ULAM’S METHOD FOR LASOTA-YORKE MAPS WITH HOLES

CHRISTOPHER BOSE, GARY FROYLAND, CECILIA GONZÁLEZ-TOKMAN,
AND RUA MURRAY

ABSTRACT. Ulam’s method is a rigorous numerical scheme for approximating invariant densities of dynamical systems. The phase space is partitioned into connected sets and an inter-set transition matrix is computed from the dynamics; an approximate invariant density is read off as the leading left eigenvector of this matrix. When a hole in phase space is introduced, one instead searches for conditional invariant densities and their associated escape rates. For Lasota-Yorke maps with holes we prove that a simple adaptation of the standard Ulam scheme provides convergent sequences of escape rates (from the leading eigenvalue), conditional invariant densities (from the corresponding left eigenvector), and quasi-conformal measures (from the corresponding right eigenvector). We also immediately obtain a convergent sequence for the invariant measure supported on the survivor set. Our approach is non-perturbative and allows us to consider relatively large holes.

1. INTRODUCTION

Dynamical systems $\hat{T} : I \to I$ typically model complicated deterministic processes on a phase space $I$. The map $\hat{T}$ induces a natural action on probability measures $\eta$ on $I$ via $\eta \mapsto \eta \circ \hat{T}^{-1}$. Of particular interest in ergodic theory are those probability measures that are $\hat{T}$-invariant; that is, $\eta$ satisfying $\eta = \eta \circ \hat{T}^{-1}$. If $\eta$ is ergodic, then such $\eta$ describe the time-asymptotic distribution of orbits of $\eta$-almost-all initial points $x \in I$. In this paper, we consider the situation where a “hole” $H_0 \subseteq I$ is introduced and any orbits of $\hat{T}$ that fall into $H_0$ terminate. The hole induces an open dynamical system $T : X_0 \to I$, where $X_0 = I \setminus H_0$. Because trajectories are being lost to the hole, in many cases, there is no $T$-invariant probability measure. One can, however, consider conditionally invariant probability measures [20], which satisfy $\eta \circ T^{-1}(I) \cdot \eta = \eta \circ T^{-1}$, where $0 < \eta \circ T^{-1}(I) < 1$ is identified as the escape rate for the open system.

We will study $\hat{T}$ drawn from the class of Lasota-Yorke maps: piecewise $C^1$ expanding maps of the interval, such that $|D\hat{T}|^{-1}$ has bounded variation. The hole $H_0$ will be a finite union of intervals. In such a setting, because of the expanding property, one can expect to obtain conditionally invariant probability measures that are absolutely continuous with respect to Lebesgue measure [5, 23, 17]. Such conditionally invariant measures are “natural” as they may correspond to the result of repeatedly pushing forward Lebesgue measure by $\hat{T}$. In the next section we will discuss further conditions due to [17] that make this precise: (i) how much of phase space can “escape” into the hole, and (ii) the growth rate of intervals that partially escape relative to the expansion of the map and the rate of escape. These conditions will also guarantee the existence of a unique absolutely continuous conditionally invariant probability measure (accim). This accim $\nu$, with density $h$, and its corresponding escape rate $\rho$ are the first two objects that we will rigorously numerically approximate using Ulam’s method. Existence and uniqueness results for subshifts of finite type with Markov holes were previously established by Collet, Martínez and Schmitt in [8]; see also [6, 7].
One may also consider the set of points \( X_\infty \subset I \) that never fall into the hole \( H_0 \). A probability measure \( \lambda \) on \( X_\infty \) can be defined as the \( n \to \infty \) limit of the accim \( \nu \) conditioned on \( X_n \). The measure \( \lambda \) will turn out to be the unique \( \hat{T} \)-invariant measure supported on \( X_\infty \) and has the form \( \lambda = h \mu \), where \( h \) is a Lebesgue integrable function and \( \mu \) is known as the quasi-conformal measure for \( \hat{T} \). We will also rigorously numerically approximate \( \mu \) and thus \( \lambda \).

Our main result, Theorem 3.2, concerns convergence properties of an extension of the well-known construction of Ulam [22], which allows for efficient numerical estimation of invariant densities of closed dynamical systems. The Ulam approach partitions the domain \( I \) into a collection of connected sets \( \{I_1, \ldots, I_k\} \) and computes single-step transitions between partition sets, producing the matrix

\[
\hat{P}_{ij} = \frac{m(I_i \cap \hat{T}^{-1}I_j)}{m(I_j)}.
\]

Li [16] demonstrated that the invariant density of Lasota-Yorke maps can be \( L^1 \)-approximated by step functions obtained directly from the leading left eigenvector of \( \hat{P} \). Since the publication of [16] there have been many extensions of Ulam’s method to more general classes of maps, including expanding maps in higher dimensions [9, 18], uniformly hyperbolic maps [10, 12], nonuniformly expanding interval maps [19, 13], and random maps [11, 14]. Explicit error bounds have also been developed, eg. [11, 4].

We will show that in order to handle open systems, the definition of \( \hat{P} \) above need only be modified to \( P \), having entries

\[
P_{ij} = \frac{m(I_i \cap X_0 \cap \hat{T}^{-1}I_j)}{m(I_j)}.
\]

As in the closed setting, one uses the leading left eigenvector to produce a step function that approximates the density \( h \) of the accim \( \nu \). However, in the open setting, the leading eigenvalue of \( P \) also approximates the escape rate \( \rho \) of \( \nu \), and the right eigenvector approximates the quasi-conformal measure \( \mu \). Note that for closed systems, \( \rho = 1 \) and \( \mu = m \).

The literature concerning the analysis of Ulam’s method is now quite large. Early work on Ulam’s method for Axiom A repellors [12] showed convergence of an Ulam-type scheme using Markov partitions for the approximation of pressure and equilibrium states with respect to the potential \( -\log |\det D\hat{T}|_{E^u} \). These results apply to the present setting of Lasota-Yorke maps provided the hole is Markov and projections are done according to a sequence of Markov partitions. Bahsoun [1] considered non-Markov Lasota-Yorke maps with non-Markov holes and rigorously proved an Ulam-based approximation result for the escape rate. Bahsoun used the perturbative machinery of [15], treating the map \( T \) as a small deterministic perturbation of the map \( \hat{T} \). Utilising the results of [17] we instead only make assumptions on the expansivity of \( T \) (large enough), the escape rate (slow enough), and the rate of generation of “bad” subintervals (small enough); from these assumptions we construct an improved Lasota-Yorke inequality that allows us to get tighter constants which make applications more plausible. Moreover, we also obtain rigorous \( L^1 \)-approximations of the accim and approximations of the quasi-conformal measure that converge weakly to \( \mu \). Finally, as our approach does not rely on the perturbation arguments of [15], we can treat relatively large holes.

An outline of the paper is as follows. In Section 2 we introduce the Perron-Frobenius operator \( \mathcal{L} \), formally define admissible and Ulam-admissible holes, and develop a strong Lasota-Yorke inequality. Section 3 introduces the new Ulam scheme, states our main
Ulam convergence result, develops some specialised results for full-branch maps, and discusses some specific example maps in detail. Section 4 contains the proof of the main result.

2. LASOTA-YORKE MAPS WITH HOLES

The following class of interval maps with holes was studied by Liverani and Maume-Deschamps in [17].

**Definition 2.1.** Let $I = [0,1]$. We call $\hat{T} : I \cap \Theta$ a Lasota-Yorke map if $\hat{T}$ is a piecewise $C^1$ map, with finite monotonicity partition $\mathcal{Z}$, there exists $\hat{\Theta} < 1$ such that $\|D\hat{T}^{-1}\|_\infty \leq \hat{\Theta}$, and $\hat{g} := |D\hat{T}|^{-1}$ has bounded variation.

The transfer operator for the map $\hat{T}$ is the bounded linear operator $\hat{\mathcal{L}}$, acting on the space $BV$ of functions of bounded variation on $I$, defined by

$$\hat{\mathcal{L}}(f)(x) = \sum_{\hat{T}^{-1}(y) = x} f(y)\hat{g}(y).$$

**Definition 2.2.** Let $\hat{T} : I \cap \Theta$ be a Lasota-Yorke map. Let $H_0 \subset I$ be a finite union of closed intervals, and let $X_0 = I \setminus H_0$. Let $T : X_0 \to I$ be the restriction $T = \hat{T}|_{X_0}$. Both $T$ and the pair $T_0 = (\hat{T}, H_0)$ are referred to as open Lasota-Yorke maps (or briefly, open systems), and their associated transfer operator is the bounded linear operator $\mathcal{L} : BV \cap BV_0$ given by

$$\mathcal{L}(f)(x) = \hat{\mathcal{L}}(1_{X_0}f).$$

For each $n \geq 1$, let $X_n = \bigcap_{j=0}^{n} T^{-j}X_0$. Thus, $X_n$ is the set of points that have not escaped by time $n$. Also, we denote by $T^n$ the function $\hat{T}^n|_{X_{n-1}}$. One can readily check that

$$\mathcal{L}^n(f) = \hat{\mathcal{L}}^n(1_{X_{n-1}}f).$$

**Definition 2.3.** Let $T$ be an open Lasota-Yorke map. A probability measure $\nu$ supported on $X_0 \subset I$ which is absolutely continuous with respect to Lebesgue measure and has density $h = \frac{d\nu}{dm}$ of bounded variation is called an absolutely continuous conditional invariant measure (accim) for $T$. Let $\mathcal{L}h = \rho h$ for some $0 < \rho \leq 1$.

A probability measure $\mu$ on $I$ which satisfies $\mu(\mathcal{L}f) = \rho \mu(f)$ for every function of bounded variation $f : I \to \mathbb{R}$, with $\rho$ as above, is called a quasi-conformal measure for $T$.

**Remark 2.4.** It is usual to define $\nu$ to be an accim if $\nu(A) = \frac{\nu(T^n A \cap X_n)}{\nu(X_n)}$ for every $n \geq 0$ and Borel measurable set $A \subset I$. The definitions are indeed equivalent; see [17, Lemma 1.1] for a proof. The same lemma shows that if $\mu$ is a quasi-conformal measure for $T$, then $\mu$ is necessarily supported on $X_\infty = \bigcap_{n \geq 0} \hat{X}_n$. It is also usual to require $\mu$ to satisfy $\mu(\mathcal{L}f) = \rho \mu(f)$ for continuous functions only. We will see this makes no difference in our setting, as this weaker requirement implies the stronger one in the previous definition.

2.1. Admissible holes and quasi-invariant measures. As in the work of Liverani and Maume-Deschamps [17], we impose some conditions on the open system in order to be able to analyze it. Let us fix some notation.

