Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

ESCAPE RATES AND PERRON-FROBENIUS OPERATORS: OPEN AND CLOSED DYNAMICAL SYSTEMS

Gary Froyland and Ognjen Stancevic

School of Mathematics and Statistics
University of New South Wales,
Sydney, NSW 2052, AUSTRALIA

(Communicated by the associate editor name)

Abstract. We study the Perron-Frobenius operator $\mathcal{P}$ of closed dynamical systems and certain open dynamical systems. We prove that the presence of a large positive eigenvalue ρ of $\mathcal{P}$ guarantees the existence of a 2-partition of the phase space for which the escape rates of the open systems defined on the two partition sets are both slower than $-\log\rho$. The open systems with slow escape rates are easily identified from the Perron-Frobenius operators of the closed systems. Numerical results are presented for expanding maps of the unit interval. We apply our technique to shifts of finite type to show that if the adjacency matrix for the shift has a large positive eigenvalue, then the shift may be decomposed into two disjoint subshifts, both of which have high topological entropies. We then extend our results to non-autonomous systems of piecewise affine expanding Markov maps, and illustrate with further examples.

1. **Introduction.** Our aim is to explore the relationship between closed dynamical systems and certain associated open dynamical systems formed by the introduction of a hole. Let $T:(X,\mathcal{B},m)\circlearrowleft$ be a dynamical system, with m a finite reference measure on X. We will call this system *closed*. We may construct an open system from a closed system by introducing a $\mathcal{B}$-measurable hole $H\subset X$. Let $A:=X\setminus H$, $T_A:=T|_A$ be the restriction of T to the set A, and $m_A:=m|_A$ be the

2000 *Mathematics Subject Classification.* Primary: 37A30, 37M25; Secondary: 37C40, 37C60, 37E05, 37B10, 37B40.

Key words and phrases. Perron-Frobenius operator, open dynamical system, escape rate, almost-invariant set, topological entropy, non-autonomous dynamical system.

The research of GF is supported in part by an ARC Discovery Project and the ARC Centre of Excellence for Mathematics and Statistics of Complex Systems (MASCOS). OS is supported by an Australian Postgraduate Award and MASCOS.

restriction of m to $\mathcal{B} \cap A$. The system $T_A : (A, \mathcal{B} \cap A, m_A) \to (X, \mathcal{B}, m)$ is called *open* as trajectories may leave A, never to return.

Pianigiani and Yorke [32] contains early work on open dynamical systems. A series of papers by Collet *et al.* [9, 10, 11] followed, studying Markov systems with Markov holes. Anosov systems with non-Markov holes [6] and open billiards [31] have also been studied. More recently, Collet *et al.* [7, 8] obtained results for a wide class of systems with holes. Lasota-Yorke maps with small holes have been extensively studied; [4, 14, 30, 35]. Bunimovich and Yurchenko [5] carried out a case study of the effect of hole position for the doubling map. Open systems as perturbations of closed systems have been considered in recent work [27, 34]. Applications of Ulam's method to open systems include [17, 1, 2]. For a survey paper with more references and discussion, see [15].

Let the *time of escape* of a point $x \in A$ be the smallest positive integer $\xi(x)$ such that $T^{\xi(x)}(x) \in H$. Define A^n to be the set of all points that stay in A up to the n^{th} iterate of T; that is, all points that have not escaped by time n

$$A^n := \{x \in A \ : \ \xi(x) > n\} = T^{-n}(A) \cap T^{-n+1}(A) \cap \cdots \cap A.$$

A natural question concerning open systems is the rate of decrease of the measure of A^n.

Definition 1.1. Let m be a finite reference measure and $A \subset X$ a measurable set. Define the upper and lower escape rates as follows:

$$\overline{E}_m(A) := -\liminf_{n\to\infty} \frac{1}{n} \log m(A^n);$$

$$\underline{E}_m(A) := -\limsup_{n\to\infty} \frac{1}{n} \log m(A^n).$$

If $\overline{E}_m(A) = \underline{E}_m(A)$, then we say that *escape rate* of the measure m from A exists and is

$$E_m(A) := -\lim_{n\to\infty} \frac{1}{n} \log m(A^n) \in [0, \infty].$$

In fact, if escape rate exists and $m(\{x : \xi(x) > 0\}) = 1$, then [5]:

$$E_m(A) = -\lim_{n\to\infty} \frac{1}{n} \log m\left\{x \ : \ \xi(x) = n\right\}.$$

Remark 1. Note that if ν and m are equivalent measures, and the density $d\nu/dm$ is bounded away from zero and infinity, then one has $E_\nu(A) = E_m(A)$.

If T is nonsingular with respect to m we may form the Perron-Frobenius operator $\mathcal{P}$ of T.

Definition 1.2. Let $T : (X, \mathcal{B}, m) \circlearrowleft$ be a closed dynamical system where (T, m) is nonsingular. The *Perron-Frobenius operator* is the unique operator $\mathcal{P} : L^1(X, \mathcal{B}, m) \circlearrowleft$ that satisfies

$$\int_B \mathcal{P}f\ dm = \int_{T^{-1}(B)} f\ dm, \quad \forall B \in \mathcal{B}, \quad \forall f \in L^1(X, \mathcal{B}, m).$$

We will throughout assume that there is a nonnegative density $\tilde{f} \in L^1$ fixed by $\mathcal{P}$. If $\mathcal{P}f = \rho f$ for some $|\rho| < 1$ and $f \in L^1$, then $\int f\ dm = \int \mathcal{P}f\ dm = \int \rho f\ dm$ implies that $\int f\ dm = 0$. By setting $A_+ := \{f > 0\}$ and $A_\ominus := \{f \leq 0\}$, we may form two open systems T_{A_+} and $T_{A_\ominus}$. Our main result is control of the escape rate for these two open systems.

Main Theorem. *Let $T : (X, \mathcal{B}, m) \circlearrowleft$ be nonsingular and suppose that ρ is a real positive eigenvalue of $\mathcal{P} : L^1(X, \mathcal{B}, m) \circlearrowleft$. Then*

$$\overline{E}_m(A_+) \leq -\log \rho, \qquad \overline{E}_m(A_\ominus) \leq -\log \rho.$$

In words, if the Perron-Frobenius operator for our *closed system* T has a real eigenvalue ρ, then we may break the phase space X into two pieces, forming *two open systems*, both of which have escape rates slower than $-\log \rho$. When ρ is an eigenvalue close to 1, the escape rate of each A_+ and $A_\ominus$ is low. As A_+ and $A_\ominus$ partition X, escape from A_+ corresponds to entry into $A_\ominus$ and vice-versa. In the closed system, for large ρ, this exchange may lead to rates of mixing slower than rates of local separation of trajectories. The use of eigenfunctions corresponding to large ρ to determine *almost-invariant sets* for the closed system has been considered in [13, 20, 18].

An outline of this paper is as follows. Section 2 introduces conditional Perron-Frobenius operators and conditionally invariant measures, proves Theorem 2.4, and begins to investigate its consequences. Section 3 considers the implications of Theorem 2.4 for Lasota-Yorke maps, discusses an example where the escape rate of both A_+ and $A_\ominus$ is slower than the rate of local separation of trajectories, compares the notions of escape rate and almost-invariance, and discusses related work. In Section 4 we develop a version of Theorem 2.4 for shifts of finite type. Section 5 considers non-autonomous systems and includes an extension of Theorem 2.4 to such systems.

2. **Closed and Open Systems.**

2.1. **Conditionally invariant measures and conditional Perron-Frobenius operators.**

The notion of conditionally invariant probability measures is central to the study of open systems.

Definition 2.1. A measure μ on A is called a *conditionally invariant probability measure* of the open system $T_A : A \to X$ if for every measurable $B \subset A$

$$\mu(T_A^{-1}(B)) = \mu(A^1)\mu(B).$$

If μ is conditionally invariant then $E_\mu(A) = -\lim_{n\to\infty} \frac{1}{n}\log(\mu(A^n)) = -\lim_{n\to\infty} \frac{1}{n}\log(\mu(A^1)^n) = -\log\mu(A^1)$.

Definition 2.2. For $T_A : A \to X$, where $A \subset X$, the *conditional Perron-Frobenius operator* $\mathcal{P}_A : L^1(X, \mathcal{B}, m) \circlearrowleft$ is defined by:

$$\int_B \mathcal{P}_A f \; dm = \int_{T_A^{-1}(B)} f \; dm, \quad \forall B \in \mathcal{B}, \quad \forall f \in L^1(X, \mathcal{B}, m).$$

Remark 2. We can write $\mathcal{P}_A f = \mathcal{P}(f\chi_A)$, where χ_A is the indicator function of A and more generally $\mathcal{P}_A^n f = \mathcal{P}(f\chi_{A^{n-1}})$. Thus $\|\mathcal{P}_A^n 1\|_1 = \|\mathcal{P}(\chi_{A^{n-1}})\|_1 = m(A^{n-1})$ and so

$$E_m(A) = -\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}_A^n 1\|_1,$$

if this limit exists.

Proposition 1 ([32]). *For $\lambda > 0$ and f nonnegative, $\mathcal{P}_A f = \lambda f$ if and only if the absolutely continuous measure μ, with density $f = d\mu/dm$, is conditionally invariant. It follows that $E_\mu(A) = -\log\lambda$.*

Define $\lambda_m(A) := \exp(-E_m(A))$. Following [4, 14], we call $\lambda_m(A)$ the *eigenvalue of the measure m* with respect to the set A.

