) is required to remove the finer striation and greater irregularity. We employ a variation-minimizing criterion described in detail in §6.1. The variation-minimizing criterion for selecting \(q\) has worked very well in all of the cases we have tested. The ability to visualise the raw eigenfunction \(g_n\) and its unwrappings \(\hat{P}^q_n g_n\) is also important for interpreting the resulting unwrapped function.

**Main Algorithm**

**Preliminaries**

1. Partition state space \(M\) into \(\{A_1, \ldots, A_n\}\).
2. Construct Ulam matrix \(P\) according to (9).
3. Compute the unique fixed left eigenvector \(p_n\) and construct the matrix \(\hat{P}\) according to (13).
4. Compute the outer spectrum of the sparse matrix \(P\).
5. Eigenvalues \(\lambda\) that lie outside the theoretical bounds for the essential spectral radius\(^9\) have a chance of approximating isolated spectral values of \(P.\) Ideally, one would compute spectra for several different fine partitions \(\{A_1, \ldots, A_n\}\) and try to observe some sort of numerical stability of

---

\(^9\) In the setting of [6], \(r_{ess}(P) \leq \sigma,\) where \(\sigma > \max\{\lambda^{-1}_u, \lambda^3\},\) \(\|DT^n(x)|_{E^u(x)}\| \leq C\lambda^{-n}\) for all \(n \geq 0\) and \(x \in M,\) and \(0 \leq \beta < 1;\) see [6] for precise details.
a particular numerical eigenvalue.
(6) Select an eigenvalue $\lambda_n$ and its corresponding eigenvector $v_n$ defining the
eigenfunction $g_n$.

Unwrapping

(1) If $T$ is hyperbolic, visual inspection of the values of $g_n$ should reveal
striation along unstable directions; see eg. Figure 3.
(2) Unwrap $g_n$ to produce $\hat{P}^q_n g_n$ for some low values of $q$ and compute the
variation of $\hat{P}^q_n g_n$ to identify a local minimum at $q = q^*$; see §6.1 for de-
tails. Visual inspection of $\hat{P}^q_n g_n$ should reveal no strong striation in either
the unstable or stable directions. If the values of $\hat{P}^q_n g_n$ are striated along
stable directions, one has unwrapped too far and $q$ should be reduced; see
eg. Figure 7. Most meaningful results are achieved when $q$ is selected so
that the variation of $\hat{P}^q_n g_n$ is minimized and there is no obvious striation
in either unstable or stable directions; see eg. Figures 5 and 6.
(3) For the selected value of $q$ visually inspect $\hat{P}^q_n g_n$. The function $\hat{P}^q_n g_n$
should be viewed as “separating” regions in $M$. The sets $\hat{A}^q_n^+$ and $\hat{A}^q_n^-$
create two regions in $M$ separated by the sign of $\hat{P}^q_n g_n$. An extension of
this idea is to consider sets $\{\hat{P}^q_n g_n > c\}$ and/or $\{\hat{P}^q_n g_n < c\}$ for $c \neq 0$. If
there are regions of $M$ highlighted by a peak or trough in the value of
$\hat{P}^q_n g_n$, these regions are important and are captured by a suitable choice
of $c$.
(4) If $\lambda_n$ is real and positive the identified regions $\hat{A}^q_m^+$ are expected to be
almost-invariant; that is,

$$\frac{\mu(\hat{A}^q_m^+ \cap T^{-1} \hat{A}^q_m^+)}{\mu(\hat{A}^q_m^+)}$$

is close to 1; the closer $\lambda_n$ is to 1, the closer this ratio will be to 1. This
is illustrated in Example 1 below.

If $\lambda_n$ is an $s$th root of a real, positive number, then the identified regions
$\hat{A}^q_n^+$ form part or all of an $s$-cycle. That is, there is a collection of regions
$\{\hat{A}_1, \ldots, \hat{A}_s\}$ such that

$$\frac{\mu(\hat{A}_i \cap T^{-1} \hat{A}_{(i+1) \ (\text{mod} \ s)})}{\mu(\hat{A}_i)}, i = 1, \ldots, s$$

is close to one. Again, the closer $|\lambda_n|$ is to one, the closer these ratios will
be to one. This is illustrated in Example 2 below.
6.1 Selecting an appropriate value for $q$

The issue of “how much to unwrap” a given regularised eigenfunction is in practice often best resolved by visual inspection of the unwrappings. We aim for the unwrapping procedure to eliminate all of the striation along unstable directions without introducing striation along stable directions. A quantitative argument sympathetic with this idea is as follows: by unwrapping we hope to convert the complicated striated numerical eigenfunction $g_n$ into a “simpler” function $\hat{P}^q g_n$ that more clearly reveals the almost-invariant dynamics. By “simpler” we mean “more regular”, and we propose to select $q$ by minimizing a quantity that measures the regularity of $\hat{P}^q g_n$. As $g_n$ and its numerical unwrappings are step functions, an obvious approach is to calculate the variation of $\hat{P}^q g_n$.