Let $(\hat{T}, H_0)$ be an open Lasota-Yorke map, which we also refer to as $T$. For each $n \geq 1$, let $D_n = \{x \in I : \mathcal{L}^n 1(x) \neq 0\}$, and let $D_\infty := \bigcap_{n \geq 1} D_n$. In what follows, we assume that $D_\infty \neq \emptyset$.

\[\text{1}^{\text{Throughout this paper, a monotonicity partition } \mathcal{Z} \text{ refers to a partition such that for every } Z \in \mathcal{Z} \text{ } \hat{T}|_Z \text{ has a } C^1 \text{ extension to } Z.\]
For each $\epsilon > 0$ (not necessarily small), we let $G_\epsilon = G_\epsilon(T)$ be the collection of finite partitions of $I$ into intervals such that $Z_\epsilon \in G_\epsilon(T)$ if (i) the interior of each $A \in Z_\epsilon$ is either disjoint from or contained in $X_0$, and (ii) for each $A \in Z_\epsilon$, $\var_A \left(1_{X_0} | DT^{-1}| \right) < \| DT^{-1} \|_\infty (1 + \epsilon)$. Since $H_0$ consists of finitely many intervals, this condition is possible to achieve, as the work of Rychlik [21, Lemma 6] shows. We call $G_\epsilon$ the collection of $\epsilon$-adequate partitions for $T$. The set of elements of $Z_\epsilon$ whose interiors are contained in $X_0$ is denoted by $Z_\epsilon^*$. Next, the elements of $Z_\epsilon^*$ are divided into good and bad. A set $A \in Z_\epsilon^*$ is good if
\[
\lim_{n \to \infty} \inf_{x \in D_n} \frac{L^{n+1}A(x)}{L^n1(x)} > 0.
\]
We point out that it is shown in [17] that the limit above always exists, as the sequence involved is increasing and bounded, and it is clearly non-negative. The set $A$ is called bad when the limit above is 0. We let
\[
Z_{\epsilon,g} = \{ A \in Z_\epsilon^* : A \text{ is good} \}, \quad \text{and} \quad Z_{\epsilon,b} = \{ A \in Z_\epsilon^* : A \text{ is bad} \}.
\]
Finally, two elements of $Z_\epsilon^*$ are called contiguous if there are no other elements of $Z_\epsilon^*$ in between them (but there may be elements of $Z_\epsilon$ that are necessarily contained in $H_0$). We let $\xi_\epsilon = \xi_\epsilon(T)$ be the infimum over $\epsilon$-adequate partitions for $T$ of the maximum number of contiguous elements in $Z_{\epsilon,b}$.

In a similar manner, we let $G^{(n)}_\epsilon = G^{(n)}_\epsilon(T)$ be the collection of finite partitions of $I$ into intervals such that $Z^{(n)}_\epsilon \in G^{(n)}_\epsilon$ if (i) the interior of each $A \in Z^{(n)}_\epsilon$ is either disjoint from or contained in $X_{n-1}$, and (ii) for each $A \in Z^{(n)}_\epsilon$, $\var_A \left(1_{X_{n-1}} | DT^{n-1}| \right) < \| DT^{n-1} \|_\infty (1 + \epsilon)$. The partitions $Z^{(n)}_\epsilon, Z^{(n)}_{\epsilon,g}, Z^{(n)}_{\epsilon,b}$ are defined analogously. We denote by $\xi_{\epsilon,n} = \xi_{\epsilon,n}(T)$ the infimum over $\epsilon$-adequate partitions for $T^n$ of the maximum number of contiguous elements in $Z^{(n)}_{\epsilon,b}$; so $\xi_\epsilon = \xi_{\epsilon,1}$.

The following quantities are relevant in what follows:
\[
\rho = \rho(T) := \lim_{n \to \infty} \inf_{x \in D_n} \frac{L^{n+1}1(x)}{L^n1(x)},
\]
\[
\tilde{\Theta} = \tilde{\Theta}(T) := \exp \left( \lim_{n \to \infty} \frac{1}{n} \log \| DT^n \|_\infty \right),
\]
\[
\hat{\xi}_\epsilon = \hat{\xi}_\epsilon(T) := \exp \left( \limsup_{n \to \infty} \frac{1}{n} \log (1 + \xi_{\epsilon,n}) \right),
\]
\[
(3) \quad \alpha_\epsilon = \alpha_\epsilon(T) := \| DT^{-1} \|_\infty (2 + \epsilon + \hat{\xi}_\epsilon).
\]

**Definition 2.5 (Admissible holes).** Let $\tilde{T} : I \circlearrowright$ be a Lasota-Yorke map, and $\epsilon > 0$. We say that $H_0 \subset I$ is:
- an $\epsilon$-admissible hole for $\tilde{T}$ if $D_\infty \neq \emptyset$ and $\hat{\xi}_\epsilon \tilde{\Theta} < \rho$,
- an admissible hole for $\tilde{T}$ if it is $\epsilon$-admissible for $\epsilon = 1$,
- an $\epsilon$-Ulam-admissible hole for $\tilde{T}$ if $D_\infty \neq \emptyset$ and $\alpha_\epsilon < \rho$.

The main result of Liverani and Maume-Deschamps [17] is concerned with the existence of the objects we intend to rigorously numerically approximate.

**Theorem 2.6 ([17, Theorem A & Lemma 3.10]).** Assume $(\tilde{T}, H_0)$ is an open system with an admissible hole. Then,

\[\text{This is the choice made in [17].}\]
(1) There exists a unique absolutely continuous conditionally invariant measure (ac-cim) \( \nu = h_m \) for \((\hat{T}, H_0)\).

(2) There exists a unique quasi-conformal measure \( \mu \) for \((\hat{T}, H_0)\), such that \( \mu(\mathcal{L}f) = \rho \mu(f) \) for every \( f \in BV \). Furthermore, this measure is atom-free, and satisfies the property that
\[
\mu(f) = \lim_{n \to \infty} \inf_{x \in D_n} \frac{\mathcal{L}^n f(x)}{\mathcal{L}^n 1(x)}
\]
for every \( f \in BV \), and \( \rho = \mu(\mathcal{L}1) \).

(3) The measure \( \lambda = h_\mu \) is, up to scalar multiples, the only \( T \)-invariant measure supported on \( X_\infty \) and absolutely continuous with respect to \( \mu \).

(4) There exist \( \kappa < 1 \) and \( C > 0 \) such that for any function of bounded variation \( f \),
\[
\left\| \frac{\mathcal{L}^n f}{\rho^m} - h_\mu(f) \right\|_\infty \leq C \kappa^n \| f \|_{BV}.
\]

Remark 2.7. It follows readily from the proof of Theorem 2.6 [17] that the same conclusion can be obtained if the hypothesis of \( H_0 \) being an admissible hole is replaced by \( H_0 \) being an \( \epsilon \)-admissible hole for some \( \epsilon > 0 \).

To close this section, we present a lemma concerning admissibility of different holes, obtained by enlarging an initial hole \( H_0 \) to \( H_m := I \setminus X_m \). This broadens the applicability of Theorem 3.2 because enlarging the holes may reduce the number of contiguous bad intervals, and also the variation remaining on the domain of the open Lasota-Yorke map without increasing the expansion.

Lemma 2.8 (Enlarging holes). Let \( T_0 = (\hat{T}, H_0) \) be an open system with an \( \epsilon \)-admissible hole, and for each \( m \geq 0 \), let \( H_m := I \setminus X_m \). Then, for each \( m \geq 0 \), \( T_m := (\hat{T}, H_m) \) is an open system with an \( \epsilon \)-admissible hole. Furthermore, let \( \rho(T_m), h(T_m) \) and \( \mu(T_m) \) be the escape rate, accim and quasi-conformal measures of \( T_m \), respectively. Then we have the following.

1. \( \rho(T_m) = \rho(T_0) \).
2. \( \mathcal{L}^m(h(T_m)) = \rho(T_0)^m h(T_0) \), and
3. \( \mu(T_m) = \mu(T_0) \).

The proof of Lemma 2.8 is presented in §4.2.

2.2. Auxiliary lemmas. Under the assumptions of Theorem 2.6, the quasi-conformal measure \( \mu \) of \((\hat{T}, H_0)\) satisfies some further properties that will be exploited in our approach. The measure \( \mu \) can be used to define a useful cone of functions in \( BV \). For each \( a > 0 \) let
\[
\mathcal{C}_a = \{ 0 \leq f \in BV : \text{var}(f) \leq a \mu(f) \}.
\]

Combining the result of Lemmas 4.2 and 4.3 from [17] with the argument in the proof of Lemma 3.7 (therein), the conditions on \( T \) imply the existence of a constant \( a_1 > 0 \) such that for any \( a > a_1 \) there is an \( \epsilon_a > 0 \) and \( N \in \mathbb{N} \) such that
\[
\mathcal{L}^N \mathcal{C}_a \subseteq \mathcal{C}_{a-\epsilon_a}.
\]

The values of \( N, a_1 \) and \( \epsilon_a \) are all computable in terms of the constants associated with \( T \). We present a modified version of these arguments, based on the classical work of Rychlik [21], that specialize to the case \( N = 1 \), and allow us to improve some of the constants involved in the estimates of [17].
Lemma 2.9. Let \((\hat{T}, H_0)\) be a Lasota-Yorke map with an \(\epsilon\)-Ulam-admissible hole. Then, there exists \(K_\epsilon > 0\) such that for every \(f \in BV\),
\[
\var(L f) \leq \alpha_\epsilon \var(f) + K_\epsilon \mu(|f|).
\]
Furthermore, there is a constant \(a_1 > 0\) such that for any \(a > a_1\) there is an \(\epsilon_a > 0\) such that
\[
(5) \quad \mathcal{LC}_a \subseteq C_{a-\epsilon_a}.
\]

The proof of Lemma 2.9 is deferred to § 4.1.

We will also make use of the following facts coming from [17].

Lemma 2.10. Let \(\mathcal{Z}^{(n)}\) be the partition into monotonicity intervals of \(\hat{T}^n\).

1. [17, Lemma 3.10] For each \(\epsilon > 0\) there exists \(n_0\) such that for each \(n \geq n_0\),
\[
\sup_{Z \in \mathcal{Z}^{(n)}} \mu(1_Z) \leq \epsilon.
\]

2. There exists \(a_0 > 0\) such that for each \(a \geq a_0\) there is a constant \(C > 0\) (depending on \(a\)) for which \(h \in C_a \Rightarrow \mu(h) \leq C\|h\|_1\).