From now on, we will assume Lebesgue measure ℓ to be the reference measure. All of our results will hold if a general finite reference measure m is used. For notational simplicity we will write $E(A)$ for $E_\ell(A)$ and $\lambda(A)$ for $\lambda_\ell(A)$. Unless otherwise stated, *escape rate* will refer to Lebesgue escape rate.

Existence of an absolutely continuous (w.r.t Lebesgue) conditionally invariant probability measure (ACCIPM) whose density is bounded away from zero and infinity implies existence of Lebesgue escape rate. Existence and uniqueness of such ACCIPM for Markov maps satisfying a suitable transitivity condition was proved in [32]. Subsequent work has largely been aimed at relaxing the Markov condition, unfortunately at the expense of limiting the size of the hole. Collet *et al.* [7, 8] proved the existence of ACCIPMs for non-Markov maps under some technical assumptions. Chernov *et al.* [6] developed some results for existence of ACCIPMs for Anosov diffeomorphisms with small holes. Liverani and Maume-Deschamps [30] proved existence of ACCIPMs for Lasota-Yorke maps with small holes using a perturbation result based on [26], as well as for maps with larger holes under assumption that the map has enough full branches outside of the hole.

Remark 3. Generally, there will exist multiple, even uncountably many [15] ACCIPMs, with different escape rates. Of course not all are of physical relevance. In an ideal case (see [15]) suppose that $\mathcal{P}_A$ is quasi-compact in an appropriate Banach space, $(\mathbb{B}, \|\cdot\|) \subset L^1(X)$, and $\mathcal{P}_A f = \lambda f$ where λ is of multiplicity 1 and of maximal modulus. Then we may write $\mathbb{B} = \{f\} \oplus \mathbb{H}$ where $\{f\}$ is the space spanned by f and $\mathcal{P}_A(\mathbb{H}) \subset \mathbb{H}$. If there exists a $g \in \mathbb{B} \setminus \mathbb{H}$ such that $C^{-1} \leq g \leq C$ for some $C > 0$, then Lebesgue escape rate exists and $E(A) = -\log \lambda$. All other absolutely continuous measures have higher escape rates.

2.2. **Connecting Closed and Open Systems.** We now restate and prove our first main result which, roughly speaking, relates eigenvalues of the operator $\mathcal{P}$ to the largest eigenvalue of a particular conditional operator $\mathcal{P}_A$.

Definition 2.3. For a function $f \in L^1(X)$ we denote by $\mathrm{supp}(f) := \{x \in X : f(x) \neq 0\}$ the *support* of f. Also define $f^+, f^- \in L^1(X)$ by $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$.

Theorem 2.4 (Main Theorem). *Let $T : X \circlearrowleft$ be a nonsingular transformation on the finite measure space $(X, \mathcal{B}, \ell)$ and let $\mathcal{P} : L^1(X, \mathcal{B}, \ell) \circlearrowleft$ be the corresponding Perron-Frobenius operator. Suppose that $\mathcal{P}$ has a real positive eigenvalue $0 < \rho < 1$, with corresponding bounded eigenfunction $-\infty < f < \infty$. Define the measurable sets $A_+, A_- \subset X$ by*

$$A_+ := \mathrm{supp}(f^+) \quad \textit{and} \quad A_- := \mathrm{supp}(f^-).$$

Then one has $\overline{E}(A_+) \leq -\log \rho$ and $\overline{E}(A_-) \leq -\log \rho$.

For the proof we will need the following lemma.

Lemma 2.5. *For a finite measure ν, let $A \subset X$ be measurable and $0 < \gamma \leq 1$.*

(i) If $\nu(A^{n+1}) \geq \gamma \nu(A^n)$ for all $n \geq 0$, then $\overline{E}_\nu(A) \leq -\log \gamma$.
(ii) If $\nu(A^{n+1}) \leq \gamma \nu(A^n)$ for all $n \geq 0$, then $\underline{E}_\nu(A) \geq -\log \gamma$;

Proof. (i) By induction it follows that $\nu(A^n) \geq \gamma^n \nu(A)$.

$$\overline{E}_\nu(A) = -\liminf_{n\to\infty} \frac{1}{n} \log \nu(A^n) \leq -\liminf_{n\to\infty} \frac{1}{n} \log(\gamma^n \nu(A)) = -\log \gamma.$$

(ii) is analogous to *(i)*. □

Proof of Theorem 2.4. Define a finite measure ν on X by

$$\nu(B) := \int_B |f| \, d\ell, \quad B \in \mathcal{B}.$$

Now note that for all $n \geq 0$ we have $f > 0$ on A_+^n. Also $A_+^{n+1} = T^{-1}(A_+^n) \cap A_+$, therefore

$$\begin{aligned} \rho\nu(A_+^n) &= \rho \int_{A_+^n} f \, d\ell \\ &= \int_{A_+^n} \mathcal{P}f \, d\ell \\ &= \int_{T^{-1}(A_+^n)} f \, d\ell \\ &= \int_{T^{-1}(A_+^n)\cap A_+} f \, d\ell + \int_{T^{-1}(A_+^n)\cap(X\setminus A_+)} f \, d\ell \\ &\leq \int_{T^{-1}(A_+^n)\cap A_+} f \, d\ell \\ &= \nu(A_+^{n+1}), \end{aligned}$$

where the inequality above is due to $f \leq 0$ on $X \setminus A_+$. Since $\nu(A_+^{n+1}) \geq \rho\nu(A_+^n)$, by (i) of Lemma 2.5 we have $\overline{E}_\nu(A_+) \leq -\log\rho$. It remains to show that $\overline{E}(A_+) \leq \overline{E}_\nu(A_+)$. Since $f \leq C$ for some constant $C > 0$, we have $\nu(A_+^n) \leq C\ell(A_+^n)$ for all $n \geq 0$. This gives $\overline{E}_\nu(A_+) = -\liminf_{n\to\infty} \frac{1}{n}\log(\nu(A_+^n)) \geq -\liminf_{n\to\infty}\frac{1}{n}\log(C\ell(A_+^n)) = \overline{E}(A_+)$. Thus $\overline{E}(A_+) \leq \overline{E}_\nu(A_+) \leq -\log\rho$. The inequality for A_- is obtained by considering $-f$ in place of f. □

Remark 4. If one wishes to create a 2-partition of X such that each element of the partition has upper escape rate lower than $-\log\rho$, then the set $\{f = 0\}$ may be absorbed into either A_+ or A_-. Enlarging A_+ does not increase $\overline{E}(A_+)$ so Theorem 2.4 also holds for $A_\oplus := X \setminus A_-$ and $A_\ominus := X \setminus A_+$. The desired 2-partition is then $\{A_+, A_\ominus\}$ or $\{A_\oplus, A_-\}$ (or any other redistribution of $\{f = 0\}$ among the two sets).

Lemma 2.6. *$E(A) = E(A^N)$ for any measurable set $A \subset X$ and integer $N \geq 0$.*

Proof. It is a simple exercise to show that $(A^N)^n = A^{N+n}$ for all $n, N \geq 0$. The result follows:

$$\begin{aligned} E(A^N) &= -\lim_{n\to\infty}\frac{1}{n}\log\ell((A^N)^n) = -\lim_{n\to\infty}\frac{1}{n}\log\ell(A^{N+n}) \\ &= -\lim_{n\to\infty}\frac{1}{n-N}\log\ell(A^n) = E(A). \end{aligned}$$

□

Remark 5. By Lemma 2.6, we may replace T_A with T_{A^1} and obtain an open system with an identical escape rate. We may think of T_{A^1} as an open system on A with hole $A \setminus T^{-1}A$. Consider now our partition $\{A_+, A_\ominus\}$ of X formed from the positive and nonpositive parts of some $f \in L^1$ satisfying $\mathcal{P}f = \rho f$, $0 < \rho < 1$. By the above remarks, the open system T_{A_+} has the same escape rate as the open system $T_{A_+^1}$, where the hole for the latter system is $A_+ \setminus T^{-1}A_+ = A_+ \cap T^{-1}A_\ominus \subset A_+$. Thus, while the hole $H = A_\ominus$ for the open system T_{A_+} is very large in measure, we may easily construct another system $T_{A_+^1}$ with the same escape rate, but a hole $H = A_+ \cap T^{-1}A_\ominus$ that is likely to be much smaller in terms of ℓ. Similarly, we may define an open system $T_{A_\ominus^1}$, with hole $A_\ominus \setminus T^{-1}A_\ominus = A_\ominus \cap T^{-1}A_+ \subset A_\ominus$; this open system has the same escape rate as $T_{A_\ominus}$.

2.3. **Spectrum of $\mathcal{P}$ in L^1.** Let $\sigma(\mathcal{P})$ denote the L^1 spectrum of $\mathcal{P}$. Ding *et al.* [16] (Corollary 3.2) state that for $(X, \mathcal{B}, \ell)$ a σ-finite measure space, T nonsingular and $\mathcal{P} : L^1(X, \ell) \circlearrowleft$, with a nonnegative fixed density of full support, if $0 \in \sigma(\mathcal{P})$, then $\sigma(\mathcal{P}) = \{z \in \mathbb{C} : |z| \leq 1\}$.

Consider $T : [0,1] \circlearrowleft$ to be piecewise monotonic and C^2 on each monotone branch. Then we may represent $\mathcal{P} : L^1([0,1], \ell) \circlearrowleft$ as

$$\mathcal{P}f(x) = \sum_{y \in T^{-1}\{x\}} \frac{f(y)}{|T'(y)|}, \tag{1}$$

where $T'(y)$ is defined by continuity along an inverse branch if $T'(y)$ fails to exist. The following Lemma shows that if 0 is an eigenvalue, then every point in the open unit disk is also an eigenvalue.