For brevity, we describe the two-dimensional case where $M$ is either boundaryless (for example, a torus) or has a boundary whose elements are aligned with the coordinate axes. Further, we assume that the partition $\mathcal{P}_n = \{ A_1, \ldots, A_n \}$ has rectangular elements with boundaries aligned with the coordinate axes; in other words $\mathcal{P}_n$ is a “grid-like” partition. This setup covers the two examples we present in this section. For $i = 1, \ldots, n$ denote by $A_i^{\text{right}}$ and $A_i^{\text{top}}$ the partition sets that share the right-hand and top boundary (respectively) with $A_i$. If the set $A_i$ is at a right-hand or upper boundary of $M$, we will say that $A_i^{\text{right}}$ and $A_i^{\text{top}}$ do not exist and in particular do not appear in (18) below. Let $x_i, x_i^{\text{right}}$ and $x_i^{\text{top}}$ represent candidate points in the interiors of $A_i, A_i^{\text{right}},$ and $A_i^{\text{top}},$ respectively and denote by $\ell(\partial^{\text{right}} A_i)$ and $\ell(\partial^{\text{top}} A_i)$ the one-dimensional length of the right-hand and upper boundaries of $A_i$, respectively. Then the variation of $\hat{P}^q g_n$ is given by

$$\text{var}(\hat{P}^q g_n) := \sum_{i=1}^{n} |\hat{P}^q g_n(x_i) - \hat{P}^q g_n(x_i^{\text{right}})| \cdot \ell(\partial^{\text{right}} A_i) + |\hat{P}^q g_n(x_i) - \hat{P}^q g_n(x_i^{\text{top}})| \cdot \ell(\partial^{\text{top}} A_i).$$

(18)

Higher-dimensional situations may be treated in an analogous way.

Example 1: The Quadbaker map

We return to the map $T$ considered in Section 2. The map $T : [0,1]^2 \to [0,1]^2$ defined below has been studied in detail in [14].
We wish to reveal the geometric structures associated with the eigenvalue \( \lambda = 1/2 \). Because \( \lambda \) is real and positive, we expect to find sets which are almost-inv\textit{a}ariant. In fact, we hope to be able to reconstruct the function \( g_0 \) described in Section 2 which perfectly detects the two almost-inv\textit{a}ariant sets \([0, 1/2] \times [0, 1] \) and \([1/2, 1] \times [0, 1] \). By producing Figures 2 and 3, we have already completed the Preliminary steps of our Main Algorithm: we have chosen \( n = 2^{14} \) and \( \{A_1, \ldots, A_n\} \) is a partition of \([0, 1]^2 \) into equally sized squares. We have computed the matrix \( P_{16384} \) using the software GAIO [7] and its outer spectrum using MATLAB. From Figure 2 it is clear that the numerical eigenvalue \( \lambda_{16384} = 0.5 \) of \( P_{16384} \) is the one we should select to estimate the isolated eigenvalue \( \lambda = 1/2 \) of \( P \).

Now to the Unwrapping part of our Main Algorithm. Figure 3 displays the expected striation along unstable directions. Figure 4 shows the result of unwrapping once \( (q = 1) \). Practically, since \( h_n = 1 \) and \( m(A_i) = m(A_j) \) for all \( i, j = 1, \ldots, n \), we have \( \tilde{P}_{16384} = P_{16384}^\top \), and so the unwrapping calculation amounts to simply multiplying the eigenvector \( g_{16384} \) by the transpose of \( P_{16384} \). There is still some striation along unstable directions, although \( \tilde{P}_{16384}g_{16384} \) is significantly more regular in stable directions than \( g_{16384} \). Calculating the variation of \( \tilde{P}_{16384}^q g_{16384} \) for \( q = 0, \ldots, 6 \) reveals a minimum at \( q = 3 \); see Figure 5. As suggested by the variation calculations, we unwrap twice more, multiplying \( \tilde{P}_{16384}g_{16384} \) by \( P_{16384}^\top \) twice to obtain \( \tilde{P}_{16384}^3 g_{16384} \). The result is shown in Figure 6. This is a good place to stop; the function \( \tilde{P}_{16384}^3 g_{16384} \) delineates two large connected regions of \([0, 1]^2 \) with no striation along unstable directions. Unwrapping further produces striation in stable directions and increasing variation of \( \tilde{P}_{16384}^q g_{16384} \); see Figures 5 and 7. Applying the heuristic described in Unwrapping step 3 to \( \tilde{P}_{16384}^3 g_{16384} \) with \( c = 0 \), we obtain the almost-inv\textit{a}ariant sets \( \tilde{A}_{16384}^+ = [0, 1/2] \times [0, 1] \) and \( \tilde{A}_{16384}^- = [1/2, 1] \times [0, 1] \); see Figure 6. These sets geometrically separate the unit square into precisely the two regions that are responsible for the isolated eigenvalue \( \lambda = 1/2 \), and for the slow dynamics associated with this eigenvalue. This slow dynamics and geometry has been detected by the Perron-Frobenius operator and encoded by the isolated eigenvalue \( \lambda = 1/2 \) and the corresponding eigendistribution \( \tilde{g} \). However, this information is only revealed in a useful geometric way when the