Proof. (2): By [17, Lemma 3.11], for each such \(a \geq a_0\) there is an \(n\) (depending on \(a\)) and a finite collection of subintervals \(\mathcal{Z}^{(n)}\) such that for any \(h \in C_a\) there is a \(Z \in \mathcal{Z}^{(n)}\) for which \(\mu h \leq 4 \min Z h dm \leq \frac{\|h\|_1}{\min_Z m(Z)}\). \(\Box\)

3. Ulam’s method for Lasota-Yorke maps with holes

3.1. The Ulam scheme. In the case of a closed system \(\hat{T}\), the Ulam method introduced in [22] provides a way of approximating the transfer operator with a sequence of finite-rank operators \(\hat{L}_k\), each coming from discretizing the interval \(I\) into \(k\) bins (which may or may not be of equal length). The only requirements are that each bin is a non-trivial interval, and that the maximum diameter of the partition elements, denoted by \(\eta_k\), goes to 0 as \(k\) goes to infinity. We call such \(k\)-bin partition \(P_k\). The operator \(\hat{L}_k\) preserves the \(k\)-dimensional subspace \(\text{span}\{\chi_j : \chi_j = 1_{I_j}, I_j \in P_k\}\). The matrix \(\hat{P}_k\) defined in the introduction represents the action of \(\hat{L}_k\) on this space, with respect to the ordered basis \((\chi_1, \ldots, \chi_k)\) [16].

In the case of an open system \((\hat{T}, H_0)\), one can still follow Ulam’s approach to define a discrete approximation \(L_k\) to the transfer operator \(L\). For a function \(f \in BV\), the operator is defined by \(L_k f = \pi_k(L f) = \pi_k \hat{L}(1_{X_0} f)\), where \(\pi_k\) is given by the formula
\[
\pi_k(f) = \sum_{j=1}^{k} \frac{1}{m(I_j)} \left( \int \chi_j f \, dm \right) \chi_j.
\]
The entries of the Ulam transition matrix \(P_k\) representing \(L_k\) in the ordered basis \((\chi_1, \ldots, \chi_k)\) are
\[
(P_k)_{ij} = \frac{m(I_i \cap X_0 \cap \hat{T}^{-1} I_j)}{m(I_j)}.
\]
(When the partition \(P_k\) is uniform\(^3\), the transition matrices \(\hat{P}_k\) defined in (1) are stochastic, and \(P_k\) are substochastic, the loss of mass being a consequence of the presence of a hole.) Since the entries of \(P_k\) are non-negative, an extension of the Perron-Frobenius

\(^3\)That is, \(m(I_i) = m(I_j), \forall i, j.\)
theorem applies and provides the existence of a non-negative eigenvalue $0 \leq \rho_k \leq 1$ of maximal absolute value for $P_k$, with associated left and right eigenvectors with non-negatives entries; see e.g. [3]. In general, these may or may not be unique. Non-negative left eigenvectors $p_k$ of $P_k$ induce densities on $I$ according to the formula

$$h_k = \sum_{j=1}^{k} [p_k]_j \chi_j,$$

(where we adopt the convention that a vector $x$ can be written in component form as $x = ([x]_1, \ldots, [x]_k)$. Non-negative right eigenvectors $\psi_k$ of $P_k$ induce measures $\mu_k$ on $I$ according to the formula

$$\mu_k(E) = \sum_{j=1}^{k} [\psi_k]_j m(I_j \cap E).$$

We conclude the section with the following.

**Lemma 3.1.** Let $P_k$ be the matrix representation of $L_k = \pi_k \circ L$ with respect to the basis $\{\chi_j\}$. If $P_k \psi_k = \rho_k \psi_k$ then the measure $\mu_k$ corresponding to $\psi_k$ satisfies $\mu_k(L_k \pi_k \varphi) = \rho_k \mu_k(\varphi)$ for every $\varphi \in L^1(m)$.

**Proof.** Let $\varphi \in L^1(m)$ and put $\varphi_k = \pi_k \varphi$. Then,

$$\mu_k(\varphi) = \int \varphi \, d\mu_k = \sum_{j=1}^{k} \int_{I_j} \varphi \, dm(\psi_k)_j = \sum_{j=1}^{k} \int_{I_j} \pi_k \varphi \, dm(\psi_k)_j = \rho_k \int \varphi \, dm(\psi_k)_j = \rho_k^{-1} \mu_k(L_k \varphi_k),$$

where the last equality in the second line follows from the fact that $P_k$ is the matrix representing $L_k$ in the basis $\{\chi_j\}$, and acts on densities by right multiplication (i.e. if $p$ is the vector representing the function $\varphi_k$, then $p^T P_k$ is the vector representing $L_k \varphi_k$). \qed

### 3.2. Statement of the main result.

The main result of this paper is the following. Its proof is presented in Section 4.3.

**Theorem 3.2.** Let $\hat{T} : I \diamond$ be a Lasota-Yorke map with an $\epsilon$-Ulam-admissible hole $H_0$. Let $h \in BV$ be the unique accim for the open system $(\hat{T}, H_0)$, and $\mu$ the unique quasi-conformal measure for the open system supported on $X_\infty$, as guaranteed by Theorem 2.6. Let $\rho$ be the associated escape rate. For each $k \in \mathbb{N}$, let $\rho_k$ be the leading eigenvalue of the Ulam matrix $P_k$. Let $h_k$ be densities induced from non-negative left eigenvectors of $P_k$ corresponding to $\rho_k$. Let $\mu_k$ be measures induced from non-negative right eigenvectors of $P_k$ corresponding to $\rho_k$. Then,

1. For $k$ sufficiently large, $\rho_k$ is a simple eigenvalue for $P_k$,
2. $\lim_{k \to \infty} \rho_k = \rho$, $\lim_{k \to \infty} h_k = h$ in $L^1(m)$, and
3. $\lim_{k \to \infty} \mu_k = \mu$ in the weak-* topology of measures.
3.3. Examples. To illustrate the adequacy of Ulam’s method, beyond the small-hole setting, we present some examples of Ulam-admissible open Lasota-Yorke systems. We deal with the case of full-branched maps in §3.3.1, and then treat some more general examples, including \( \beta \)-shifts, in §3.3.2. We conclude the section with a general result relating admissibility of holes with Ulam-admissibility in §3.3.3. This result, together with Lemma 2.8, broadens the scope of applicability of Theorem 3.2 by allowing to (i) replace the map by an iterate (Lemma 3.13), or (ii) enlarge the hole in a dynamically consistent way (Lemma 2.8).

Given a Lasota-Yorke map with holes, \((\hat{T},H_0)\) with monotonicity partition \(Z\), we let \(Z_h = \{ Z \in Z : Z \subseteq H_0 \} \), \(Z_f = \{ Z \in Z : Z \cap H_0 = \emptyset, T(Z) = I \} \) and \(Z_u = \{ Z \in Z : Z \notin Z_h \cup Z_f \} \). Thus, the elements of \(Z_f\) are precisely the ones contained in \(X_0\) that are full branches for \(T\), and those of \(Z_u\) are the remaining ones.

3.3.1. Full-branched maps.

**Definition 3.3.** A full-branched map with holes, \((\hat{T},H_0)\), is a Lasota-Yorke map with holes, such that \(Z_u = \emptyset\).

For piecewise linear maps, the situation is rather simple.

**Lemma 3.4.** Let \(T_0 = (\hat{T},H_0)\) be a piecewise linear full-branched map with holes. Then, for every \(\epsilon > 0\) the following holds: \(\xi_\epsilon(T_0) = 0\),

\[
\rho(T_0) = 1 - \text{Leb}(H_0), \quad \text{and} \quad \alpha_\epsilon(T_0) = \max_{Z \in Z_f} \text{Leb}(Z)(2 + \epsilon).
\]

**Proof.** If \(T_0\) is a piecewise linear full-branched map, then each interval \(Z \in Z_f\) is good. Observing that an interval being good is equivalent to having non-zero \(\mu\) measure, and using the fact that \(\mu\) is atom-free, each \(Z\) may be split into two good intervals \(Z_-, Z_+\), in such a way that there is at most one discontinuity of \(g\) on each \(Z_-, Z_+\). Thus, \(\text{var}_{Z_-}(g), \text{var}_{Z_+}(g) \leq \|DT_0^{-1}\|_\infty\). Therefore \(\xi_\epsilon(T_0) = 0\). Also,

\[
\mathcal{L}_0(1)(x) = \sum_{y \in Z \in Z_f, T_0(y) = x} \frac{1}{|DT_0(y)|} = \sum_{Z \in Z_f} \text{Leb}(Z) = 1 - \text{Leb}(H_0).
\]

On the other hand, \(\sup_{x \in Z \in Z_f} \frac{1}{|DT_0(x)|} = \max_{Z \in Z_f} \text{Leb}(Z)\).

In fact, in the piecewise linear, full branched setting, a direct calculation shows that Lebesgue measure is an accim for the open system. For perturbations of these systems, explicit estimates are not generally available. However, we have the following bounds.

**Lemma 3.5.** Let \(T_0 = (\hat{T},H_0)\) be a full-branched map with holes. Then, for every \(\epsilon > 0\), there exists some computable \(m \in \mathbb{N}\) such that \(\xi_\epsilon(T_m) = 0\), where \(T_m := (\hat{T},H_m)\) is obtained from \(T_0\) by enlarging the hole, as in Lemma 2.8. Furthermore,

\[
\rho(T_m) = \rho(T_0) \geq \inf_{x \in I} \sum_{y \in Z \in Z_f, T_0(y) = x} \frac{1}{|DT_0(y)|} =: \rho_0 \quad \text{and} \quad \alpha_\epsilon(T_m) \leq \sup_{x \in Z \in Z_f} \frac{1}{|DT_0(x)|}(2 + \epsilon) =: \alpha_\epsilon,0.
\]

An immediate consequence is the following.
Corollary 3.6. In the setting of Lemma 3.5, if \( \rho_0 > \alpha_{\epsilon, 0} \), then \( H_m \) is \( \epsilon \)-Ulam admissible for \( \hat{T} \). In this case, Lemma 2.8 allows one to approximate the escape rate, accim and quasi-conformal measure for \( T_0 \) via Theorem 3.2 applied to \( T_m \).

Proof of Lemma 3.5. First, let us note that for any map with \( Z_f \neq \emptyset \), we have that \( D_\infty \neq \emptyset \), as the map has at least one fixed point outside the hole. If \( m \) is sufficiently large, each interval \( I \) is either (i) contained in \( H_{m-1} \), and thus not in \( Z^{(m)} \) or (ii) \( T_0^m(Z) = I \) and \( \varphi_Z(\hat{g}1_{X_m}) < \|\hat{g}1_{X_m}\|_\infty (1 + \epsilon) \). In the latter case, \( Z \) is a good interval for \( T_0 \), because \( \mu_0(Z) = \rho_0^m \mu_0(\mathcal{L}_0^m) \geq \rho_0^m \|DT_0^m\|_\infty \mu_0(I) > 0 \). Since good intervals for \( T_0 \) and for \( T_m \) coincide (see beginning of proof of Lemma 2.8), we get that \( \xi(T_m) = 0 \).

