1. Let $M$ be an $R$-module. Show that $\text{Hom}_R(M, -)$ is covariantly functorial and additive.

2. Let $G$ be a finite group. Show that we have a covariant functor $(-)^G : \text{Mod} - kG \to \text{Mod} - kG$ defined on objects by $M^G$ is the fixed submodule and on morphisms $f : M \to N$ by the restricted map $f^G = f|_{M^G} : M^G \to N^G$.

3. Let $\phi : R \to S$ be a ring homomorphism and $f : M \to N$ be an $S$-linear map. Show that $f$ is also $R$-linear in the sense that it defines an $R$-module homomorphism $f : M_R \to N_R$. Show that we have a change of scalars functor $F : \text{Mod} - S \to \text{Mod} - R$ defined on objects by $F(M) = M_R$ and on morphisms by $F(f) = f$.

4. Let $C$ be a category. We say that a morphism $\phi \in \text{Hom}_C(M, N)$ is an isomorphism if there exists $\psi \in \text{Hom}_C(M, N)$ with $\psi \phi = \text{id}_M$, $\phi \psi = \text{id}_N$. Prove that if $\phi$ is an isomorphism and $F : C \to D$ is a functor, then $F(\phi)$ is also an isomorphism.

5. Let $M$ be a right $R$-module and $N' \leq N$ be left $R$-modules. By functoriality, the inclusion $N' \hookrightarrow N$ and projection $N \to N/N'$ homomorphisms induce homomorphisms $\iota : M \otimes_R N' \hookrightarrow M \otimes_R N$, $\pi : M \otimes_R N \to M \otimes_R N/N'$. Show that $\pi$ is surjective with kernel $\text{im} \ i$. Hint: Use the universal property to construct an homomorphism $M \otimes_R N/N' \to (M \otimes_R N)/\text{im} \ i$ (which is inverse to the isomorphism given by the first isomorphism theorem).

6. Show that the set of isomorphism classes of 1-dimensional $kG$-modules forms a group with multiplication given by the tensor product. Show that this group is naturally isomorphic to the group $\hat{G} = \text{Hom}_{\mathbb{Z}}(G_{ab}, k^\times)$ of 1-dimensional representations.

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7. Let $V$ be the irreducible 2-dimensional $\mathbb{C}S_3$-module. Using Fourier/character theory or otherwise, decompose $V \otimes \mathbb{C} V$ into a sum of irreducibles.

8. Let $V$ be a finite dimensional $\mathbb{C}G$-module and $g \in G$ be an element of finite order. Show that $\chi_{V^*}(g) = \overline{\chi(g)}$. Hint: Consider the eigenvalues of an appropriate matrix $\rho_V(g)$.

9. Let $G$ be a dihedral group of order $2n$ where $n$ is odd. Show using character theory or otherwise, that any finite dimensional $\mathbb{C}G$ is isomorphic to its dual.

10. Let $G$ be the dihedral group of order 8. Compute the character table for $G$ over $\mathbb{C}$ as follows. Find the 4 1-dimensional representations of $G$ and the corresponding 4 rows of the character table. Use orthogonality to compute the remaining row.

11. Consider the representation of $S_4$ given by $\rho_W : S_4 \to GL_4(\mathbb{C})$ where 
   
   \[ \rho_W(\sigma) = (e_{\sigma(1)} \quad e_{\sigma(2)} \quad e_{\sigma(3)} \quad e_{\sigma(4)}) \]

   and $e_i \in \mathbb{C}^4$ are the standard basis vectors. This makes $W = \mathbb{C}^4$ into a $\mathbb{C}S_4$-module and hence also a $\mathbb{C}A_4$-module. Decompose $W_{\mathbb{C}A_4}$ into a direct sum of simple modules using Fourier/character theory or otherwise.