Lemma 2.7. *Suppose there is a nonzero $\hat{f} \in L^1([0,1], \ell)$ satisfying $\mathcal{P}\hat{f} = 0$. Every $\rho \in \{z \in \mathbb{C} : |z| < 1\}$ is an eigenvalue of $\mathcal{P}$.*

Proof. This proof appears in a slightly different context in the proof of Theorem 1.5 (7) [3]. If $\rho = 0$ we are done. Let $\rho > 0$. Then $f := \sum_{n=0}^{\infty} \rho^n \hat{f} \circ T^n \in L^1$ is an eigenfunction with eigenvalue ρ. To see this, we note that $f \in L^1$ and compute

$$\begin{aligned}
\mathcal{P}f(x) &= \sum_{y \in T^{-1}\{x\}} \sum_{n=0}^{\infty} \rho^n \hat{f} \circ T^n(y)/|T'(y)| \\
&= \sum_{y \in T^{-1}\{x\}} \hat{f}(y)/|T'(y)| + \sum_{y \in T^{-1}\{x\}} \sum_{n=1}^{\infty} \rho^n \hat{f} \circ T^n(y)/|T'(y)| \\
&= 0 + \rho \sum_{y \in T^{-1}\{x\}} \sum_{n=0}^{\infty} \rho^n \hat{f} \circ T^n(x)/|T'(y)| \\
&= \rho \sum_{n=0}^{\infty} \rho^n \hat{f} \circ T^n(x) \sum_{y \in T^{-1}\{x\}} 1/|T'(y)| = \rho f.
\end{aligned}$$

□

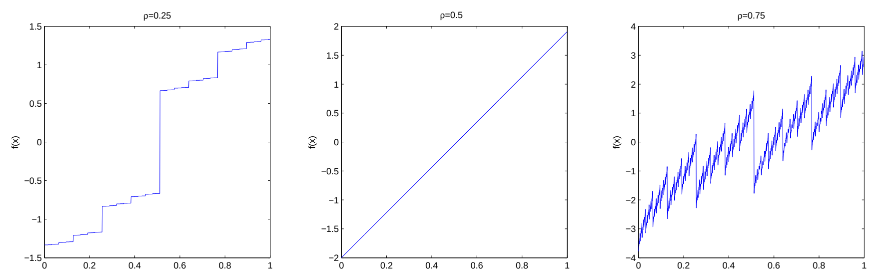

FIGURE 1. Graphs of L^1 eigenfunctions for the circle map $x \mapsto 2x$ for $\rho = 0.25, 0.5$, and 0.75.

Example 2.8. Figure 1 shows three eigenfunctions for the doubling map $x \mapsto 2x$ on S^1. We may apply Theorem 2.4 to any one of these eigenfunctions to obtain two open systems, both of which have escape rates slower than $-\log \rho$. Each eigenfunction produces a very large hole, and Theorem 2.4 says that one may set ρ as close to unity as one wishes, to obtain very slow escape rates. The penalty that one pays for producing escape rates less than $\log 2$ are sets A_+ that may be very complicated. We discuss this further in the next section.

3. **Application to Lasota-Yorke maps.** Let us focus on the case where $I := [0,1]$ and $T : I \circlearrowleft$ is piecewise monotone and C^2; that is, there is a finite partition $\{a_0, a_1, \ldots, a_n\}$ with $a_0 = 0, a_n = 1$ so that T is monotone and C^2 on the interior each interval (a_{i-1}, a_i), $i = 1, \ldots, n$. Furthermore we assume that T is expanding, that is $\tau := \inf |T'| >$

1 where the infimum is taken over all points in $[0,1]$ for which the derivative exists. Such maps are known as *Lasota-Yorke maps*.

We begin this section by stating that we can expect the L^1 spectrum to be the entire unit disk for interesting Lasota-Yorke maps.

Lemma 3.1. *Let T be a Lasota-Yorke map and suppose that there are two monotone branches $T_i := T|_{(a_{i-1},a_i)}, T_j := T|_{(a_{j-1},a_j)}$, $i \neq j$, for which $T_i((a_{i-1},a_i)) \cap T_j((a_{j-1},a_j)) \neq \emptyset$. Then $0 \in \sigma(\mathcal{P})$.*

Proof. We construct a nonzero $f \in L^1$ with $\mathcal{P}f = 0$. As T_i, T_j are monotonic and expanding, $T_i((a_{i-1},a_i)) \cap T_j((a_{j-1},a_j))$ is an interval, which we denote (x_1,x_2). Let $f(x) = 0$ for $x \in [0,1] \setminus (T_i^{-1}(x_1,x_2) \cup T_j^{-1}(x_1,x_2))$, and $f(x) = 1$ for $x \in T_i^{-1}(x_1,x_2)$. We now determine the value of $f(x)$ for $x \in T_j^{-1}(x_1,x_2)$.

By (1), for $x \in (x_1,x_2)$ we have

$$\mathcal{P}f(x) = 1/|T_i'(T_i^{-1}(x))| + f(T_j^{-1}(x))/|T_j'(T_j^{-1}(x))|.$$

Equating the RHS with zero and rearranging, we obtain $f(T_j^{-1}(x)) = -|T_j'(T_j^{-1}(x))|/|T_i'(T_i^{-1}(x))|$, defining $f(x)$ for $x \in T_j^{-1}(x_1,x_2)$. For $x \notin (x_1,x_2)$ $\mathcal{P}f(x)$ is clearly also zero by the definition of f. □

3.1. **Spectrum of $\mathcal{P}$ on $BV(I)$.** By replacing $(L^1(X), \|\cdot\|_1)$ with $(\mathrm{BV}(I), \|\cdot\|_{\mathrm{BV}})$, the space of functions of bounded variation where $\|\cdot\|_{\mathrm{BV}} = \max\{\mathrm{var}_I(\cdot), \|\cdot\|_1\}$, the operator $\mathcal{P} : (\mathrm{BV}, \|\cdot\|_{\mathrm{BV}}) \circlearrowleft$ becomes *quasi-compact* (see eg. [3]). Eigenfunctions of $\mathcal{P}$ that lie in BV give rise to sets A_+ with a reasonably simple structure.

Definition 3.2. Let $\mathcal{I}$ be the family of sets $A \subset I$ such that A can be written as a countable union of intervals (where singleton sets $\{x\} = [x,x]$ are included).

Proposition 2 ([28])**.** *If $f \in BV$ then* $\mathrm{supp}(f) \in \mathcal{I}$.

Corollary 1. *If $f \in BV$ then $f^+, f^- \in BV$ also, so in the BV setting both sets A_+ and A_- from Theorem 2.4 belong to $\mathcal{I}$.*

Example 3.3. Returning to the doubling map $x \mapsto 2x$, considered as a map on I, the spectrum of $\mathcal{P} : \mathrm{BV}(I) \circlearrowleft$ is contained in $\{|z| \leq 1/2\} \cup \{1\}$. Thus, all BV eigenfunctions corresponding to eigenvalues $0 < \rho < 1$ must in fact have $\rho \leq 1/2 = 1/\tau$. In particular, this excludes the third, more irregular L^1 eigenfunction in Figure 1.

Thus, for the doubling map in the BV setting, we can only guarantee that Theorem 2.4 provides open subsystems defined on reasonably regular domains if the escape rates are less than $\log 2 = \log\tau$. The following section investigates a map for which one can find open systems on regular domains with escape rates slower than $\log\tau$.

3.2. **A map with escape rate slower than** $\log \tau$**.** In this section we exhibit a map for which we may identify two disjoint open subsystems, both of which have an escape rate slower than $\log \tau$. The sets A_+ and A_- constructed in Theorem 2.4 are one good way to define such open systems. Via numerical exploration, we investigate whether there are other decompositions into open systems with even slower escape rates than the decomposition identified by Theorem 2.4.

As an objective means of comparison, given a closed system, we propose to maximise the following quantity

$$\psi(A) := \min(\lambda(A), \lambda(I \setminus A)), \quad A \in \mathcal{I}.$$

Example 3.4. Consider the following piecewise affine map $T : I \circlearrowleft$ [19].

$$T(x) = \begin{cases} 4x, & x \in [0, 1/8); \\ 4x - 1/2, & x \in [1/8, 2/8); \\ 4x - 1, & x \in [2/8, 4/8); \\ 4x - 2, & x \in [4/8, 6/8); \\ 4x - 5/2, & x \in [6/8, 7/8); \\ 4x - 3, & x \in [7/8, 1]. \end{cases}$$

The graph of T is shown in Figure 2. The Perron-Frobenius operator of T has an isolated second largest eigenvalue $\rho_2 = 1/2$ with the corresponding eigenfunction $f_2 \in \mathrm{BV}$, shown in Figure 3.

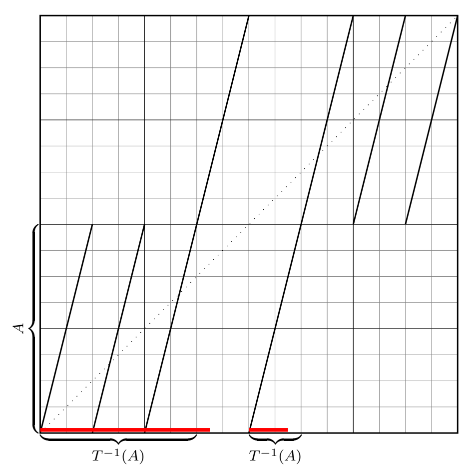

FIGURE 2. Graph of T in Example 3.4. The set $A = [0, \frac{1}{2}]$ and its pre-image are shown.