\[
T(x, y) = \begin{cases} 
(x/4, 4y), & x \in [0, 1/2), y \in [0, 1/4); \\
(x/4 + 1/8, 4y - 1), & x \in [0, 1/2), y \in [1/4, 1/2); \\
(x/4 + 1/4, 4y - 2), & x \in [0, 1/2), y \in [1/2, 3/4); \\
(x/4 + 1/2, 4y - 3), & x \in [0, 1/2), y \in [3/4, 1]; \\
((x - 1)/2)/4 + 3/8, 4y - 3), & x \in [1/2, 1], y \in [3/4, 1]; \\
((x - 1)/2)/4 + 5/8, 4y), & x \in [1/2, 1], y \in [0, 1/4); \\
((x - 1)/2)/4 + 3/4, 4y - 1), & x \in [1/2, 1], y \in [1/4, 1/2); \\
((x - 1)/2)/4 + 7/8, 4y - 2), & x \in [1/2, 1], y \in [1/2, 3/4).
\end{cases}
\]
Fig. 4. Result of one application of the unwrapping procedure applied to the numerical eigenfunction shown in Figure 3. The shade of the figure denotes the value of the function, with white corresponding to +1 and black to −1.

Fig. 5. Determining a suitable value $q$ for the Quadbaker map by calculation of variation of $\hat{\mathbf{P}}_{16384}^q g_{16384}$ approximating eigenfunction $g_n$ is unwrapped to form $\hat{\mathbf{P}}_n^q g_n$.

For this example we obtain the following result that states that for a suitably chosen partition, we can exactly recover the desired function $g_0$ as an unwrapping of the numerical eigenfunction $g_n$. 
Theorem 1. Let \( \{I_1, \ldots, I_{8^q}\} \) equipartition the unit interval, \( q \geq 0 \), and let \( \{J_1, \ldots, J_S\} \) partition the unit interval. Partition the unit square via \( \{I_p \times J_s\}_{p=1,\ldots,8^q; s=1,\ldots,S} := \{A_i\}_{i=1,\ldots,8^q \cdot S} \) and construct

\[
P_{8^q \cdot S, ij} = \frac{m(A_i \cap T^{-1} A_j)}{m(A_i)}.
\]

Compute the left eigenvector \( v_{8^q \cdot S} \) corresponding to the eigenvalue \( 1/2 \) and construct the function \( g_{8^q \cdot S} = \sum_{i=1}^{8^q \cdot S} v_{8^q \cdot S, i} \chi_{A_i} \). Then \( g_0 = P_{8^q \cdot S}^{q+1} g_{8^q \cdot S} \), where \( g_0 \) is defined by (3).

**Proof.** See proofs section. \( \square \)

**Example 2: The Standard map**

We consider the Standard map \( S : \mathbb{T}^2 \), defined by

\[
S(x, y) = (x + y, y + a \sin(x + y)) \pmod{2\pi}.
\]

We choose the parameter \( a = 8 \); for such a choice, Lebesgue measure appears numerically to be ergodic and \( S \) to be non-uniformly hyperbolic, in contrast
with the invariant KAM curves existing at lower parameter values (see, for example, [1], p.477). We choose \( n = 2^{16} \) and partition the flattened 2-torus into equally sized squares. The matrix \( P_{65536} \) is constructed and its outer spectrum computed. Figure 8 displays the 100 largest (in magnitude) eigenvalues of a matrix \( P_{65536} \). The four largest real eigenvalues are approximately 1, 0.7032, 0.6817, and -0.6763. There appears to be a six-fold symmetry present in the outer spectrum; the real eigenvalues 0.6817 and -0.6763 form part of a six-member set of eigenvalues that are approximately multiples of \( e^{i\pi/3} \) of one other. We select the eigenvalue \( \lambda_{65536} \approx 0.6817 \) and expect that some almost-6-cyclic behaviour is encoded in the numerical eigenfunction \( g_{65536} \) shown in Figure 9. In Figure 9 and subsequent figures, we have conjugated \( S \) with a vertical translation \((x, y) \mapsto (x, y + \pi)\) to make the six-fold symmetry more apparent. Figure 10 indicates that by setting \( q = 2 \) we can achieve a significant reduction in the variation of \( \hat{P}_q g_n \) over that of \( g_n \). Unwrapping twice \((q = 2)\) we obtain \( \hat{P}_{65536}^2 g_{65536} \) shown in Figure 11. The six-fold symmetry is significantly more apparent in Figure 11 than Figure 9. The unwrapping procedure has resulted in a localisation of two collections of highlighted regions; one highlighted in black, the other in gray. We focus on the black highlighted

---

\(^{10}\)We remark that we make no claims of the existence of formal eigenvalues and eigendistributions for \( P \); indeed a functional analytic formalism for this setting has not been developed. Our intention is to perform a numerical study, supported by the results of Section 5.
Fig. 8. The 100 largest (in magnitude) spectral values obtained from the matrix $P_{65536}$ based upon 65536 grid sets and 1000 test points per grid set.