Furthermore,

\[
\rho(T) = \rho(T_0)\mu_0(1) = \mu_0(\mathcal{L}_0(1)) \geq \inf_{x \in I} \mathcal{L}_0(1)(x) = \inf_{y \in Z, T_0(y) = \hat{t}} \sum_{y \in Z} \frac{1}{|DT_0(y)|},
\]

The bound on \( \alpha(T_m) \) follows directly from the definition. \( \square \)

The following is an interesting consequence of Lemmas 3.4 and 3.5.

Corollary 3.7. Let \((\hat{T}, H_0)\) be a piecewise linear full-branched map with holes. Assume that \( \text{Leb}(H_0) < 1 - 2\max_{Z \in Z_f} \text{Leb}(Z) \). Then, if \( \epsilon > 0 \) is sufficiently small, \( H_0 \) is \( \epsilon \)-Ulam-admissible for any full-branched map \((\hat{S}, H_0)\) that is a sufficiently small \( C^{1+\text{Lip}} \) perturbation of \((\hat{T}, H_0)\) (where the \( C^{1+\text{Lip}} \) topology is defined, for example, by the norm given by the maximum of the \( C^{1+\text{Lip}} \) norms of each branch). In particular, Theorem 3.2 applies.

Proof. The statement for \((\hat{T}, H_0)\) follows from Remark 3.4. For perturbations, the statement follows from Lemma 3.5, by observing that the quantities \( \rho_0 \) and \( \alpha_{\epsilon, 0} \), as well as the variation of \( 1/|D\hat{T}| \) on each interval depend continuously on \( \hat{T} \), with respect to the \( C^{1+\text{Lip}} \) topology. \( \square \)

Corollary 3.7 applies to examples of maps with arbitrarily large holes for which the Ulam method provides a good approximation of accims, quasi-conformal measures and escape rates.

Example 3.8 (Arbitrarily large holes). Let \( \delta > 0 \), \( H_0 = [\delta, 1 - \delta] \), and

\[
T_\delta(x) = \begin{cases} 
\frac{2}{5}x & \text{if } x < \frac{\delta}{2}, \\
1 - \frac{2}{5}(x - \frac{\delta}{2}) & \text{if } \frac{\delta}{2} \leq x < \delta, \\
\frac{2}{5}(x - 1 + \delta) & \text{if } 1 - \delta \leq x < 1 - \frac{\delta}{2}, \\
1 - \frac{2}{5}(x - 1) & \text{if } 1 - \frac{\delta}{2} \leq x < 1.
\end{cases}
\]

Then, \( \text{Leb}(H_0) = 1 - 2\delta < 1 - 1 = 1 - 2\max_{Z \in Z_f} \text{Leb}(Z) \) and the hypotheses of Corollary 3.7 are satisfied. Thus, Ulam’s method converges for sufficiently small \( C^{1+\text{Lip}} \) perturbations of \( T_\delta \) that are full-branched.

Other examples of this type may be found in [1] and [2]. Bahsoun established rigorous computable bounds for the errors in the Ulam method, which allowed him to find rigorous bounds on the escape rate for open Lasota-Yorke maps. Bose and Bahsoun related the escape rate to the Lebesgue measure of the hole. Both results rely on the existence of Lasota-Yorke type inequalities, relating \( BV \) and \( L^1(m) \) norms. Such inequalities may be obtained by exploiting the full-branched structure of the map.
Lemma 3.10. Let example is closely related to \[17, 6.2 \& 6.3\]. That Ulam’s method provides rigorous approximations in specific systems. The following objects of interest (escape rates, accims and quasi-conformal measures) exactly. We show that when non-full branches are present, the dynamics is typically non-Markovian. Thus, nearly piecewise linear maps with enough full branches.

3.3.2. Nearly piecewise linear maps with enough full branches. When non-full branches are present, the dynamics is typically non-Markovian. Thus, even in the piecewise linear setting there may not be direct ways to find the various objects of interest (escape rates, accims and quasi-conformal measures) exactly. We show that Ulam’s method provides rigorous approximations in specific systems. The following example is closely related to \[17, 6.2 \& 6.3\].

Lemma 3.10. Let \(\hat{T}, H_0\) be a piecewise linear Lasota-Yorke map with holes, and assume \(Z_f \neq \emptyset\). Let \(c_u\) be the maximum number of contiguous elements in \(Z_u\). If \(\|DT^{-1}\|_{\infty}(3 + c_u) < \rho\), then \(H_0\) is \((1+\epsilon)\)-Ulam-admissible for \(\hat{T}\), for every \(\epsilon > 0\) sufficiently small. Thus, the hypotheses of Theorem 3.2 are satisfied.

Proof. For any map with \(Z_f \neq \emptyset\), we have that \(D_\infty \neq \emptyset\), as the map has at least one fixed point outside the hole. Furthermore, for each \(Z \in \mathcal{Z}\), one has that \(\text{var}_Z(g) \leq \|g\|_{\infty}\), so \(Z\) is a \((1 + \epsilon)\)-adequate partition for \(T\). Also, it follows from the definition of \(Z_g\) that \(Z_f \subseteq Z_g\). Thus, \(Z_h \subseteq Z_{\hat{u}}\), and \(\xi_{1+\epsilon} \leq c_u\). Therefore, \(\alpha_\epsilon \leq \|DT^{-1}\|_{\infty}(3 + \epsilon + c_u) < \rho\), provided \(\epsilon > 0\) is sufficiently small. \(\square\)

A concrete example where the previous lemma applies is that of \(\beta\)-shifts.

Example 3.11. Let \(\beta > 1\), and \(\hat{T}_\beta\) be the \(\beta\)-shift, \(\hat{T}_\beta(x) = \beta x \mod 1\). Let \(H_0 \subseteq I\) be a finite union of closed intervals, and let \(f\) be the number of full branches of \(\hat{T}_\beta\) outside \(H_0\). Then, for the open system (\(\hat{T}_\beta, H_0\)), we have that \(\rho \geq \frac{1}{\beta}\). Then, the hypotheses of Lemma 3.10 are satisfied, provided \(f > 3 + c_u\). This happens, for example, when \(\beta \geq 5\) and \(H_0\) is a single interval of the form \([\frac{\beta}{\beta}, y]\) or \([y, 1]\), with \(\frac{\beta}{\beta} < y < 1\). Also, when \(\beta \geq 6\) and \(H_0\) is a single interval contained in \([\frac{\beta}{\beta}, 1]\); or when \(\beta \geq 7\) and \(H_0\) is any interval leaving at least 7 full branches in \(X_0\) (recall from Subsection 2.1 that two bad elements of \(Z_u\) are contiguous if there are no good elements of \(Z_f \cup Z_u\) between them, but there may be elements of \(Z_h\) in between).

We include Figures 1-6, obtained from numerical experiments for \(\beta = 5, 9\), and two different choices of holes. They include approximations to the densities of accims and cumulative distribution functions of the quasi-conformal measures for systems with a hole, as well as the acim and conformal measure for the closed system.

Remark 3.12. Using lower bounds on \(\rho\) such as those of Lemma 3.5, one can extend the conclusion of Lemma 3.10 as in Corollary 3.7, to cover small \(C^{1+\text{Lip}}\) perturbations of piecewise linear maps that respect the partition \(\hat{Z}_h \cup \hat{Z}_f \cup \hat{Z}_u\).
3.3.3. Admissibility vs. Ulam-admissibility.

We establish a relation between admissibility and Ulam-admissibility of holes. This ensures that several of the examples in the literature can be treated with our method; in particular, all the examples presented in [17].
Figure 4. Plot of $\mu_k([0,x])$ vs. $x$, $k = 10000$ for $\hat{T}_\beta$, $\beta = 5.9$, $H_0 = [0.8476,1]$ (shown in red). Note that $\mu_k$ is not supported on $H_0$.

Figure 5. Graph of $h_k$, $k = 10000$ for $\hat{T}_\beta$, $\beta = 5.9$, $H_0 = [0.9001,1]$ (shown in red). The computed value of $\rho_k$ is 0.908501 to six decimal places.

Figure 6. Plot of $\mu_k([0,x])$ vs. $x$, $k = 10000$ for $\hat{T}_\beta$, $\beta = 5.9$, $H_0 = [0.9001,1]$ (shown in red). Note that $\mu_k$ is not supported on $H_0$.

**Lemma 3.13** (Admissibility and Ulam-admissibility). If $H_0$ is an $\epsilon$-admissible hole for $\hat{T}$, there is some $n \in \mathbb{N}$ such that $H_{n-1} := I \setminus X_{n-1}$ is $\epsilon$-Ulam-admissible for $T^n$.

**Proof.** Assume $H_0$ is an $\epsilon$-admissible hole for $\hat{T}$. Then, $T^n := (\hat{T}^n,H_{n-1})$ is an open Lasota-Yorke map. Fix $\Theta < \eta < \rho$ so that for all $n$ sufficiently large,

$$\exp(\frac{1}{n} \log \|(DT^n)^{-1}\|_\infty) \exp(\frac{1}{n} \log(1 + \xi_{\epsilon,n})) < \eta.$$ 

Then, $\|(DT^n)^{-1}\|_\infty \xi_{\epsilon,n} < \eta^n$. By possibly making $n$ larger, we can assume that $(2 + \epsilon)\|(DT^n)^{-1}\|_\infty < \eta^n$, and that $2\eta^n < \rho^n$. Then, $\|(DT^n)^{-1}\|_\infty(2 + \epsilon + \xi_{\epsilon,n}) < \rho^n$. 