By considering where f_2 is positive and where it is negative, we can partition the domain of T into two sets, $A_- = [0, 1/2)$ and $A_+ =$

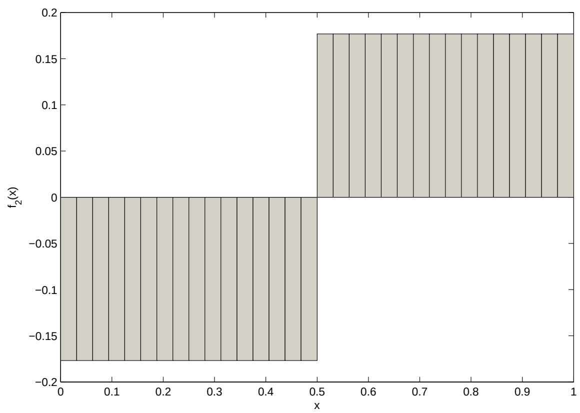

FIGURE 3. Graph of second eigenfunction f_2 of $\mathcal{P}$.

$[1/2, 1]$. The escape rate of both of these sets is relatively low: $E(A_-) = E(A_+) = -\log 3/4$ and both satisfy the inequality of Theorem 2.4. Sets with lower escape rates do exist (for example, take $A = [0, 1-\epsilon]$ for small enough ϵ), however it is not immediately obvious that there exists a set $A \in \mathcal{I}$ with $\psi(A) > 3/4$ (note the escape rate of $X \setminus A = X \setminus [0, 1-\epsilon] = (1-\epsilon, 1]$ is $-\log \frac{1}{4}$).

Intervals of length $1/2$. First, we will maximise $\psi(A)$ over the class of all intervals of length $1/2$. Let $I_{\alpha,1/2}$ be an interval of length $1/2$ centered at $x = \alpha$. Figure 4 suggests that $\psi(I_{\alpha,1/2})$ is maximised when $\alpha = 1/4$, that is $I_{\alpha,1/2} = [0, 1/2]$, coinciding with the set A_- identified by Theorem 2.4.

Intervals of varying length. We also considered intervals $I_{\alpha,l}$ with centres and lengths $\alpha, l \in \{i/512\}_{i=0,\dots 255}$. Again, we found that $\psi(I_{\alpha,l}) \leq 3/4$ for all α, l considered, with the maximum achieved by $I_{1/4,1/2}$.

Finite unions of intervals. We may also consider A to be a finite union of elements from an interval partition of I. We maximise $\psi(A)$ over all unions of intervals in the partition $\mathcal{I}_{16} := \{[i/16, (i+1)/16) : i = 0, \dots, 15\}$ and find $\psi(A) \approx 0.799$ for $A = [0, \frac{7}{16}) \cup [\frac{1}{2}, \frac{9}{16})$. If we repeat on the finer partition $\mathcal{I}_{32} := \{[i/32, (i+1)/32) : i = 0, \dots, 31\}$ we obtain maximal $\psi(A) \approx 0.8198$ for $A = [0, \frac{13}{32}) \cup [\frac{16}{32}, \frac{19}{32})$. This set is coloured in red in Figure 2.

If we allow more complicated sets than those in $\mathcal{I}$, then combining Theorem 2.4, Lemma 3.1, and Lemma 2.7 we see that $\sup\{\psi(A) : A \subset X\} = 1$ as per the discussion in Section 2.3 for the doubling map.

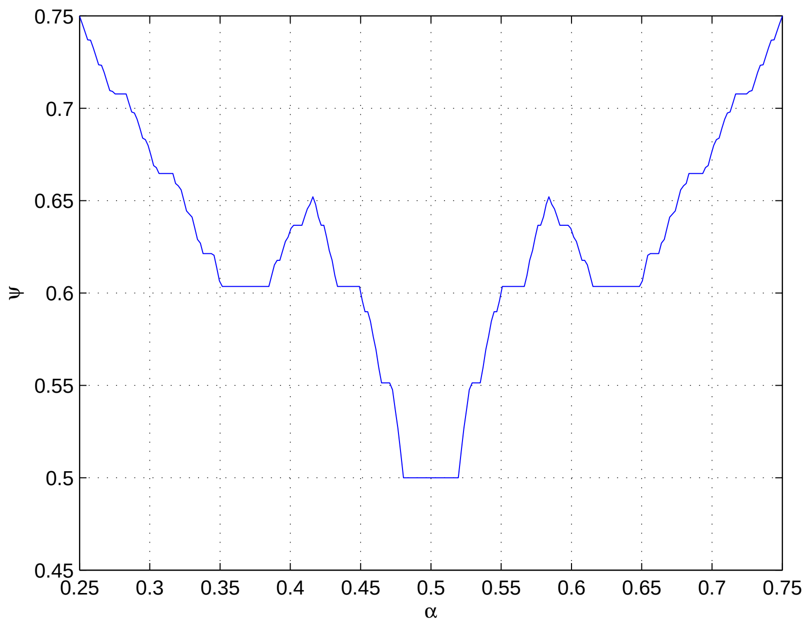

FIGURE 4. Graph of $\psi(I_{\alpha,1/2})$ where $I_{\alpha,1/2}$ is an interval of length $1/2$ with varying center point α.

3.3. **Escape Rate and Almost-Invariant Sets.** *Almost-invariant sets* [13, 20, 18] are sets for which the invariance ratio

$$\varrho(A) := \frac{\ell(T^{-1}(A) \cap A)}{\ell(A)}$$

is close to 1. Dynamical systems that are close to nonergodic typically have a decomposition into nontrivial sets, each of which has a high invariance ratio. The identification of such almost-invariant sets is often very difficult; see [23] for a recent computational study. Application areas include molecular dynamics [33], ocean dynamics [24, 25].

The construction of A_+ and A_- in Theorem 2.4 is based on an algorithm in [13] for determining almost-invariant sets. In the Lasota-Yorke map setting, with $\mathcal{P} : BV \circlearrowleft$, almost-invariant sets have formally been associated with isolated spectral points of $\mathcal{P}$ [12]. In such a setting, if the map is additionally Markov and one restricts oneself to searching for almost-invariant sets that are unions of Markov partition sets, then lower and upper bounds for the largest possible almost-invariance ratio are given by the second largest eigenvalue of an associated Markov chain [18].

Thus, there is a strong connection between almost-invariant sets and the construction we have used to define our slow escape sets A_+ and A_-. One might therefore naively expect that sets with low escape rate should have a high invariance ratio and vice-versa. However, escape rate is an asymptotic quantity, while almost-invariance measures escape over just one iteration of a map. We give examples below to demonstrate that a set may simultaneously have (i) high almost-invariance and high escape rate and (ii) low almost-invariance and low escape rate.

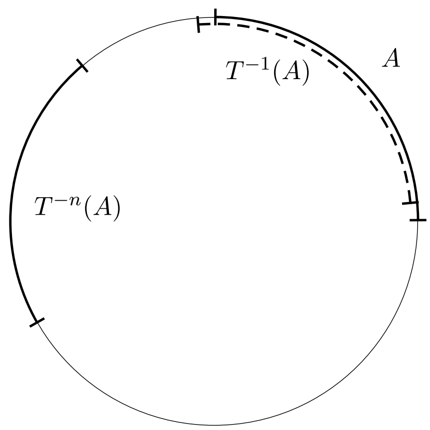

FIGURE 5. Illustration of Example 3.5.

Example 3.5 (High almost-invariance, infinite escape rate)**.** Let $T : S^1 \circlearrowleft$ be the irrational rotation of the circle, $T(x) := x + 2\pi\alpha$ where α is small. Let $A = [0, \frac{\pi}{2}]$. The pre-image of A is given by $T^{-1}(A) = [2\pi\alpha, \frac{\pi}{2} + 2\pi\alpha]$. Thus the invariance ratio of A with respect to Haar measure on the circle is $(\frac{\pi}{2} - 2\pi\alpha)/(\frac{\pi}{2}) \approx 1$. However, for $\frac{1}{4\alpha} < n < \frac{3}{4\alpha}$ we have $T^{-n}(A) \cap A = \emptyset$, therefore escape rate from A with respect to any measure is infinite. See Figure 5.

Example 3.6 (Low almost-invariance, arbitrarily low escape rate)**.** Let $T : [0, 1) \circlearrowleft$ be defined as follows:

$$T(x) = \begin{cases} (1+\epsilon)x, & 0 \le x \le 1/4; \\ 2x \pmod 1, & 1/4 < x < 1. \end{cases}$$

Let $A = [0, 1/2]$. The invariance ratio of A with respect to Lebesgue measure equals to $1/2$. However its escape rate is $\log(1+\epsilon) \approx 0$. See Figure 6.

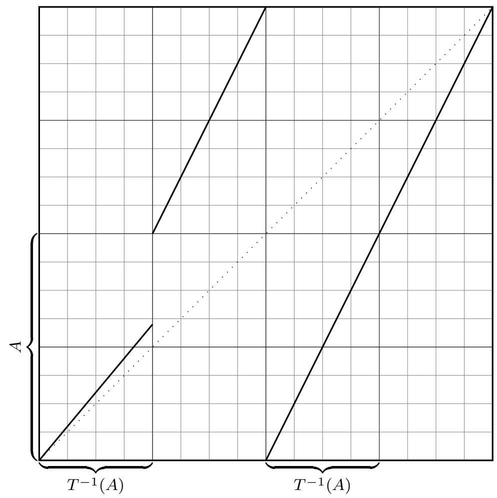

FIGURE 6. Graph of T in Example 3.6.