Fig. 9. Standard map: Numerical eigenfunction of $\pi_n P$ corresponding to numerical eigenvalue 0.6817. Constructed from an eigenvector of a transition matrix $P_{65536}$ using 65536 grid sets and 1000 points per grid set. The shade of the figure denotes the value of the function with black corresponding to $+1$ and white to $-1$. 
Fig. 10. Determining a suitable value $q$ for the Standard map by calculation of the variation of $\hat{P}^q_{65536965536}$

Fig. 11. Standard Map: Result of two applications of the unwrapping procedure applied to the eigenfunction of $\pi_n P$ shown in Figure 9. The shade of the figure denotes the value of the function with black corresponding to $+1$ and white to $-1$.

regions as the gray regions do not appear in the other 5 eigenfunctions in our family of 6 eigenfunctions. We define sets $A_1, \ldots, A_6$ as the connected components of the set $\{P^2_{65536965536} > c\}$ for a value of $c$ that extracts the highlighted regions. Of course the choice of $c$ is somewhat arbitrary; a rule of thumb is to choose $c$ as a natural “break point” where the gradient of $\hat{P}^2_{65536965536}$ is large;
in this example, we choose $c$ so as to extract the regions that our eye sees as “highlighted”.

By direct computation (filling the highlighted regions with test points and iterating those points under $S$) we can verify that the six highlighted regions form an almost-cycle; that is, one iteration of $S$ “almost-permutes” the collection $A_1, \ldots, A_6$ whereby under suitable indexing (modulo 6), $S(A_i) \approx A_{i+1}$. This is precisely what we would expect from a six-fold symmetry in the outer spectrum of $P_{65536}$.

Probing further the cause of this almost-6-cycle, we investigate whether the regions $A_1, \ldots, A_6$ contain a period 6 orbit. We search for period 6 points in a neighbourhood of $\bigcup_{i=1}^{6} A_i$. We define such a neighbourhood by $\{P^2_{65536} g_{65536} > c'\}$, where $c' < c$; the neighbourhood used is shown in Figure 12.

![Fig. 12. Initial search neighbourhood in which we look for period 6 points.](image)

Using a set-based approach [17] we find partition boxes in the search neighbourhood that could possibly contain a period 6 orbit. Those that cannot are discarded; the remaining sets are refined and the test repeated. Proceeding until the partition sets are $2^{-50}$ the area of the entire torus, we obtain Figure 13. These identified regions very closely match the sets $A_1, \ldots, A_6$. Zooming in on one collection reveals nine separate sets; see Figure 14. Zooming further reveals that each of the nine sets is in fact a connected collection of the boxes of area $2^{-50}$ that of the torus. The dots shown in Figure 15 are centre points of these regions. In summary, we can conclude rigorously that the only possible locations for period 6 points in the region shown in Figure 12 are contained in the tiny boxes shown in Figure 13. These numerical experiments strongly
Fig. 13. The regions in which numerical experiments strongly suggest period 6 points may lie. With certainty, there are at least nine $\delta$-pseudo periodic orbits of prime period 6 in these regions with $\delta$ of the order of $2^{-25}$.

Fig. 14. A zoom of the rectangular region indicated in Figure 13. We can clearly see nine distinct regions.

suggest that there is a family of 9 period six orbits appearing in clusters that are precisely highlighted by the unwrapped eigenfunction $\hat{P}_{65536965536}$; compare
Fig. 15. A zoom of one of the regions in Figure 14. The dots are centre points of boxes that are $2^{-50}$ the area of the 2-torus. The connected nature of this region at such a small scale suggests a single periodic point is contained in these boxes.

Figures 11 and 13. If these are not true periodic orbits, we can say with certainty that there are at least $9\delta$-pseudo-periodic orbits of prime period six with $\delta \leq 2^{-25} \cdot 2\pi \sqrt{2}(\max_{x \in \mathbb{T}^2} \|DT(x)\| + 1)$. Again, this information, while encoded in the raw eigenfunction $g_{65536}$ is only revealed in a useful geometric way when the unwrapping procedure is applied to the eigenfunction.

7 Conclusion and final remarks

We have described a new method called “unwrapping” by which eigendistributions of the Perron-Frobenius operator can be transformed into functions that more directly reveal almost-invariant structures in hyperbolic maps. This extends existing work for one-dimensional expanding maps and provides a new dynamical interpretation for eigendistributions of the Perron-Frobenius operator for hyperbolic systems.