12
We remark that \( \xi_c(T^n) = \xi_{c,n}(T) \). Thus \( \alpha_c(T^n) = \| (DT^n)^{-1} \|_\infty (2 + \epsilon + \xi_{c,n}) \). Furthermore, in view of Theorem 2.6, \( \rho(T^n) = \lim_{m \to \infty} \inf_{x \in D_m} \frac{\mathcal{L}^{(m+1)}(x)}{\mathcal{Z}^{(m)}(x)} = \mu(\mathcal{L}^1) = \rho^n \). \( \square \)

4. Proofs

4.1. Proof of Lemma 2.9. Let \( \mathcal{Z} \) be the monotonicity partition for \( \hat{T} \). Define \( \hat{g} : I \to \mathbb{R} \) by \( \hat{g}(x) = |D\hat{T}(x)|^{-1} \) for every \( x \in \left( I \setminus \bigcup_{Z \in \mathcal{Z}} \partial Z \right) \cup \{0,1\} \), and \( \hat{g}(x) = 0 \) otherwise. We obtain the following Lasota-Yorke inequality by adapting the approach of Rychlik [21, Lemmas 4-6]. Let \( \mathcal{Z}_c \in \mathcal{G}_c \). Then,

\[
\var(\hat{L}f) \leq \var(\hat{g} \hat{g}) \leq (2 + \epsilon)\|D\hat{T}^{-1}\|_\infty \var(f) + \|D\hat{T}^{-1}\|_\infty (1 + \epsilon) \sum_{A \in \mathcal{Z}_c} \inf_A |f|.
\]

We slightly modify \( \hat{g} \) to account for the jumps at the hole \( H_0 \), and define \( g : I \to \mathbb{R} \) by \( g = 1_{X_0} \hat{g} \). Now, only elements of \( \mathcal{Z}_c^* \) contribute to the variation of \( \hat{L}f \), and we get

\[
\var(\hat{L}f) = \var(\hat{L}(1_{X_0} g)) \leq \var(f(1_{X_0} \hat{g})) = \sum_{A \in \mathcal{Z}_c^*} \var(f(1_{X_0} \hat{g}))
\]

\[
\leq \sum_{A \in \mathcal{Z}_c^*} \var(f) \|1_{X_0} \hat{g}\|_\infty + \|1_A f\|_\infty \var(1_{X_0} \hat{g})
\]

\[
\leq \sum_{A \in \mathcal{Z}_c^*} \var(f) \|D\hat{T}^{-1}\|_\infty + \left( \inf_A |f| + \var(f) \right) \var(g).
\]

Thus, since for every \( A \in \mathcal{Z}_c^* \), \( \var_A(g) \leq \|D\hat{T}^{-1}\|_\infty (1 + \epsilon) \), one has that

\[
(6) \quad \var(\hat{L}f) \leq (2 + \epsilon)\|D\hat{T}^{-1}\|_\infty \var(f) + \sum_{A \in \mathcal{Z}_c^*} \|D\hat{T}^{-1}\|_\infty (1 + \epsilon) \inf_A |f|.
\]

Now we proceed as in the proof of [17, Lemma 2.5], and observe that there exists \( \delta > 0 \) such that if \( A \in \mathcal{Z}_{c,g} \), then

\[
(7) \quad \inf_A |f| \leq \delta^{-1} \mu(1_A |f|),
\]

whereas if \( A \in \mathcal{Z}_{c,b} \), we let \( A' \in \mathcal{Z}_{c,g} \) be the nearest good partition element\(^4\), and get

\[
\inf_A |f| \leq \inf_{A'} |f| + \var_{I(A,A')} (f),
\]

where \( I(A,A') \) is an interval that contains \( A \) and has as an endpoint \( x_{A'} \in A' \), fixed in advance, such that, after possibly redefining \( f \) at the discontinuity points of \( f \), \( |f(x_{A'})| = \inf_{A'} |f| \). Notice that either \( I(A,A') \subseteq I_-(A') \) or \( I(A,A') \subseteq I_+(A') \), where \( I_+(A') \) is the union of \( A'_+ := A' \cap \{ x : x \geq x_{A'} \} \) with the contiguous elements of \( \mathcal{Z}_{c,b} \) on the right of \( A' \), and \( I_-(A') \) is defined in a similar manner. Thus,

\[
(8) \quad \sum_{A \in \mathcal{Z}_{c,b}} \inf_A |f| \leq \xi_c \var(f) + 2\xi_c \sum_{A' \in \mathcal{Z}_{c,g}} \inf_{A'} |f|,
\]

where the factor 2 appears due to the fact that a single good interval could have at most \( \xi_c \) bad intervals on the left and \( \xi_c \) bad intervals on the right. Combining equations (7) and (8), we get

\[
\sum_{A \in \mathcal{Z}_c^*} \inf_A |f| \leq \xi_c \var(f) + \delta^{-1}(1 + 2\xi_c) \sum_{A' \in \mathcal{Z}_{c,g}} \mu(1_{A'} |f|).
\]

\(^4\)It is shown in [17, Lemma 2.4] that whenever \((T,H_0)\) is an open system with an admissible hole, then \( \mathcal{Z}_{c,g} \neq \emptyset \).
Plugging back into (6), we get
\[
\text{var}(\mathcal{L}f) \leq \|DT^{-1}\|_{\infty}(2 + \epsilon + \xi_\epsilon) \text{var}(f) + \|DT^{-1}\|_{\infty}(1 + \epsilon)\delta^{-1}(1 + 2\xi_\epsilon)\mu(|f|).
\]

We get the first part of the lemma by choosing \(K_\epsilon = \|DT^{-1}\|_{\infty}(1 + \epsilon)\delta^{-1}(1 + 2\xi_\epsilon)\). For the second part, we recall that \(\mu(\mathcal{L}f) = \rho \mu(f)\), so for every \(f \in \mathcal{C}_a\), we have that
\[
\frac{\text{var}(\mathcal{L}f)}{\mu(\mathcal{L}f)} \leq \frac{\alpha_\epsilon a + K_\epsilon}{\rho}.
\]

Thus, \(\mathcal{L}f \in \mathcal{C}_a\), provided \(a > \frac{K_\epsilon}{\rho - \alpha_\epsilon} =: a_1\).

4.2. **Proof of Lemma 2.8.** Let \(\mathcal{L}_m\) be the transfer operator associated to \(T_m\). That is, \(\mathcal{L}_m(f) = \hat{\mathcal{L}}(1_{X_m} f)\). Then, \(\mathcal{L}^n_m(f) = \hat{\mathcal{L}}^n(1_{X_{m+n}} f)\), and therefore,
\[
(9) \quad \hat{\mathcal{L}}^m \circ \mathcal{L}_m^n = \mathcal{L}^{m+n}_m.
\]

Hence, an interval is good for \(T_0\) if and only if it is good for \(T_m\) for every \(m\). In the rest of this proof we will say an interval is good if it is good for either (and therefore all) \(T_m\).

Let \(Z_0 = Z \cup H_0\), where \(H_0\) is the partition of \(H_0\) into intervals, and we recall that \(Z\) is the monotonicity partition of \(\hat{T}\). Let \(\mathcal{G}_\epsilon\) be an \(\epsilon\)-adequate partition for \(T_0\). Then, a partition \(\mathcal{G}_{\epsilon,m}\) may be constructed by cutting each element of \(\mathcal{G}_\epsilon \cup Z_0^{(m)}\) in at most \(K\) pieces, where \(K\) is independent of \(m\), in such a way that the variation requirement \(\text{max}_{Z \in \mathcal{G}_{\epsilon,m}} \text{var}_Z(\hat{g}1_{X_m}) \leq \|DT^{-1}\|_{\infty}(1 + \epsilon)\) is satisfied, and thus \(\mathcal{G}_{\epsilon,m}\) is an \(\epsilon\)-adequate partition for \(T_m\). Indeed, \(K = 2 + \left\lceil \|\hat{g}\|_\infty / \text{essinf}(\hat{g}) \right\rceil\) is a possible choice. The term 2 allows one to account for possible jumps at the boundary points of \(H_m\), as there are at most two of them in each \(Z \in \mathcal{G}_\epsilon \cup Z_0^{(m)}\).

The term \(M = \left\lceil \|\hat{g}\|_\infty / \text{essinf}(\hat{g}) \right\rceil\) allows one to split each interval \(Z \in \mathcal{G}_\epsilon \cup Z_0^{(m)}\) into at most \(M\) subintervals \(Z_1, \ldots, Z_M\), in such a way that for every \(1 \leq j \leq M\), \(\text{var}_{\text{int}(Z_j)}(\hat{g}1_{X_m}) \leq (1 + \epsilon)\|\hat{g}1_{X_m}\|_\infty\). The chosen value of \(M\) is necessary to account for the possible discrepancy between \(\|\hat{g}1_{X_0}\|_\infty\) and \(\|\hat{g}1_{X_m}\|_\infty\).

(Recall also that \(\hat{g}\) is continuous on each \(\text{int}(Z_j)\).)

Now, let \(b = \#Z_0\). Then, each bad interval of \(\mathcal{G}_\epsilon\) gives rise to at most \(Kb^{m}\) (necessarily bad) intervals in \(\mathcal{G}_{\epsilon,m}\). When a good interval of \(\mathcal{G}_\epsilon\) is split, it also gives rise to at most \(Kb^{m}\) intervals in \(\mathcal{G}_{\epsilon,m}\). In this case some of the intervals may be bad, but it is guaranteed that at least one of them remains good, as being good is equivalent to having non-zero \(\mu_0\) measure. Thus, the number of contiguous bad intervals in \(\mathcal{G}_{\epsilon,m}\) is at most \(Kb^{m}(B+2)\), where \(B\) is the number of contiguous bad intervals in \(\mathcal{G}_\epsilon\). Therefore, \(\xi_{\epsilon}(T_m) = \exp\left(\limsup_{n \to \infty} \frac{1}{n} \log(1 + \xi_{\epsilon,n}(T_m))\right) \leq \xi_{\epsilon}(T_0)\).

Clearly, \(\hat{\Theta}(T_m) \leq \hat{\Theta}(T_0)\). Finally, we will show that \(\rho(T_0) \leq \rho(T_m)\). Recall that \(\rho_j\) is the leading eigenvalue of \(\mathcal{L}_j\). Let \(f \in BV\) be nonzero and such that \(\mathcal{L}_0 f = \rho_0 f\).

We claim that \(\mathcal{L}_m(1_{X_{m-1}} f) = \rho_0 1_{X_{m-1}} f\), which yields the inequality, because necessarily \(1_{X_{m-1}} f \neq 0\). Indeed,
\[
\rho_0 1_{X_{m-1}} f = 1_{X_{m-1}} \mathcal{L}_0 f = 1_{X_{m-1}} \mathcal{L}_m f = \mathcal{L}_m f = \mathcal{L}_m(1_{X_{m-1}} f),
\]
where the second equality follows from the fact that \(\mathcal{L}_0(1_{H_m})\) is supported on \(T(H_m) = H_{m-1}\). The third one, from the fact that \(\mathcal{L}_m f\) is supported on \(T(X_m) \subseteq X_{m-1}\). The last one, because \(\mathcal{L}_m(1_{H_{m-1}} f) = 0\).

The first statement of the lemma follows. The relations between escape rates, accims and quasi-conformal measures follow from comparing via Equation (9) the statements of part (4) of Theorem 2.6 applied to \(T_0\) and \(T_m\).
4.3. Proof of the main result. In this section, we present the proof of Theorem 3.2. We begin with a few auxiliary lemmas.