3.4. **Related work.** Bunimovich and Yurchenko [5] study the doubling map $x \mapsto 2x$ on the circle with reference measure Lebesgue and consider Markov holes. They show that the escape rate is related to the first return time of a positive measure subset of the hole: longer return time to the hole implies faster escape rate into the hole. More precisely, for times longer than the return time, longer return time to the hole implies smaller survivor sets. Unfortunately, the proofs rely heavily on combinatorial arguments based upon the full 2-shift (or k-shift) structure, and thus are specific to the doubling map and systems metrically conjugate to the doubling map. Even for reasonably simple systems such as piecewise affine expanding Markov maps, similar results are not known. Numerical investigations such as Figure 4 clearly display the dependence of escape rate on the position of the hole, and support our observation that the holes identified by Theorem 2.4 are positioned so as to form open systems with very low escape rates.

Keller and Liverani [27] study the escape rates of systems with very small holes. They consider Lasota-Yorke maps with possibly countably many branches and a family of compact interval holes I_ϵ shrinking to a point z as $\epsilon \to 0$. To each ϵ is associated a conditional Perron-Frobenius operator with leading eigenvalue λ_ϵ. Formulae are provided for the ratio of $1 - \lambda_\epsilon$ to the size of the hole I_ϵ for periodic and non-periodic z. Holes shrinking to a fixed interval are also discussed. Results on exchange rates, similar to that of [13], are obtained when T has two mixing ergodic components that are joined into a single ergodic component by the addition of smooth noise.

Tokman *et al.* [34] study Lasota-Yorke maps that possess two invariant subsets of positive Lebesgue measure and exactly two ergodic absolutely continuous invariant probability measures (ACIPMs). They perturb such maps slightly to destroy the two invariant subsets and show that the (now unique) ACIPM may be approximated by a convex combination of the two initial ergodic ACIPMs. The holes considered in [34] are the holes $A_+ \cap T^{-1}A_\ominus \subset A_+$ and $A_\ominus \cap T^{-1}A_+ \subset A_\ominus$ discussed in Remark 5. Our results may be viewed as generalised converses to [34], who study the particular setting of Lasota-Yorke maps and require very precise knowledge on the initial closed dynamical system. In contrast, we begin with a closed system about which we know very little, apart from the existence of eigenvalues for its Perron-Frobenius operator. From the eigenvalue and eigenfunction information, we are able to *determine* two holes and form two open systems from which the rate of escape is guaranteed to be slower than the rate given by the eigenvalue. In general, the identification of such open systems is far from obvious. Our approach may handle very general settings (only non-singularity is required to define the Perron-Frobenius operator), and provides useful information even for macroscopic holes when the closed system may be far from nonergodic.

4. **Shifts of finite type.** Let us introduce some common notation and well known results (see e.g. [29]). Let Z_K be a finite *alphabet* of length K and let $X = Z_K^{\mathbb{Z}}$ be the space of all bi-infinite sequences of elements of Z_K. We denote an element of X by $x = (x_i)_{i\in\mathbb{Z}} = \ldots x_{-2}x_{-1}.x_0x_1x_2\ldots$. Define the *left shift map* $\sigma : X \circlearrowleft$ by $(\sigma x)_i = x_{i+1}$. A *block* of length k is a finite sequence of k elements from Z_K. A *shift of finite type*, denoted $X_\mathcal{F}$, is a σ-invariant subspace of X where $\mathcal{F}$ is a finite collection of *forbidden blocks*. A shift $X_\mathcal{F}$ is said to be of memory 1 if all forbidden blocks in $\mathcal{F}$ are of length 2. It is always possible to *recode* a shift of finite type into a conjugate shift $\tilde{X}_\mathcal{F}$ of memory 1, so without loss of generality we may assume that this has already been done. $X_\mathcal{G}$ is a *subshift* of $X_\mathcal{F}$ if $\mathcal{F} \subset \mathcal{G}$. Define the *adjacency matrix* of a memory 1 shift $X_\mathcal{F}$ to be the 0–1 matrix M such that $M_{ij} = 0$ if and only if ij is a forbidden block. If M is the adjacency matrix of $X_\mathcal{F}$ and M' is the adjacency matrix of its subshift, then $M'_{ij} = 1 \Rightarrow M_{ij} = 1$. The *topological entropy* $h(X_\mathcal{F})$ of a shift $(X_\mathcal{F}, \sigma)$ is determined by the largest eigenvalue $r(M)$ in the following way: $h(X_\mathcal{F}) = \log r(M)$.

We now restate Theorem 2.4 for shifts of finite type. The aim is to identify two disjoint subshifts of $X_\mathcal{F}$, both of which have high entropy.

Theorem 4.1. *Let $(X_\mathcal{F}, \sigma)$ be a memory 1 shift of finite type, with corresponding $K \times K$ adjacency matrix M. By the Perron-Frobenius* *theorem for nonnegative matrices, M has a real eigenvalue equal to the spectral radius $r(M)$. Let $0 < \rho < r(M)$ be another real eigenvalue of M with eigenvector $v \in \mathbb{R}^K$. Define A_+ and A_- to be the two sets of indices for which v is positive and negative, respectively:*

$$A_+ := \{i \in Z_K \ : \ v_i > 0\}\,, \quad A_- := \{i \in Z_K \ : \ v_i < 0\}\,.$$

Let M_{A_+} and M_{A_-} be the restrictions of M to indices in A_+ and A_- respectively. These matrices define two disjoint subshifts of $X_{\mathcal{F}}$, denote them by X_{A_+} and X_{A_-}. Then we have $h(X_{A_+}) \geq \log \rho$ and $h(X_{A_-}) \geq \log \rho$.

Proof. It is sufficient to show that $r(M_{A_+}) \geq \rho$, where $r(M_{A_+})$ is the spectral radius of M_{A_+}. For every $i \in A_+$

$$\begin{aligned} \rho v_i &= \sum_{j \in Z_K} M_{ij} v_j \\ &= \sum_{j \in A_+} M_{ij} v_j + \sum_{j \notin A_+} M_{ij} v_j \\ &\leq \sum_{j \in A_+} (M_{A_+})_{ij} v_j. \end{aligned}$$

It follows that $(\rho^n v^+)_i \leq (v^+ M_{A_+}^n)_i$ for all $n \geq 1$ and $i \in A_+$, where v^+ is the restriction of v to A_+, therefore $r(M_{A_+}) \geq \rho$. Thus $h(X_{A_+}) = \log r(M_{A_+}) \geq \log \rho$. By considering $-v$ in place of v we obtain $h(X_{A_-}) \geq -\log \rho$. □

Remark 6. For a memory 1 shift $X_{\mathcal{F}}$ and a subshift $X_{\mathcal{G}}$, define the set $H = \{x \in X_{\mathcal{F}} : x_0 x_1 \in \mathcal{G} \setminus \mathcal{F}\}$. The set $H \subset X$ may be regarded as a hole. It was shown in [10] that when m is a measure of maximal entropy for $X_{\mathcal{F}}$, the escape rate into H from its complement exists and

$$E_m(X_{\mathcal{F}} \setminus H) = h(X_{\mathcal{F}}) - h(X_{\mathcal{G}}).$$

Thus, for shifts of finite type, escape rate is loss in entropy, and we may view Theorem 4.1 as a version of Theorem 2.4.

Example 4.2. Let $X_{\mathcal{F}} \subset Z_3^{\mathbb{Z}}$ be a shift space with forbidden blocks $\mathcal{F} := \{001, 010, 022, 101, 110, 121, 202, 211, 212, 221\}$. The graph of memory 1 recoded shift $\tilde{X}_{\mathcal{F}}$ is shown in Figure 7. We wish to identify two disjoint subshifts of $X_{\mathcal{F}}$, namely X_{A_+} and X_{A_-}. The adjacency matrix of $\tilde{X}_{\mathcal{F}} \subset Z_9^{\mathbb{Z}}$ is

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

M has largest eigenvalue $\rho_1 \approx 1.92$, and second largest eigenvalue $\rho_2 \approx 1.42$. The eigenvector v corresponding to ρ_2 is shown in Figure 8, suggesting that we take $A_- = \{01, 11, 12, 20, 22\}$ and $A_+ = \{00, 02, 10, 21\}$. This corresponds to breaking the connections between vertices 20 and 02 in Figure 7, and taking each of the two connected components to be the graphs of X_{A_-} and X_{A_+}. We calculate the topological entropy of each of the newly obtained subshifts and get $h(X_{A_-}) \approx \log 1.76$ and $h(X_{A_+}) \approx \log 1.47$, each being greater than $\log \rho_2$, as is implied by Theorem 4.1.

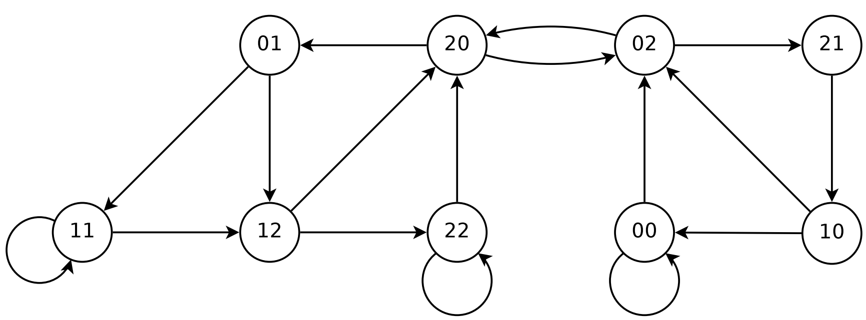

FIGURE 7. Transition graph of $\tilde{X}_{\mathcal{F}}$.

5. **Non-autonomous systems.**

We extend Theorem 2.4 to non-autonomous systems. This is motivated by recent extensions of isolated spectral values and almost-invariant sets to such systems; see [21] for fundamental theory and [22] for a computational study.