We demonstrated our numerical method rigorously in the case of a piecewise linear map of the square and successfully identified the geometric structures responsible for the slow geometric rate of correlation decay. By applying the unwrapping method to the Standard map, we were able to pinpoint clusters of period 6 orbits.
Total variation, a measure of regularity, was used to determine how much unwrapping is required in particular examples. Numerically we observed a single local minimum in irregularity and this is intuitively reasonable. An interesting question is whether one can prove that the variation (or a similar measure of regularity) of $\hat{P}^q g$ must have the form shown in Figures 5 and 10, namely a single local minimum.

In order to make the method rigorous in greater generality, it will likely be necessary to make progress on the question of rigorous numerical approximation of eigendistributions of the Perron-Frobenius operator in a suitable functional analytic setting. This question is still an area of intense research. It is hoped that the new dynamical interpretation of the eigendistributions of the Perron-Frobenius operator presented here will further motivate this research effort.

8 Proofs

Proof of Lemma 1.

\[
\int |g| \, dm = \int \frac{|g|}{h} \, d\mu
= \int \frac{|g \circ T^q|}{h \circ T^q} \, d\mu
= \int \frac{h|g \circ T^q|}{h \circ T^q} \, dm
= \int |\hat{P}^q g| \, dm
\]

\[
\square
\]

Proof of Proposition 1.

\[
C_{\hat{P}^q f/h, \hat{P}^q g/h}(k) = \left| \int (\hat{P}^q f/h) \circ T^k \cdot (\hat{P}^q g/h) \, d\mu \right|
= \left| \int (f/h) \circ T^{k+q} \cdot (g/h) \circ T^q \, d\mu \right|
= \left| \int (f/h) \circ T^k \cdot (g/h) \, d\mu \right|
= C_{f/h, g/h}(k)
\]

\[
\square
\]

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Proof of Proposition 2. (1)

\[(\mathcal{P}^q \hat{\mathcal{P}}^q g)(x) = \mathcal{P}^q(h \cdot \mathcal{U}^q(g/h))(x)\]

\[= \frac{h(T^{-q}x)(\mathcal{U}^q(g/h)(T^{-q}x))}{|\det DT^q(T^{-q}x)|}\]

\[= \frac{h(T^{-q}x)g(x)}{h(x)|\det DT^q(T^{-q}x)|}\]

\[= g(x),\]

since \(h\) satisfies \(h \circ T^{-q} / (h \cdot |\det DT^q(T^{-q})|) \equiv 1\).

(2)

\[\hat{\mathcal{P}}^q \mathcal{P}^q g = h \cdot \mathcal{U}^q(g \circ T^{-q} / (h \cdot |\det DT^q \circ T^{-q}|))\]

\[= h \cdot (g / (h \circ T^q \cdot |\det DT^q|))\]

\[= g,\]

since \(h / (h \circ T^q \cdot |\det DT^q|) \equiv 1\).

\[\square\]

Proof of Proposition 3. (1) Let \(g^+ = \max\{g,0\}\). Note that

\[(\hat{\mathcal{P}}^q g)^+ = (h \cdot \mathcal{U}^q(g/h))^+ = h \cdot (\mathcal{U}^q(g/h))^+ = h \cdot ((g \circ T^q) / (h \circ T^q))^+ = (h / (h \circ T^q)) \cdot (g \circ T^q)^+.\]

Thus \(\mathcal{A}^q^+ = \{\hat{\mathcal{P}}^q g > 0\} = \{g \circ T^q > 0\} = T^{-q} \{g > 0\} = T^{-q} \mathcal{A}^+,\) as required.

(2) This follows directly from (i) by \(T\)-invariance of \(\mu\).

\[\square\]

Sublemma 1 (Duality of \(\pi_n \mathcal{U}\) and \(\pi_n \mathcal{P}\)). Let \(f, g \in L^1(M,m)\).

\[
\int \pi_n \mathcal{U} \left( \frac{\pi_n f}{h_n} \right) \cdot \frac{\pi_n g}{h_n} \, d\mu_n = \int \frac{\pi_n f}{h_n} \cdot \pi_n \mathcal{P}(\pi_n g) \, dm \tag{20}
\]

Proof. Let \(\tilde{f}_k = (1/m(A_k)) \int_{A_k} f \, dm\) and \(\tilde{g}_i = (1/m(A_i)) \int_{A_i} g \, dm\).
\[
\int \pi_n U \left( \frac{\pi_n f}{h_n} \right) \cdot \frac{\pi_n g}{h_n} \, d\mu_n = \sum_{i=1}^{n} \sum_{k=1}^{n} \left( \frac{\bar{f}_k}{p_k} \right) U_{ki} \cdot \bar{g}_i m(A_i)
\]
\[
= \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\bar{f}_k}{p_k} P_{ik} \frac{m(A_k)}{m(A_i)} \cdot \bar{g}_i m(A_i)
\]
\[
= \sum_{k=1}^{n} \bar{f}_k \cdot \left( \sum_{i=1}^{n} \bar{g}_i P_{ik} \right) m(A_k)
\]
\[
= \int \frac{\pi_n f}{h_n} \cdot \pi_n P(\hat{P}_n g) \, dm
\]

\[
\square
\]