**Lemma 4.1.** \( \lim_{k \to \infty} \max_{J \in P_k} \mu(1_J) = 0. \)

**Proof.** Lemma 2.10(1) implies that for every \( \epsilon > 0 \), there exists \( m_0 \in \mathbb{N} \) such that

\[
\max_{Z \in Z^{(m_0)}} \mu(1_Z) \leq \frac{\epsilon}{2}.
\]

Furthermore, if \( k \) is sufficiently large, each \( J \in P_k \) is contained in the union of at most two intervals in \( Z^{(m_0)} \). Therefore,

\[
\mu(1_J) \leq 2 \max_{Z \in Z^{(m_0)}} \mu(1_Z) \leq \epsilon.
\]

\[\square\]

**Lemma 4.2.** For each \( \epsilon > 0 \) and \( a > 0 \) there exists \( k_0 \in \mathbb{N} \) such that for every \( k \geq k_0 \),

\[
\pi_k C_a \subset C_{a+\epsilon}.
\]

**Proof.** For each \( k \in \mathbb{N} \), each \( J \in P_k \), and each \( x \in J \), we have that

\[
|f(x) - \pi_k f(x)| \leq \sum_{J \in P_k} (\text{var} f) 1_J(x).
\]

Therefore,

\[
\mu(|f - \pi_k f|) \leq \left( \sum_{J \in P_k} \text{var} f \right) \max_{J \in P_k} \mu(1_J) \leq \text{var}(f) \max_{J \in P_k} \mu(1_J).
\]

Hence, for \( f \in C_a \),

\[
\mu(\pi_k f) \geq \mu(f) - \mu(|f - \pi_k f|) \geq \text{var}(f) \left( \frac{1}{a} - \max_{J \in P_k} \mu(1_J) \right) \geq \frac{\text{var}(f)}{a + \epsilon}
\]

for all large enough \( k \) by Lemma 4.1. The conclusion follows because \( \pi_k \) does not increase variation. \[\square\]

**Lemma 4.3.** For every \( a > a_1 \) (from Lemma 2.9) there is a \( k_0 \in \mathbb{N} \) (depending on \( a \)) with the property that for every \( k \geq k_0 \), the Ulam scheme preserves the cone \( C_a \), i.e. \( L_k C_a \subset C_a \).

**Proof.** By Lemma 2.9, there is an \( \epsilon > 0 \) for which \( \rho_k \subset C_a \epsilon \).

The conclusion now follows from Lemma 4.2 by noting that \( L_k = \pi_k \circ L \). \[\square\]

**Lemma 4.4.** If \( a > 0 \), \( f \in C_a \) and \( \mu(f) = 0 \) then \( f = 0 \).

**Proof.** Since \( \text{var}(f) \leq a \mu(f) = 0 \), \( f = c \) for some \( c \in \mathbb{R} \). Since \( \mu \) is a probability measure, \( c = \mu(f) = 0 \). \[\square\]

**Lemma 4.5.** Let \( \mu^* \) be a weak-* limit point of \( \{\mu_k\}_{k \in \mathbb{N}} \), say along a subsequence \( \{k_j\}_{j \in \mathbb{N}} \), and suppose \( \liminf_{j \to \infty} \rho_{k_j} \geq \rho \). Then \( \mu^* \) is non-atomic and \( L^* \mu_{k_j} \xrightarrow{\text{weak-*}} L^* \mu^* \).

The proof of Lemma 4.5 is deferred until §4.4.
4.3.1. Proof of Theorem 3.2(II). Fix \( a > \max\{a_0, a_1\} \), where \( a_0 \) is as in Lemma 2.10, and \( a_1 \) is as in Lemmas 4.3 and 2.9. The Ulam scheme introduced in Section 3.1 provides a sequence of (Lebesgue normalized) approximations \( \{h_k\}_{k \in \mathbb{N}} \) to \( h \). Let \( q_k = (m(I_1), \ldots, m(I_k)) \). For each \( k \), the set of vectors \( p = ([p]_1, \ldots, [p]_k) \) satisfying

\[
\Delta_k = \left\{ p \geq 0 : \sum_{i=1}^{k} [p]_i x_i \in C_a, p^T q_k = 1 \right\}
\]

is compact, convex and non-empty, since \( 1 \in C_a \) for every \( a > 0 \). The map \( p \mapsto p^T p_k / (p^T q_k) \) maps \( \Delta_k \) to itself and it is continuous, in view of the following.

**Sublemma 4.6.** If \( p \in C_a \setminus \{0\} \) then \( p^T p_k q_k > 0 \).

**Proof.** By matrix multiplication, \( [P_k q_k]_j = m(I_j) \cap X_0 \cap \hat{T}^{-1} I \). Thus \( p^T p_k q_k = 0 \) only if \( \text{supp}(\sum_{i} [p]_i X_i) \) is disjoint from \( X_0 \cap \hat{T}^{-1} I \). But \( \text{supp}(\mu) \subseteq X_0 \cap \hat{T}^{-1} I \), so this possibility would imply \( \mu(\sum_{i} [p]_i X_i) = 0 \), contradiction to membership of \( C_a \).

Thus, by Brouwer’s fixed point theorem, there is a \( p_k \in \Delta_k \) such that \( p_k = p_k^T p_k q_k = p_k^T q_k \).

For each \( k \geq k_0 \), let \( h_k \in C_a \) be obtained from this left eigenvector of the Ulam matrix \( P_k \). We point out that the facts that the eigenvalue \( \tilde{\rho}_k \) associated to \( h_k \) is the leading eigenvalue of \( P_k \), and that this eigenvector is unique for sufficiently large \( k \) may not be evident at this point. However, these facts are not used until they are established in §4.3.3.

Now we show that \( \lim_{k \to \infty} h_k = h \) in \( L^1(m) \). Let \( \rho_* \) be a limit point of \( \{\tilde{\rho}_k\}_{n \in \mathbb{N}} \). Since \( \|h_k\|_1 = 1 \), Lemma 2.10(2) yields that \( \sup \mu(h_k) < \infty \), so \( \{\text{var}(h_k)\}_{k \in \mathbb{N}} \) is uniformly bounded, and so is \( \{h_k\}_{k \in \mathbb{N}} \). By Helly’s selection theorem, there exists a subsequence \( k_j \) and \( h_* \in BV \) such that \( \tilde{\rho}_{k_j} \to \rho_* \), and \( h_{k_j} \to h_* \), where the last convergence is both pointwise and in \( L^1(m) \). In particular \( \|h_*\|_1 = 1 \) and \( Lh_* = \rho_* h_* \). Indeed,

\[
\|L(h_* - h_{k_j})\|_1 \leq \|L(h_* - h_{k_j})\|_1 + \|(I - \pi_{k_j})Lh_{k_j}\|_1 \to 0, \tag{10}
\]

where we have used again that \( \{\text{var}(h_k)\}_{k \in \mathbb{N}} \) is uniformly bounded to control the last term.

The Lebesgue dominated convergence theorem implies that \( \mu(h_n) = \lim_{j \to \infty} \mu(h_{k_j}) \). We claim that \( \mu(h_*) > 0 \). Otherwise, \( \lim_{j \to \infty} \mu(h_{k_j}) = 0 \), and since \( h_{k_j} \in C_a \), we would also have that \( \lim_{j \to \infty} \text{var}(h_{k_j}) = 0 \). In such a case, there would be a constant \( c \in \mathbb{R} \), such that \( h_{k_j} \to h_* = c \) in \( L^\infty(m) \). But since \( \|h_*\|_1 = 1 \), it would be necessarily the case that \( c = 1 \). Thus, \( \mu(h_{k_j}) \to \mu(h_*) = 1 \), contradicting the assumption.

Since both \( h_* \neq 0 \) and \( \mu(h_*) \neq 0 \), part (4) of Theorem 2.6 implies \( \rho_* = \rho \). So by uniqueness of the accim for the open system, \( h_* = h \). Therefore, \( \lim_{n \to \infty} h_k = h \).

4.3.2. Proof of Theorem 3.2(III). For each \( k \), let \( \mu_k \) be the probability measure induced by a non-negative right eigenvector of \( P_k \) of leading eigenvalue \( \rho_k \). We will show that \( \lim_{k \to \infty} \mu_k = \mu \) in the weak-* topology of measures.

First, since \( \rho_k \) is the leading eigenvalue, \( \rho_k \geq \tilde{\rho}_k \), where each \( \tilde{\rho}_k \) is as in §4.3.1, and hence \( \liminf_{k \to \infty} \rho_k \geq \rho \) (since \( \rho = \lim \tilde{\rho}_k \) by §4.3.1). Without loss of generality assume that \( \rho^* = \lim_{k \to \infty} \rho_k \) and let \( \mu^* \) be any weak-* limit of the sequence \( \{\mu_k\} \).

We will show that \( \mu^*(Lf) = \rho^* \mu^*(f) \) for all continuous \( f \). Applying Theorem 2.6(4) with \( f = 1 \), gives \( (\rho^* / \rho)^n \to \mu^*(h) \mu(1) \), so that \( \rho = \rho^* \). An approximation argument (using the non-atomicity of \( \mu^* \)-Lemma 4.5) then shows that \( \mu^*(Lf) = \rho^* \mu^*(f) = \rho \mu^*(f) \).
for all \( f \in BV \). Since \( \mu \) is the unique probability measure with that property, the proof will be complete. To this end, let \( f \) be continuous. Then,

\[
\mu^*(\mathcal{L}f) = \lim_{k \to \infty} \mu_k(\mathcal{L}f) = \lim_{k \to \infty} \mu_k(\mathcal{L}_k f) = \lim_{k \to \infty} \mu_k(\mathcal{L}_k \pi_k f) + \lim_{k \to \infty} \mu_k(\mathcal{L}_k (f - \pi_k f)),
\]

where the first equality relies on \( \mu^* \) being atom-free. The second equality holds because \( \mu_k \) is uniform on each interval of \( P_k \), thus \( \mu_k(\phi) = \mu_k(\pi_k \phi) \) for every continuous function \( \phi \); we also recall that \( \mathcal{L}_k = \pi_k \mathcal{L} \). For term (I), \( \mu_k(\mathcal{L}_k \pi_k f) = \rho_k \mu_k(\mathcal{L} f) \to \rho^* \mu^*(f) \), whereas for (II), note that since \( f \) is uniformly continuous, \( \|f - \pi_k f\|_\infty \to 0 \) as \( k \to \infty \). Since \( \|\mathcal{L} \phi\|_\infty \leq \#\text{branches of } T \times \|g\|_\infty \|\phi\|_\infty \), we have (II) \( \to 0 \).

4.3.3. Proof of Theorem 3.2(I). In this section we show that for sufficiently large \( k \), the density \( h_k \in \mathcal{C}_a \) selected by Brouwer’s theorem in §4.3.1 corresponds to the leading eigenvalue of \( P_k \) (Lemma 4.8). Furthermore, we show that this eigenvalue is simple for sufficiently large \( k \) (Lemma 4.9). We start with an auxiliary lemma.

Lemma 4.7. There exists \( n > 0 \) such that \((P_k^n)_{ij} > 0 \) for all \( i, j \) satisfying \( \mu(I_i) > 0 \) and \( \int_{I_j} h \, dm > 0 \).