Definition 5.1. For a shift of finite type $\Omega \subset Z_K^{\mathbb{Z}}$, let $(\Omega, \mathcal{H}, p)$ be a σ-ergodic probability space and let $(X, \mathcal{B}, \ell)$ be a finite measure space. A *(one sided) cocycle* over σ is a function $\Phi : \mathbb{Z}^+ \times \Omega \times X \to X$ such that for all $x \in X$ and $\omega \in \Omega$:

- $\Phi(0, \omega, x) = x$;

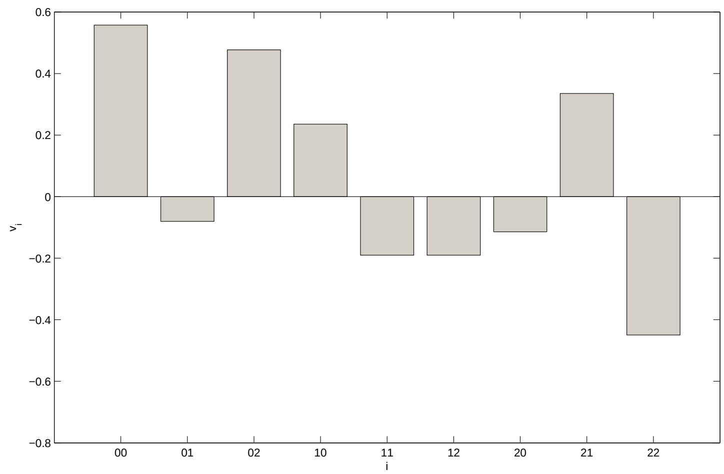

FIGURE 8. Second eigenvector of M.

- for all $m, n \in \mathbb{Z}^+$, $\Phi(m+n, \omega, x) = \Phi(m, \sigma^n\omega, \Phi(n, \omega, x))$.

The *generator* of a cocycle Φ is the mapping $\tilde{\Phi} : \Omega \to \mathrm{End}(X)$ given by $\tilde{\Phi}(\omega) = \Phi(1, \omega, \cdot)$.

Throughout this section, we set $X = I = [0,1]$ and $\{T_i\}_{i\in Z_K}$ to be a collection of Lebesgue measure preserving, piecewise affine, expanding Markov maps of the interval I sharing a common Markov partition Π of $\#\Pi$ elements. Each T_i is monotone on elements of Π. Let $\chi(\Pi)$ denote the space of step functions spanned by $\{\chi_{\pi_i}\}_{\pi_i\in\Pi}$. We denote by $\mathcal{P}_i$ the Perron-Frobenius operator associated to T_i.

Definition 5.2. The maps $\{T_i\}_{i\in Z_K}$ induce a cocycle $\Phi : \mathbb{Z}^+ \times \Omega \times I \to I$ whose generator is $\tilde{\Phi}(\omega) := T_{\omega_0}$. The *Perron-Frobenius cocycle* $\mathcal{P} : \mathbb{Z}^+ \times \Omega \times \mathrm{BV}(I) \to \mathrm{BV}(I)$ associated to Φ is the one-sided cocycle with generator $\tilde{\mathcal{P}}(\omega) = \mathcal{P}_{\omega_0}$. For simplicity of notation we will write T_ω for T_{ω_0}, and $\mathcal{P}_\omega$ for $\mathcal{P}_{\omega_0}$.

Definition 5.3 (Escape Rate)**.** For $\omega \in \Omega$, let $\mathcal{A}(\omega) := \{A(\sigma^n\omega)\}_{n\in\mathbb{Z}^+}$ be a sequence of measurable sets $A(\sigma^n\omega) \subset I$. For every $n \in \mathbb{Z}^+$ define $A^n(\omega) \subset I$ to be

$$A^n(\omega) := \bigcap_{i=0}^{n-1} \Phi(i, \omega, \cdot)^{-1}(A(\sigma^i\omega)).$$

The Lebesgue escape rate from $\mathcal{A}(\omega)$ is defined to be

$$E(\mathcal{A}(\omega)) := -\lim_{n\to\infty} \frac{1}{n} \log \ell(A^n(\omega))$$

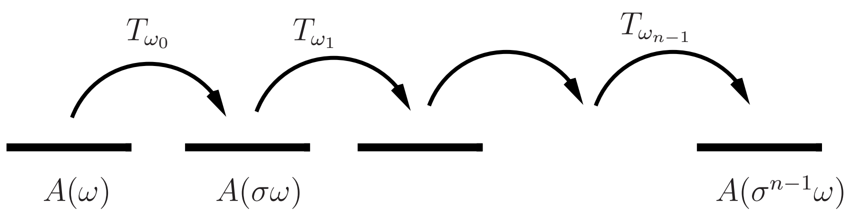

FIGURE 9. Sets of $\mathcal{A}(\omega)$ and the action of the maps $\{T_i\}_{i\in Z_K}$.

if the limit exists. In analogy to Definition 1.1, using lim inf and lim sup we define the upper and lower escape rates $\overline{E}(\mathcal{A}(\omega))$ and $\underline{E}(\mathcal{A}(\omega))$.

The collection $\mathcal{A}(\omega)$ defines a sequence of open domains upon which the sequence of maps $T_{\omega_{n-1}}, \ldots, T_{\omega_0}$ acts. The escape rate $E(\mathcal{A}(\omega))$ measures the asymptotic decrease in mass of the set $A^n(\omega)$, which is the set of points that have not fallen out of the sequence of open domains by time n. See Figure 9.

In the non-autonomous setting, we no longer have the notion of eigenvalue for our Perron-Frobenius cocycle. The analogous objects are *Lyapunov exponents*:

Definition 5.4 ([21])**.** For $f \in BV(I)$ and $\omega \in \Omega$ define the Lyapunov exponent of f with respect to ω to be

$$\Lambda(\omega, f) := \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}_{\sigma^{n-1}\omega}\mathcal{P}_{\sigma^{n-2}\omega}\ldots\mathcal{P}_\omega f\|_{\mathrm{BV}} \le 0$$

where $\|\cdot\|_{\mathrm{BV}} = \max\{\|\cdot\|_{L^1}, \mathrm{var}_I(\cdot)\}$.

We restate a version of (Corollary 4.1)[21] for our current purposes.

Theorem 5.5 ([21])**.** *There is a forward invariant full p-measure subset $\tilde{\Omega} \subset \Omega$, $0 = \Lambda_1 > \Lambda_2, \ldots, > \Lambda_r$ and $m_1, \ldots, m_r \in \mathbb{N}$ satisfying $m_1 + \cdots + m_r \le \#\Pi$ such that for all $\omega \in \tilde{\Omega}$:*

1. *there exist subspaces $\mathcal{W}_i(\omega) \subset \chi(\Pi), i = 0, \ldots, r, \dim \mathcal{W}_i(\omega) = m_i$;*
2. *$\mathcal{P}_\omega \mathcal{W}_i(\omega) = \mathcal{W}_i(\sigma\omega)$;*
3. *$f \in \mathcal{W}_i(\omega) \setminus \{0\} \Rightarrow \Lambda(\omega, f) = \Lambda_i$.*

Proposition 3. *For $f \in \mathcal{W}_i(\omega)$ we have*

$$\Lambda(\omega, f) = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}_{\sigma^{n-1}\omega}\mathcal{P}_{\sigma^{n-2}\omega}\ldots\mathcal{P}_\omega f\|_1.$$

Proof. It is sufficient to show that there exists a $C > 0$ such that for all $f \in \chi(\Pi)$, $\mathrm{var}_I(f) \le C\|f\|_1$. Assume that $f \in \chi(\Pi)$, then

$f = \sum_{i=0}^{\#\Pi-1} a_i \chi_{\pi_i}$. The result follows:

$$\mathrm{var}_I(f) = \sum_{i=0}^{\#\Pi-1} |a_i - a_{i-1}| \leq 2 \sum_{i=0}^{\#\Pi-1} |a_i| \leq \frac{2}{\min_i\{\ell(\pi_i)\}} \|f\|_1.$$

□

Now we state and prove the non-autonomous version of Theorem 2.4.

Theorem 5.6. *For $\omega \in \tilde{\Omega}$ let $f_\omega \in \mathcal{W}_i(\omega)$ such that $\Lambda_i < 0$ and let $f_{\sigma^n\omega} := \mathcal{P}_{\sigma^{n-1}\omega}\mathcal{P}_{\sigma^{n-2}\omega} \dots \mathcal{P}_\omega f_\omega = \mathcal{P}(n, \omega, f_\omega)$. Define two sequences of sets, $\mathcal{A}_+(\omega) := \{A_+(\sigma^n\omega)\}_{n\in\mathbb{Z}^+}$ and $\mathcal{A}_-(\omega) := \{A_-(\sigma^n\omega)\}_{n\in\mathbb{Z}^+}$ where*

$$A_+(\sigma^n\omega) := \mathrm{supp}(f^+_{\sigma^n\omega}), \quad \text{and} \quad A_-(\sigma^n\omega) := \mathrm{supp}(f^-_{\sigma^n\omega}).$$

Then one has $\underline{E}(\mathcal{A}_+(\omega)) \leq -\Lambda_i$ and $\underline{E}(\mathcal{A}_-(\omega)) \leq -\Lambda_i$.