**Proof of Proposition 4.** First, we prove (15). By Sublemma 1,

\[
C_{\pi_n f/h_n, \hat{P}_n g/h_n, \nu} (1) = \int \pi_n U \left( \frac{\hat{P}_n f}{h_n} \right) \cdot \frac{\hat{P}_n g}{h_n} \, d\mu_n
\]
\[
= \int \hat{P}_n f \cdot \pi_n P(\hat{P}_n g) \, dm \quad (21)
\]
\[
= \int \pi_n U \left( \frac{\pi_n f}{h_n} \right) \cdot \pi_n P(\hat{P}_n g) \, dm. \quad (22)
\]

Now,

\[
C_{\pi_n f/h_n, \pi_n g/h_n, \nu} (1) = \int \pi_n U \left( \frac{\pi_n f}{h_n} \right) \cdot \frac{\pi_n g}{h_n} \, d\mu_n
\]
\[
= \int \pi_n U \left( \frac{\pi_n f}{h_n} \right) \cdot \pi_n g \, dm
\]
\[
= \int \pi_n U \left( \frac{\pi_n f}{h_n} \right) \cdot g \, dm
\]

Thus,

\[
|C_{\pi_n f/h_n, \hat{P}_n g/h_n, \nu} (1) - C_{\pi_n f/h_n, \pi_n g/h_n, \nu} (1)| \leq \left\| \pi_n U \left( \frac{\pi_n f}{h_n} \right) \right\|_{\infty} \cdot \left\| \pi_n P \hat{P}_n g - g \right\|_1
\]

Since \( \left\| \pi_n U \right\|_{\infty} \leq 1, \left\| f \right\|_{\infty} \leq F < \infty \) and \( h_n \geq a \) for some \( a > 0 \), the result will follow from Proposition 5.

Now we show (16).
\[|C_{\pi_n f/h_n, \pi_n g/h_n, n}(1) - C_{f/h, g/h}(1)|\]
\[
= \int \pi_n \mathcal{U} \left( \frac{\pi_n f}{h_n} \right) \cdot \frac{\pi_n g}{h_n} d\mu_n - \int \mathcal{U} \left( \frac{f}{h} \right) \cdot \frac{g}{h} d\mu \\
= \left| \int \left( \frac{\pi_n f}{h_n} \right) \cdot \pi_n \mathcal{P}(\pi_n g) \, dm - \int \left( \frac{f}{h} \right) \cdot \mathcal{P} g \, dm \right| \\
\text{by Sublemma 1 and duality of } \mathcal{P} \text{ and } \mathcal{U} \\
\leq \int \left( \frac{\pi_n f}{h_n} \right) \cdot (\pi_n \mathcal{P}(\pi_n g) - \mathcal{P} g) \, dm + \int \left( \frac{\pi_n f}{h_n} - \frac{f}{h} \right) \cdot \mathcal{P} g \, dm \\
\leq \left\| \frac{\pi_n f}{h_n} \right\|_\infty \|\pi_n \mathcal{P}(\pi_n g) - \mathcal{P} g\|_1 + \left\| \frac{\pi_n f}{h_n} - \frac{f}{h} \right\|_1 \|\mathcal{P} g\|_\infty. \quad (23)\]

Considering the first term, we have \( h_n \geq a \) for all \( n \geq 1 \) and \( \|f\|_\infty \leq F \). Furthermore,

\[
\|\pi_n \mathcal{P}(\pi_n g) - \mathcal{P} g\|_1 \leq \|\pi_n \mathcal{P}(\pi_n g) - \pi_n \mathcal{P} g\|_1 + \|\pi_n \mathcal{P} g - \mathcal{P} g\|_1 \\
\leq \|\pi_n \mathcal{P}\|_1 \|\pi_n g - g\|_1 + \|\pi_n(\mathcal{P} g) - \mathcal{P} g\|_1 \quad (24)\]

Noting that \( \|\pi_n g - g\|_1 \to 0 \) for \( g \in L^1(M, m) \) as \( n \to \infty \) (Lemma 2.2 [18]), both terms in (24) approach zero as \( n \to \infty \), and hence the first term in (23) approaches zero as \( n \to \infty \). For the second term of (23), by hypothesis \( \|g\|_\infty \leq F \) and since \( |\det DT| \geq c > 0 \) almost everywhere, we have \( \|\mathcal{P} g\|_\infty \leq F/c \) almost everywhere. The fact that \( \|\pi_n f/h_n - f/h\|_1 \to 0 \) as \( n \to \infty \) follows similarly to the bounding of (27) in the proof of Proposition 5. This concludes the proof. \( \square \)