Proof. Fix \( i, j \) satisfying the hypotheses. By Theorem 2.6, \( \|(\mathcal{L}^{n_i} \chi_i)/\rho^n - \mu(I_i)h\|_\infty \to 0 \) as \( n \to \infty \). Choose \( n_{ij} \) large enough so that \( \int_{I_j} \mathcal{L}^{n_{ij}} \chi_i \, dm > 0 \). Because there are a finite number of \( I_i \) and \( I_j \) we can put \( n = \max_{i,j} n_{ij} \) and obtain \( \int_{I_j} \mathcal{L}^{n_{i}} \chi_i \, dm > 0 \) for all \( i, j \) satisfying the hypotheses. Note that this implies \( \int_{I_j} (\pi_k \mathcal{L})^{n} \chi_i \, dm > 0 \) because the support of the integrand is possibly enlarged by taking Ulam projections. This now implies \((P_k^n)_{ij} > 0 \). \( \square \)

Lemma 4.8. For large enough \( k \), \( \hat{\rho}_k \), the eigenvalue of \( h_k \in \mathcal{C}_a \), is the leading eigenvalue \( \rho_k \) of \( P_k \).

Proof. Since \( \mu \) is non-atomic, \( \mu(f) \) is defined for every \( f \in BV \) and in fact, \( \mu_k(f) \to \mu(f) \) for every \( f \in BV \), where \( \mu_k \) is the probability measure corresponding to a leading right eigenvector of \( P_k \). (The proof of this follows from an approximation argument, similar to that in the last paragraph of §4.4.)

In particular, since \( \mu(h) > 0 \), we have that \( \mu_k(h) > 0 \) for every sufficiently large \( k \). For any such \( k \), the vector \( \psi_k \) with components \( [\psi_k]_l = \mu_k(I_l)/m(I_l), \ l = 1, \ldots, k \), satisfies \( P_k^n \psi_k = \rho_k^n \psi_k \) for \( n \geq 1 \). Let \( I_j \) be a cell in the \( k \)th Ulam partition such that \( \int_{I_j} h \, dm > 0 \); hence \( \int_{I_j} h \, dm > 0 \) and \( \mu_k(I_j) > 0 \). By Lemma 4.7, if \( \mu(I_j) > 0 \) there is an \( n \) such that \((P_k^n)_{ij} > 0 \). Hence, selecting such an \( i \),

\[
\mu_k(I_i) = m(I_i)[\psi_k]_i = m(I_i)\rho_k^{-n}[P_k^n \psi_k]_i \\
\geq m(I_i)\rho_k^{-n}(P_k^n)_{ij}[\psi_k]_j = \frac{m(I_i)}{m(I_j)}\rho_k^{-n}(P_k^n)_{ij}\mu_k(I_j) > 0.
\]

Hence \( \text{supp}(\mu) \subseteq \text{supp}(\mu_k) \). Since \( h_k \neq 0 \) and \( h_k \in \mathcal{C}_a \), \( \mu(h_k) > 0 \) and therefore \( \mu_k(h_k) > 0 \) (recall that \( h_k \) is piecewise constant). Thus,

\[
\hat{\rho}_k \mu_k(h_k) = \mu_k(\hat{\rho}_k h_k) = \mu_k(\mathcal{L} h_k) = \rho_k \mu_k(h_k),
\]

and the claim follows, since \( \rho_k \) is the largest eigenvalue of \( P_k \) by construction. \( \square \)

Lemma 4.9. For sufficiently large \( k \), \( \rho_k \) is a simple eigenvalue for \( P_k \).
Proof. Let $B_k = \bigcup_{\{p_k: \bar{p}^T_k \geq 0, \bar{p}_k = p_k \bar{p}_k \}} \text{supp}(p_k)$, and $C_k = \bigcap_{\{\psi_k: \psi_k \geq 0, p_k \psi_k = \rho_k \psi_k\}} \text{supp}(\psi_k)$. We know from the proof of Lemma 4.8 that for large enough $k$, supp($\mu$) $\subseteq$ supp($\psi_k$) for all leading non-negative right eigenvectors $\psi_k$ of $P_k$. Hence, supp($\mu$) $\subseteq$ $C_k$.

Let $p_k$ be a non-negative left eigenvector for $P_k$ of eigenvalue $\rho_k$ and maximal support, i.e. supp($p_k$) = $B_k$. Let $B_k^c$ be the complement of $B_k$. Let us note that $B_k^c$ is invariant under $P_k$ in the sense that if $\gamma \in \mathbb{R}^k$ is such that supp($\gamma$) $\subseteq$ $B_k^c$, then supp($P_k \gamma$) $\subseteq$ $B_k^c$. Indeed, $0 = p_k^T \gamma = \rho_k^{-1} p_k^T P_k \gamma$.

The rest of the proof consists of three steps.

**Step 1.** The modulus of the leading eigenvalue of $P_k|_{B_k^c}$ is strictly less than $\rho_k$.

Proof. We know it is at most $\rho_k$. Assume it is equal to $\rho_k$. Then, the Perron-Frobenius theorem ensures the existence of a non-negative $\gamma \in \mathbb{R}^d$, with supp($\gamma$) $\subseteq$ $B_k^c$ and $P_k \gamma = \rho_k \gamma$. Since supp($\mu$) $\subseteq$ $C_k$ $\subseteq$ supp($\gamma$) $\subseteq$ $B_k^c$, then $\mu(h_k) = 0$ for all $h_k$ associated to a non-negative left eigenvector of $P_k$ of eigenvalue $\rho_k$. In such a case, no $h_k$ could belong to a cone $C_a$, contradicting what was established in §4.3.1.

**Step 2.** $P_k$ has no non-trivial Jordan blocks of eigenvalue $\rho_k$.

Proof. Assume the size of the largest Jordan block for $\rho_k$ is $J > 1$. Then the sequence of matrices $\left\{ \frac{1}{n^J} \sum_{j=0}^{n-1} \rho_k^{-n} P_k \right\}_{n \in \mathbb{N}}$ has a non-zero, non-negative limit, say $A_k$. Since lim$_{\gamma \to \infty} \frac{1}{n^J} \sum_{j=0}^{n-1} \rho_k^{-n} P_k \gamma = \lim_{\gamma \to \infty} \frac{1}{n^J} p_k^T A_k = 0$, we have that $p_k^T A_k = 0$. Since $p_k$ is a non-negative eigenvector of $P_k$ of maximal support and $A_k$ is non-negative, then $p^T A_k = 0$ for every $p$ such that supp($p$) $\subseteq$ $B_k$. On the other hand, since the leading eigenvalue of $P_k|_{B_k^c}$ is strictly less than $\rho_k$, one can also verify that $q^T A_k = 0$ for every $q$ such that supp($q$) $\subseteq$ $B_k^c$. Hence $A_k = 0$, which is a contradiction.

**Step 3.** The geometric multiplicity of $\rho_k$ is one.

This step is further subdivided into two parts.

- If the geometric multiplicity of $\rho_k$ is greater than one, then there exist non-negative right eigenvectors $\psi_k, \psi'_k$ such that supp($\psi_k$) $\cap$ supp($\psi'_k$) $\cap$ $B_k = \emptyset$.

Proof. Let $\gamma, \gamma'$ be two linearly independent right eigenvectors for $P_k$ of eigenvalue $\rho_k$. Let

$$\phi = (\gamma - \gamma')^+ 1_{B_k}, \quad \text{and} \quad \phi' = (\gamma - \gamma')^- 1_{B_k},$$

where $f^+ = \max\{f, 0\}$, and $f^- = -\min\{f, 0\}$. Then, supp($\phi$) $\cap$ supp($\phi'$) $\emptyset$. We note that $(\gamma - \gamma')^+ 1_{B_k} \neq 0$; otherwise $\gamma - \gamma'$ would be an eigenvector of $P_k$ of eigenvalue $\rho_k$ supported on $B_k^c$, contradicting Step 1. In fact, we may assume that neither $\phi$ nor $\phi'$ are identically zero; this amounts to possibly rescaling $\gamma$ or $\gamma'$.

Then,

$$\begin{align*}
  p_k^T \phi &= p_k^T \phi' = p_k^T (\phi - \phi') = p_k^T (\gamma - \gamma') = \rho_k^{-1} p_k^T P_k (\gamma - \gamma') \\
  &= \rho_k^{-1} p_k^T P_k (\phi - \phi') \leq \rho_k^{-1} p_k^T P_k \phi \leq \rho_k^{-1} p_k^T P_k \phi = p_k^T \phi.
\end{align*}$$

Thus, all inequalities must be equalities, and in particular $p_k^T P_k \phi = (p_k - p_k^T P_k) \phi = 0$. This shows that supp($(P_k \phi - \rho_k \phi) 1_{B_k} \subseteq$ supp($\phi$). On the other hand, because $(P_k y) 1_{B_k} = 0$ whenever supp$(y) \subseteq B_k^c$, we have $(P_k (\phi - \phi')) 1_{B_k} = (P_k (\gamma - \gamma')) 1_{B_k} = \rho_k (\phi - \phi')$ (recall that $\gamma - \gamma'$ is an eigenvector
with eigenvalue $\rho_k$). Hence $(P_k\phi)\mathbf{1}_{B_k} - \rho_k\phi = (P_k\phi')\mathbf{1}_{B_k} - \rho_k\phi'$, and the latter vector is supported in $\text{supp}(\phi')$ by a similar argument to above. Since the supports of $\phi$ and $\phi'$ are disjoint, this is possible only if $(P_k\phi)\mathbf{1}_{B_k} = \rho_k\phi$.

By Step 2, the sequence of matrices $\left\{\frac{1}{n}\sum_{j=0}^{n-1} P_k^n \right\}_{n \in \mathbb{N}}$ is bounded. Hence, the sequence $\left\{\frac{1}{n}\sum_{j=0}^{n-1} \rho_k^{-n} P_k^n \mathbf{1}_{B_k} \right\}_{n \in \mathbb{N}}$ converges to a limit, say $A'_k$. Let $\psi_k = A'_k\phi$ and $\psi'_k = A'_k\phi'$. Note that $\psi_k\mathbf{1}_{B_k} = \phi$ and $\psi'_k\mathbf{1}_{B_k} = \phi'$ by the previous paragraph.

It is also direct to check that $\psi_k$ and $\psi'_k$ are non-negative right eigenvectors for $P_k$ of eigenvalue $\rho_k$, and also that $\text{supp}(\psi_k) \cap \text{supp}(\psi'_k) \cap B_k = \emptyset$, as required.

• Conclusion.

If there exist eigenvectors $\psi_k, \psi'_k$ as above, then $\mu(\sum_{n \geq 2} L_k n) \leq C_k \leq \mu(\sum_{n \geq 2} \text{supp}(\psi_k) \cap \text{supp}(\psi'_k)) \subset B_k$. Hence, $\mu(h_k) = 0$ for all $h_k$ associated to a non-negative left eigenvector of $P_k$ of eigenvalue $\rho_k$. As before, in this case no $h_k$ could belong to a cone $C_n$, contradicting what was established in §4.3.1. Therefore, the geometric multiplicity of $\rho_k$ is one.