Lemma 5.7. *For $\Lambda_i < 0$ and $f_{\sigma^n\omega}$ defined as in Theorem 5.6, we have for every $n \in \mathbb{Z}^+$*

$$\int_{A_+(\sigma^n\omega)} f_{\sigma^n\omega}\, d\ell = \frac{1}{2}\|f_{\sigma^n\omega}\|_1.$$

Proof. First, we show that for all n

$$\int_I f_{\sigma^n\omega}\, d\ell = 0.$$

Suppose that

$$\int_I f_\omega\, d\ell = L \neq 0.$$

As each $\mathcal{P}_{\sigma^n\omega}$ preserves Lebesgue measure, one has

$$\int_I f_{\sigma^n\omega}\, d\ell = L$$

for all $n \geq 0$. Thus $\|f_{\sigma^n\omega}\|_1 \geq |L|$ for all $n \geq 0$. Using Proposition 3,

$$\Lambda_i = \lim_{n\to\infty} \frac{1}{n} \log \|f_{\sigma^n\omega}\|_1 \geq \lim_{n\to\infty} \frac{1}{n} \log |L| = 0.$$

This is a contradiction as $\Lambda_i < 0$, therefore $\int_I f_{\sigma^n\omega}\, d\ell = 0$. Now we have

$$0 = \int_{A_+(\sigma^n\omega)} f_{\sigma^n\omega}\, d\ell + \int_{I\setminus A_+(\sigma^n\omega)} f_{\sigma^n\omega}\, d\ell$$

and

$$\|f_{\sigma^n\omega}\|_1 = \int_{A_+(\sigma^n\omega)} f_{\sigma^n\omega}\, d\ell - \int_{I\setminus A_+(\sigma^n\omega)} f_{\sigma^n\omega}\, d\ell$$

from which the result follows. □

Proof of Theorem 5.6. Notice that $A_+^n(\omega) = A_+(\omega) \cap T_\omega^{-1}(A_+^{n-1}(\sigma\omega))$. From the definition of $\mathcal{P}_\omega$ for every measurable $B \subset X$:

$$\begin{aligned}\int_B f_{\sigma\omega}\, d\ell = \int_B \mathcal{P}_\omega f_\omega\, d\ell &= \int_{T_\omega^{-1}B} f_\omega\, d\ell \\ &= \int_{T_\omega^{-1}B} f_\omega \chi_{A_+(\omega)}\, d\ell + \int_{T_\omega^{-1}B} f_\omega \chi_{X\setminus A_+(\omega)}\, d\ell \\ &\leq \int_{T_\omega^{-1}B} f_\omega \chi_{A_+(\omega)}\, d\ell \\ &= \int_{T_\omega^{-1}B\cap A_+(\omega)} f_\omega\, d\ell.\end{aligned}$$

Now letting $B = A_+^{n-j}(\sigma^j\omega)$ and considering $f_{\sigma^{j-1}\omega}$ we have for all $j \geq 0$:

$$\int_{A_+^{n-j}(\sigma^j\omega)} f_{\sigma^j\omega}\, d\ell \leq \int_{A_+^{n-j+1}(\sigma^{j-1}\omega)} f_{\sigma^{j-1}\omega}\, d\ell,$$

so the following series of inequalities hold

$$\int_{A_+^0(\sigma^n\omega)} f_{\sigma^n\omega}\, d\ell \leq \int_{A_+^1(\sigma^{n-1}\omega)} f_{\sigma^{n-1}\omega}\, d\ell \leq \cdots \leq \int_{A_+^n(\omega)} f_\omega\, d\ell.$$

Thus we can compare the first term and the last term, keeping in mind that $A_+^0(\sigma^n\omega) = A_+(\sigma^n\omega)$:

$$\int_{A_+(\sigma^n\omega)} f_{\sigma^n\omega}\, d\ell \leq \int_{A_+^n(\omega)} f_\omega\, d\ell.$$

Now by Lemma 5.7 $\int_{A_+(\sigma^n\omega)} f_{\sigma^n\omega}\, d\ell = \frac{1}{2}\|f_{\sigma^n\omega}\|_1$ and since $f_\omega \in BV$ is bounded we have $\int_{A_+^n(\omega)} f_\omega\, d\ell \leq C\ell(A_+^n(\omega))$ for some constant $C < \infty$. Thus

$$\frac{1}{2}\|f_{\sigma^n\omega}\|_1 \leq C\ell(A_+^n(\omega)).$$

Taking the logarithm of both sides, dividing by n and taking limit supremum we obtain the required inequality $\underline{E}(\mathcal{A}_+(\omega)) \leq -\Lambda_i$. By considering $-f_\omega$ in place of f_ω we obtain $\underline{E}(\mathcal{A}_-(\omega)) \leq -\Lambda_i$. □

We show two examples illustrating some consequences of Theorem 5.6. The first example is for a periodic case, while the second is for an aperiodic case. Both of these examples are taken from [21].

Example 5.8. Let T_1, T_2, T_3 be the piecewise affine maps shown in Figure 10, with a common Markov partition and let $\Omega \subset Z_3^{\mathbb{Z}}$ be the shift space generated by the periodic point $\omega = \ldots 123.123123\ldots$. The space Ω consists of three periodic points of period three, namely

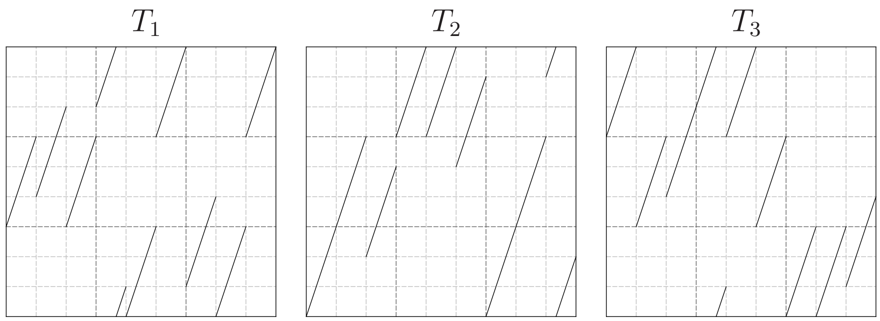

FIGURE 10. Graphs of maps T_1, T_2 and T_3.

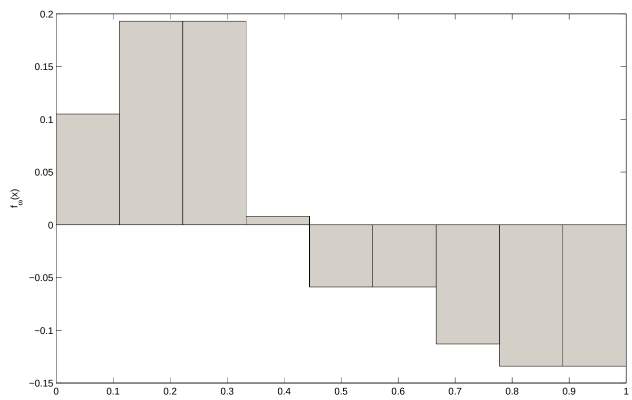

FIGURE 11. Graph of f_ω.

$\omega, \sigma\omega, \sigma^2\omega$. Note that $\tilde{\Omega} = \Omega$. Our aim is to find a collection of sets $\mathcal{A}_+(\omega) = \{A_+(\omega), A_+(\sigma\omega), A_+(\sigma^2\omega)\}$ such that the escape rate, $E(\mathcal{A}_+(\omega))$, is small. In [21] it was found that $\Lambda_2 \approx \log 0.8153$ and the space $\mathcal{W}_2(\omega) = span(f_\omega)$ where f_ω is shown in Figure 11. Pushing f_ω forward by $\mathcal{P}_2$ and then by $\mathcal{P}_3$ we obtain $f_{\sigma\omega}$ and $f_{\sigma^2\omega}$. Following the construction from Theorem 5.6 we have $\mathcal{A}_+(\omega) = \{[0, \frac{4}{9}), [\frac{1}{3}, \frac{2}{3}), [\frac{2}{3}, 1]\}$. The escape rate with respect to Lebesgue measure of $\mathcal{A}_+(\omega)$ is calculated to be $E(\mathcal{A}_-(\omega)) \approx -\log 0.8911 < -\log 0.8153 \approx -\Lambda_2$. We also calculate the escape rate from $\mathcal{A}_-(\omega) = \{[\frac{4}{9}, 1], [0, \frac{1}{3}) \cup [\frac{2}{3}, 1], [0, \frac{2}{3})\}$ giving $E(\mathcal{A}_-(\omega)) \approx -\log 0.9071 < -\Lambda_2$.

Example 5.9. Let $\{T_i\}_{i=1}^{6}$ be the collection of piecewise affine maps shown in Figures 10 and 12. Let $\Omega \subset Z_6^{\mathbb{Z}}$ be the vertex shift of the graph in Figure 13. In [21] $\mathcal{W}_2(\omega)$ was approximated for a non-periodic

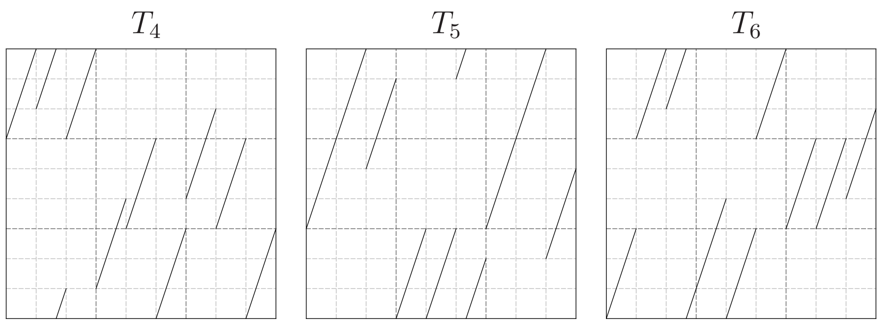

FIGURE 12. Graphs of maps T_4, T_5 and T_6.