**Proof of Proposition 5.** (1)

\[
\|g - \pi_n \mathcal{P}\hat{\mathcal{P}}_n g\|_1 = \|\mathcal{P}(h\mathcal{U}(g/h)) - \pi_n \mathcal{P}(h_n \pi_n \mathcal{U}(\pi_n g/h_n))\|_1 \\
\leq \|\mathcal{P}(h\mathcal{U}(g/h)) - \mathcal{P}(h\mathcal{U}(\pi_n g/h))\|_1 + \|\mathcal{P}(h\mathcal{U}(\pi_n g/h))\|_1 \\
\leq \|g - \pi_n g\|_1 + \|(1 - \pi_n)\pi_n g\|_1 \\
\leq \|g - \pi_n g\|_1 + \|\pi_n\|_1 \|h\mathcal{U}(\pi_n g/h) - h_n \pi_n \mathcal{U}(\pi_n g/h_n)\|_1 \\
\leq \|\pi_n\|_1 \cdot \|(h\mathcal{U}(\pi_n g/h)) - h_n \pi_n \mathcal{U}(\pi_n g/h_n)\|_1 \\
\leq 1 \cdot \|(h\mathcal{U}(\pi_n g/h)) - h_n \pi_n \mathcal{U}(\pi_n g/h_n)\|_1 \quad (25)\]

The second term is zero and the first term approaches 0 as \( n \to \infty \) by Lemma 2.2 [18]. We now tackle the third term. Noting that \( \|\pi_n\|_1 = 1 \),

\[
\|\pi_n\|_1 \cdot \|(h\mathcal{U}(\pi_n g/h)) - h_n \pi_n \mathcal{U}(\pi_n g/h_n)\|_1 \\
\leq 1 \cdot \|(h\mathcal{U}(\pi_n g/h)) - h_n \pi_n \mathcal{U}(\pi_n g/h_n)\|_1 \\
+ \|h_n \pi_n \mathcal{U}(\pi_n g/h) - h_n \pi_n \mathcal{U}(\pi_n g/h)\|_1 \quad (26)\]

\[
+ \|h_n \pi_n \mathcal{U}(\pi_n g/h) - h_n \pi_n \mathcal{U}(\pi_n g/h)\|_1 \quad (27)\]

We deal with (25)–(27) in turn.
Firstly (25):

\[(25) \leq \|h\|_\infty \|(1 - \pi_n)(\mathcal{U}((\pi_n g)/h))\|_1\]

Proof of the following sublemma will take care of (25).

**Sublemma 2.** \(\|(1 - \pi_n)\mathcal{U}\left(\frac{\pi_n g}{h}\right)\|_1 \to 0 \text{ as } n \to \infty.\)

**Proof.** Let \(g_c \in C^0(M, \mathbb{R})\) be a continuous \(L^1\) approximation of \(g\). Note that \(\|\mathcal{U}g\|_1 \leq \|\det DT^{-1}\|_\infty \|g\|_1\).

\[
\left\| (1 - \pi_n)\mathcal{U}\left(\frac{\pi_n g}{h}\right) \right\|_1 
\leq \left\| \mathcal{U}\left(\frac{\pi_n g}{h}\right) - \mathcal{U}\left(\frac{g_c}{h}\right) \right\|_1 
+ \left\| \mathcal{U}\left(\frac{g_c}{h}\right) - \pi_n \mathcal{U}\left(\frac{g_c}{h}\right) \right\|_1 
+ \left\| \pi_n \mathcal{U}\left(\frac{g_c}{h}\right) - \pi_n \mathcal{U}\left(\frac{\pi_n g}{h}\right) \right\|_1 
\leq \|\det DT^{-1}\|_\infty \|\pi_n g/h - g_c/h\|_1 
+ \|(1 - \pi_n)\mathcal{U}(g_c/h)\|_1 
+ \|\det DT^{-1}\|_\infty \|g_c/h - \pi_n g/h\|_1
\]

Since \(\mathcal{U}(g_c/h) \in L^1(m)\), the second term approaches zero as \(n \to \infty\) by Lemma 2.2 [18]. Finally, \(\|\pi_n g/h - g_c/h\|_1 \leq \|\pi_n g/h - g/h\|_1 + \|g/h - g_c/h\|_1\) approaches zero as \(n \to \infty\) by Lemma 2.2 [18] and the fact that we may make \(\|g/h - g_c/h\|_1\) as small as required. \(\square\)

Now (26):

\[(26) \leq \|h - h_n\|_1 \|\mathcal{U}(\pi_n g/h)\|_\infty 
\leq \|h - h_n\|_1 \|\mathcal{U}\|_\infty \|(\pi_n g)\|_\infty \|1/h\|_\infty 
\to 0 \text{ as } n \to \infty,\]

since \(\|h - h_n\|_1 \to 0\) and the other terms are all bounded. Lastly (27):

\[(27) \leq \|h_n\|_\infty \|\pi_n\|_1 \|\mathcal{U}((\pi_n g)/h - (\pi_n g)/h_n)\|_1 
\leq \|h_n\|_\infty \|\det DT^{-1}\|_\infty \|(\pi_n g)\|_\infty \|1/h - 1/h_n\|_1 
\to 0 \text{ as } n \to \infty,\]

since \(\|h - h_n\|_1 \to 0\), \(h\) and \(h_n\) are bounded uniformly away from zero (\(M\) is compact), and the other terms are all bounded.