4.4. Proof of Lemma 4.5. For convenience, we present the proof assuming that in fact $\liminf_{k \to \infty} \rho_k \geq \rho$ and $\mu_k \overset{weak^*}{\rightharpoonup} \mu^\ast$. The same argument would remain applicable if taking subsequences was necessary.

Let $Z^n$ be the partition of $I$ into monotonicity intervals of $\hat{T}^n$. The lack of atoms for $\mu^\ast$ will hold if for every $\epsilon > 0$ there exists an $n_0$ such for all $n \geq n_0$ and $Z \in Z^n$

\begin{equation}
\limsup_{k \to \infty} \mu_k(Z) \leq \epsilon.
\end{equation}

In view of Lemma 3.1, for every $\varphi \in L^1(m)$,

\[\mu_k(\mathcal{L}_k \pi_k \varphi) = \rho_k \mu_k(\pi_k \varphi) = \rho_k \mu_k(\varphi).\]

By iterating the above expression, we have for any measurable $Z$,

\[\mu_k(Z) = \mu_k(1_Z) = (\rho_k)^{-n} \mu_k(\mathcal{L}_k^n \pi_k 1_Z)\]

so that for any $k, n, Z$,

\begin{equation}
\mu_k(Z) \leq \|\mathcal{L}_k^n \pi_k 1_Z\|_\infty \rho_k^{-n}.
\end{equation}

Now let $\epsilon > 0$ be given. The Ulam-admissibility assumption yields $2\|g\|_\infty < \rho$, where $g = |DT|^{-1}$. Choose $n_0$ and $\rho_0 \in (2\|g\|_\infty, \rho)$ such that $2(2\|g\|_\infty/\rho_0)^{n_0} < \epsilon$. Choose $K_0$ such that $\rho_k > \rho_0$ for $k \geq K_0$. Thus for any $k \geq K_0$ and $n \geq n_0$,

\begin{equation}
2(2\|g\|_\infty/\rho_0)^n < \epsilon.
\end{equation}

Let $n \geq n_0$ and choose $\delta = \min\{\text{diam}(Z) : Z \in Z^l \text{ for some } l = 1, \ldots, n\}$. Next, notice that if $I$ is a subinterval of some $J \in Z^l$ then the projection operator $\pi_k$ increases the support of the characteristic function of any interval by at most two subintervals of length $\eta_k$, where $\eta_k = \max\{m(I_j) : I_j \in P_k\}$. Thus,

\[
\text{diam}(\text{supp}(\mathcal{L}_k 1_J)) \leq 2\eta_k + \text{diam}(\text{supp}(\mathcal{L}_1 1_J)) \leq 2\eta_k + \text{diam}(\text{supp}(\mathcal{1}_{\hat{T}^l(I_J)})) \leq 2\eta_k + \|\hat{T}^l\|_\infty \text{diam}(I).
\]

Choose $K_n \geq K_0$ such that $(2 + \|\hat{T}^l\|_\infty)\eta_k < \delta$ for $k \geq K_n$. This choice will shortly allow us to control the growth of supports of $\mathcal{L}_k^n 1_Z$. The application of $\pi_k$ can also
transfer mass across discontinuity points of $\hat{T}$, so that even if $1_I$ and $\hat{L}1_I$ have connected supports, supp$\hat{\pi}_k 1_I$ may be disconnected; we call this splitting. We present an inductive construction that, at the $(l+1)$st-stage, allows us to keep track of the supports and $L^\infty$ norms of $((\pi_k \mathcal{L})^l \pi_k 1_Z)$, for $Z \in \mathcal{Z}^n$. This will be used to establish Equation (11), via Equation (12).

**First splitting:** Let $Z \in \mathcal{Z}^n$. Then $I_1 := \text{supp}(1_Z) = Z$ is an element of $\mathcal{Z}^n$. Write

\[
\text{supp}(\pi_k 1_Z) = I_{0L} \cup I_1 \cup I_{0R},
\]

where $I_{0L}, I_{0R}$ each have diameter bounded by $\eta_k$ and belong to elements of $\mathcal{Z}^n$ which are adjacent to $I_1$. Put

\[
\varphi_1 := \pi_k 1_Z |_{I_1}, \quad \varphi_{0L} := \pi_k 1_Z |_{I_{0L}}, \quad \varphi_{0R} := \pi_k 1_Z |_{I_{0R}},
\]

and $I_1 = \{1, 0L, 0R\}$. Then

\[
\pi_k 1_Z = \sum_{i \in I_1} \varphi_i,
\]

and each of the terms $|\varphi_i|$ is bounded above by 1.

$(l+1)$st splitting $(1 \leq l \leq n - 1)$: Suppose that $I_l$ is an index set where each $i \in I_l$ has the form $i = dx_1 \cdots x_{l-d}$ ($0 \leq d < l$) where each $x_j \in \{L, R\}$, or may be simply $i = l$. Each $I_l$ is an interval wholly contained in an element of $\mathcal{Z}^{n-(l-1)}$. When $i = l$ the interval $I_l$ is an entire element of $\mathcal{Z}^{n-(l-1)}$; in all other cases the interval $I_l$ has diameter bounded by $(2 + ||T'||_\infty)^{l-1} \eta_k$, and at least one of its endpoints is a boundary point of an element of $\mathcal{Z}^{n-(l-1)}$. Moreover,

\[
(L_k)^{l-1} \pi_k 1_Z \leq (\pi_k \hat{\mathcal{L}})^{l-1} \pi_k 1_Z = \sum_{i \in I_l} \varphi_i
\]

where supp$(\varphi_i) \subseteq I_l$ and $|\varphi_i| \leq (||g||_\infty)^{l-1}$. Now, upon applying $\pi_k \hat{\mathcal{L}}$ to each $\{\varphi_i\}$, each of $\hat{\mathcal{L}} \varphi_i$ has support in $\hat{T} I_1$, but the application of $\pi_k$ can cause leakage across boundary points of $\mathcal{Z}^{n-l}$. The application of $\pi_k$ to $1_{\hat{T} I_l}$ produces

\[
\text{supp}(\pi_k \hat{\mathcal{L}} \varphi_i) = I_{lL} \cup I_{l+1} \cup I_{lR},
\]

where $I_{lL}, I_{lR}$ each have diameter bounded by $\eta_k$ and belong to elements of $\mathcal{Z}^{n-l}$ which are adjacent to $I_{l+1} := \hat{T} I_l \in \mathcal{Z}^{n-l}$. For the other intervals $I_l$, notice that $\hat{T} I_l$ has diameter bounded by $(2 + ||T'||_\infty)^{l} \eta_k < \delta - \eta_k$ and shares a boundary point with $\mathcal{Z}^{n-l}$. Since each interval in $\mathcal{Z}^{n-l}$ has diameter bounded below by $\delta$, supp$(\pi_k \hat{\mathcal{L}} \varphi_i) = I_{lL} \cup I_{lR}$, where each of the intervals $I_{lx}$ has length bounded by $(2 + ||T'||_\infty)^{l} \eta_k$ and shares a boundary point with $\mathcal{Z}^{n-l}$. Put each $\varphi_{lx} = (\pi_k \hat{\mathcal{L}} \varphi_i) |_{I_{lx}}$ (where $x \in \{L, R\}$) and $\varphi_{l+1} = (\pi_k \hat{\mathcal{L}} \varphi_i) |_{I_{l+1}}$. Let $I_{l+1}$ be the new index set. Then

\[
(\pi_k \hat{\mathcal{L}})^{l} \pi_k 1_Z = \sum_{i \in I_{l+1}} \varphi_i.
\]

All the terms are wholly supported in an element of $\mathcal{Z}^{n-l}$ and are individually bounded by $(||g||_\infty)^{l}$.

*When $l = n$: The index set $I_n$ allows $(\pi_k \hat{\mathcal{L}})^{n-1} \pi_k 1_Z$ to be written as a sum of terms $\varphi_i$ where each $\varphi_i$ is supported in an element of $\mathcal{Z}$ (the monotonicity partition of $\hat{T}$), and*
each term is bounded by \((\|g\|_{\infty})^{n-1}\). Moreover, there are at most \(2^n - d\) indices of the form \(i = dx_1 \cdots x_{n-d}\) \((0 \leq d \leq n)\) in \(I_n\). Thus,

\[
\hat{L}(\pi_k \hat{L})^{n-1} \pi_k 1_Z = \sum_{i \in I_n} \hat{L} \varphi_i \leq \sum_{d=0}^{n-1} 2^{n-d} (\|g\|_{\infty})^{n} = (2^{n+1} - 1) (\|g\|_{\infty})^{n}.
\]

This inequality is preserved by the application of \(\pi_k\), and hence

\[
\hat{L}^n \pi_k 1_Z \leq (\pi_k \hat{L})^{n} \pi_k 1_Z \leq 2 (\|g\|_{\infty})^{n}.
\]

Equation (11) now follows from this final estimate because of equations (12) and (13). Finally, if \(f\) is continuous then \(\mathcal{L} f\) is discontinuous at only finitely many points (since \(g 1_{X_0}\) has only finitely many discontinuities). By the construction above, for any \(\epsilon > 0\) there are open intervals \(Z_1, \ldots, Z_M\) containing these discontinuities such that \(\mu_k(Z_1 \cup \cdots \cup Z_M) < \epsilon/\|\mathcal{L} f\|_{\infty}\) for all large enough \(k\). Then, \(\mathcal{L} f\) can be approximated by a continuous function \(\tilde{f}\) such that \(\|\mathcal{L}f - \tilde{f}\| \leq \|\mathcal{L} f\|_{\infty} 1_{Z_1 \cup \cdots \cup Z_M}\) and

\[
|\mathcal{L}^n \mu_k(f) - \mathcal{L}^n \mu^*(f)| = |\mu_k(\mathcal{L} f) - \mu^*(\mathcal{L} f)| \leq 2\epsilon + |\mu_k(\tilde{f}) - \mu^*(\tilde{f})|
\]

for all large enough \(k\). The last part of the lemma follows.

Acknowledgments. The authors thank Banff International Research Station (BIRS), where the present work was started, for the splendid working conditions provided. CB’s work is supported by the Pacific Institute for the Mathematical Sciences (PIMS) and NSERC. RM thanks the Department of Mathematics and Statistics at the University of Victoria for hospitality. CGT is partially supported by the UNSW School of Mathematics and an ARC Discovery Project (DP110100068), and thanks the Department of Mathematics and Statistics (University of Victoria) for hospitality during part of the period when this paper was written.

References