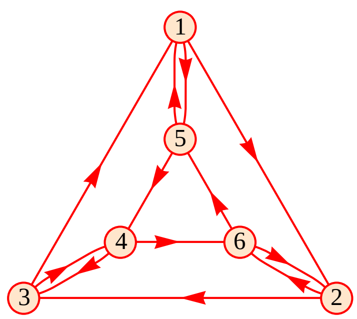

FIGURE 13. Graph of Ω.

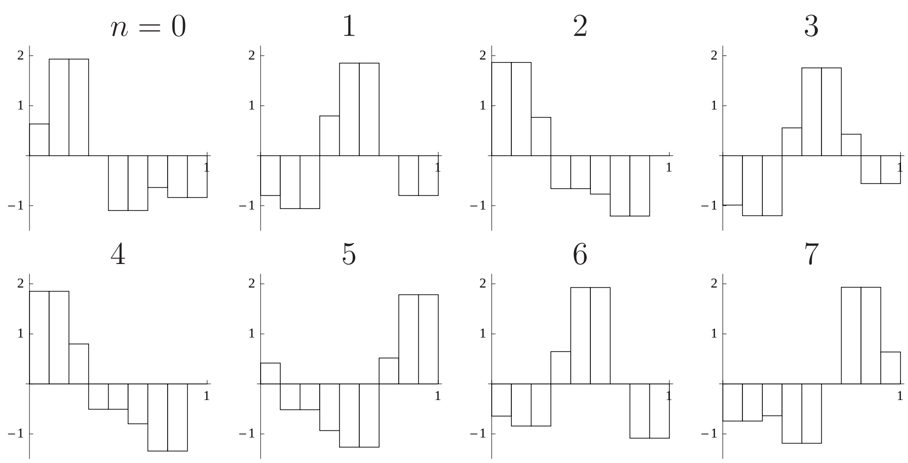

FIGURE 14. Graphs of the $f_{\sigma^n \omega}$ for $n = 0, \ldots, 7$.

point $\omega \in \Omega$

$$\omega = \dots 546231543.1515462651 \dots .$$

Note that this time we do not know whether $\omega \in \tilde{\Omega}$, however we may still check if results of Theorem 5.6 hold. The graphs of $f_{\sigma^n\omega}$ for $n = 0, \dots, 7$ are shown in Figure 14. The sequence of sets $\mathcal{A}_+(\omega) = \{A_+(\sigma^n\omega)\}_{i\in\mathbb{Z}^+}$ can then be approximated by following the construction of Theorem 5.6. We obtain

$$\mathcal{A}_+(\omega) = \Big\{ [0, \frac{1}{3}) \, , [\frac{1}{3}, \frac{7}{9}), [0, \frac{1}{3}), [\frac{1}{3}, \frac{7}{9}), [0, \frac{1}{3}), \\ [0, \frac{1}{9}) \cup [\frac{2}{3}, 1], [\frac{1}{3}, \frac{2}{3}), [\frac{2}{3}, 1] \dots \Big\} .$$

The escape rate of these sets is $E(\mathcal{A}_+(\omega)) \approx -\log 0.88 < -\log 0.81 \approx -\Lambda_2$. Similarly, we obtain $E(\mathcal{A}_-(\omega)) \approx -\log 0.91$.

REFERENCES

[1] W. Bahsoun, *Rigorous numerical approximation of escape rates*, Nonlinearity, **19**(11) (2006), 2529–2542.

[2] W. Bahsoun and C. Bose, *Quasi-invariant measures, escape rates and the effect of the hole*, preprint arXiv:0906.5375.

[3] V. Baladi, "Positive transfer operators and decay of correlations," volume 16 of *Advanced Series in Nonlinear Dynamics*, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

[4] H. Bruin, M. Demers, and I. Melbourne, *Existence and convergence properties of physical measures for certain dynamical systems with holes*, preprint arXiv:0705.4271v2.

[5] L. Bunimovich and A. Yurchenko. *Where to place a hole to achieve a maximal escape rate*, preprint arXiv:0811.4438.

[6] N. Chernov, R. Markarian, and S. Troubetzkoy, *Conditionally invariant measures for Anosov maps with small holes*, Ergodic Theory Dynam. Systems, **18**(5) (1998), 1049–1073.

[7] P. Collet, S. Martínez, and V. Maume-Deschamps, *On the existence of conditionally invariant probability measures in dynamical systems*, Nonlinearity, **13**(4) (2000), 1263–1274.

[8] P. Collet, S. Martínez, and V. Maume-Deschamps, *Corrigendum: "On the existence of conditionally invariant probability measures in dynamical systems"*, Nonlinearity, **17**(5) (2004), 1985–1987.

[9] P. Collet, S. Martínez, and B. Schmitt, *The Yorke-Pianigiani measure and the asymptotic law on the limit Cantor set of expanding systems*, Nonlinearity, 7(5) (1994), 1437–1443.

[10] P. Collet, S. Martínez, and B. Schmitt, *The Pianigiani-Yorke measure for topological Markov chains*, Israel J. Math, **97** (1997) 61–70.

[11] P. Collet, S. Martínez, and B. Schmitt, *On the enhancement of diffusion by chaos, escape rates and stochastic instability*, Trans. Amer. Math. Soc., **351**(7) (1999), 2875–2897.

[12] M. Dellnitz, G. Froyland, and S. Sertl, *On the isolated spectrum of the Perron-Frobenius operator*, Nonlinearity, **13**(4) (2000), 1171–1188.

[13] M. Dellnitz and O. Junge, *On the approximation of complicated dynamical behavior*, SIAM J. Numer. Anal., **36**(2) (1999), 491–515.

[14] M. F. Demers, *Markov extensions for dynamical systems with holes: an application to expanding maps of the interval*, Israel J. Math., **146** (2005), 189–221.

[15] M. F. Demers and L.-S. Young, *Escape rates and conditionally invariant measures*, Nonlinearity, **19**(2) (2006), 377–397.

[16] J. Ding, Q. Du, and T. Y. Li, *The spectral analysis of Frobenius-Perron operators*, J. Math. Anal. Appl., **184**(2) (1994), 285–301.

[17] G. Froyland, *Using Ulam's method to calculate entropy and other dynamical invariants*, Nonlinearity, **12**(1) (1999), 79–101.

[18] G. Froyland, *Statistically optimal almost-invariant sets*, Phys. D, **200**(3-4) (2005), 205–219.

[19] G. Froyland, *On Ulam approximation of the isolated spectrum and eigenfunctions of hyperbolic maps*, Discrete and Continuous Dynamical System, Series A,**17**(3) (2007), 671–689.

[20] G. Froyland and M. Dellnitz, *Detecting and locating near-optimal almost-invariant sets and cycles*, SIAM J. Sci. Comput., **24**(6) (2003) 1839–1863.

[21] G. Froyland, S. Lloyd, and A. Quas, *Coherent structures and isolated spectrum for Perron-Frobenius cocycles*, Ergodic Theory and Dynamical Systems, to appear, (2009), arXiv:0804.1426.

[22] G. Froyland, S. Lloyd, and N. Santitissadeekorn, *Coherent sets for nonautonomous dynamical systems*, Submitted, 2008.

[23] G. Froyland and K. Padberg, *Almost-invariant sets and invariant manifolds – connecting probabilistic and geometric descriptions of coherent structures in flows*, Physica D, **238**(16) (2009), 1507–1523.

[24] G. Froyland, K. Padberg, M. H. England, and A. M. Treguier, *Detection of coherent oceanic structures via transfer operators*, Physical Review Letters, **98**(22) (2007), 224503.

[25] G. Froyland, M. Schwalb, K. Padberg, and M. Dellnitz, *A transfer operator based numerical investigation of coherent structures in three-dimensional southern ocean circulation*, in "Proceedings of the 2008 International Symposium on Nonlinear Theory and its Applications, Budapest, September 2008", 313–316.

[26] G. Keller and C. Liverani, *Stability of the spectrum for transfer operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **28**(1) (1999), 141–152.

[27] G. Keller and C. Liverani, *Rare events, escape rates and quasistationarity: some exact formulae*. Journal of Statistical Physics, **135**(3) (2009) 519–534.

[28] T. Y. Li and J. A. Yorke, *Ergodic transformations from an interval into itself*, Trans. Amer. Math. Soc., **235** (1978) 183–192.

[29] D. Lind and B. Marcus, "An introduction to symbolic dynamics and coding", Cambridge University Press, Cambridge, 1995.

[30] C. Liverani and V. Maume-Deschamps, *Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set*, Ann. Inst. H. Poincaré Probab. Statist., *39*(3) (2003), 385–412.

[31] A. Lopes and R. Markarian, *Open billiards: invariant and conditionally invariant probabilities on Cantor sets*, SIAM J. Appl. Math., **56**(2) (1996) 651–680.

[32] G. Pianigiani and J. A. Yorke, *Expanding maps on sets which are almost invariant. Decay and chaos* Trans. Amer. Math. Soc., **252** (1979) 351–366.

[33] C. Schütte, W. Huisinga, and P. Deuflhard, *Transfer operator approach to conformational dynamics in biomolecular systems*. in "Ergodic theory, analysis, and efficient simulation of dynamical systems", Springer, Berlin, 2001, 191–223.

[34] C. G. Tokman, B. R. Hunt, and P. Wright, *Approximating invariant densities of metastable systems*, preprint, arXiv:0905.0223.

[35] H. van den Bedem and N. Chernov, *Expanding maps of an interval with holes* Ergodic Theory Dynam. Systems, **22**(3) (2002) 637–654.

E-mail address: g.froyland@unsw.edu.au

E-mail address: o.stancevic@unsw.edu.au