(2) Part (1) of Proposition 5 states that in terms of matrices,

\[
\left\| \sum_{k=1}^{n} \left(\tilde{g}_n(\text{Id}_n - \hat{P}_n P_n)\right) \chi_{A_k} \right\|_1 \to 0.
\]
Noting that

\[
\left\| \sum_{k=1}^{n} \left( \hat{g}_n(\text{Id}_n - \hat{P}_n P_n) \right) \chi_{A_k} \right\|_1 = \left\| \hat{g}_n(\text{Id} - \hat{P}_n P_n) \right\|_{1,n},
\]

we are done.

\[\square\]

**Proof of Proposition 6.** First note that \( \{ \pi_n g > 0 \} \subset B_{\delta_n}(\{ g > 0 \}) \). Secondly, as in the proof of Proposition 3, \( \{ \mathcal{U} \pi_n g > 0 \} = T^{-1}\{ \pi_n g > 0 \} \).

Combining these, \( \{ \hat{P}_n g > 0 \} = \{ \pi_n \mathcal{U} \pi_n g > 0 \} \subset B_{\delta_n}(\{ \mathcal{U} \pi_n g > 0 \}) = B_{\delta_n}(T^{-1}\{ \pi_n g > 0 \}) \subset B_{\delta_n}(T^{-1}B_{\delta_n}(\{ g > 0 \})). \)

\[\square\]

**Proof of Theorem 1.** Let \( T_s : [0,1] \to [0,1] \) be defined by

\[
T_s(x) = \begin{cases}
4x, & x \in [0,1/8); \\
4x - 1/2, & x \in [1/8,1/4); \\
4x - 1, & x \in [1/4,1/2); \\
4x - 2, & x \in [1/2,3/4); \\
4x - 5/2, & x \in [3/4,7/8); \\
4x - 3, & x \in [7/8,1].
\end{cases}
\] (28)

The expanding interval map \( T_s \) is dual to the contracting dynamics of \( T \) and is linked with the full map \( T \) via the observation [14]:

\[
\mathcal{P}(g(x)1(y)) = g(T_s(x))1(y).
\] (29)

Denote by \( \{ I_1, \ldots, I_k \} \) the equipartition \( \{ [0,1/8), \ldots, [7/8,1] \} \) of \([0,1]\). In the proof of Theorem 3 [14] it is demonstrated that the eigenfunction \( g_{8.4^q.S} \) of \( \pi_{8.4^q.S} \mathcal{P} \) is given by \( g_{8.4^q.S}(x,y) = G_q(x)1(y) \) where\(^{11}\)

\[
G_q(x) = \begin{cases}
2^q, & \text{if } x \in [i_0 \cdots i_q] \text{ and } i_q \in \{1,2,3,5\}; \\
-2^q, & \text{if } x \in [i_0 \cdots i_q] \text{ and } i_q \in \{4,6,7,8\}.
\end{cases}
\]

and the cylinder \([i_0 \cdots i_q]\) denotes the set \( I_{i_0} \cap \cdots \cap T_{s}^{-q}I_{i_q} \).

Since \( g_{8.4^q.S} \) is independent of \( S \) we may without loss assume that \( S = 1 \). In the proof of Proposition 2 [14] it is noted that \( G_q = 2^q G_0 \circ T_s^q \). Thus

\(^{11}\)In [14], \( G_q \) is denoted by \( g_q \) and \( g_{8.4^q.S} \) by \( h_q \).
\[ U^q g_{8.4v}(x, y) = U^q (G_0(x) \cdot 1(y)) \]
\[ = U^q (2^q G_0 \circ T^q_s(x) \cdot 1(y)) \]
\[ = 2^q (G_0 \circ T^q_s \cdot 1) \circ T^q(x, y). \]

Now, (29) implies \( (G_0 \cdot 1) \circ T^{-q}(x, y) = G_0(T^q_x(s))1(y) \). Composing with \( T^q \) we obtain \( G_0(x)1(y) = (G_0(T^q_x(s))1(y)) \circ T^q \). We can now conclude that \( U^q g_{8.4v}(x, y) = 2^q G_0(x)1(y) = 2^q g_8(x, y) \). It is straightforward to check directly that one further application of \( U \) to \( g_8 \) will yield \( 2g_0 \) (in an abuse of indexing notation where \( g_0 \) is defined by (3)). \( \square \)

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References


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