Throughout, $k$ will denote some field, $R$ a ring and $n$ some positive integer. This problem sets concerns lectures 15-20.

1. This question gets you to prove proposition 2 in lecture 16. Let $R = R_1 \times R_2 \times \ldots \times R_n$ and $\pi_j : R \rightarrow R_j$ be the canonical projection onto the $i$-th component. Note that $\pi_j$ is a ring homomorphism, so by change of scalars, any $R_j$-module is also naturally an $R$-module. Let $\iota_j : R_j \rightarrow R$ be the canonical injection into the $i$-th component.

(a) Let $e_j = \iota_j(1_{R_j}) \in R$. Show that $\{e_1, \ldots, e_n\}$ is a complete set of orthogonal idempotents in the centre of $R$.

(b) Show that $R_j = Re_j$. More precisely

$$Re_j = 0 \times \ldots \times 0 \times R_j \times 0 \times \ldots \times 0.$$

(c) Let $M$ be an $R$-module. Show that $M$ can be gotten by change of scalars from an $R_j$-module if and only if right multiplication by $e_j$ acts as the identity on $M$, or equivalently, $M(1 - e_j) = 0$.

(d) Show that $Me_j$ is an $R$-submodule of $M$ which comes from an $R_j$-module structure. Moreover, show $Me_j$ contains all other such $R_j$-submodules of $M$.

(e) Show that $M = Me_1 \oplus \ldots \oplus Me_n$ and that this is the unique way of writing $M$ as an internal direct sum $M = M_1 \oplus \ldots \oplus M_n$ with $M_j$ an $R_j$-module.

2. This question gets you to prove proposition 3 of lecture 16. Let $R$ be a commutative ring and $G$ be a group. Prove that the following defines a one-to-one correspondence between the class of $R$-linear representations of $G$ and the class of $RG$-modules. Given a representation $\rho : G \rightarrow \text{Aut}_R V$, the corresponding $RG$-module is the abelian group $V$ with scalar multiplication given by

$$(\sum_{g \in G} r_g g)v = \sum_g r_g \rho(g)v.$$

for any $v \in V, r_g \in R$. 

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\footnote{by Daniel Chan}
3. Let $G = \langle \sigma \rangle$ be a cyclic group of order $n$. We identify the vector space $\mathbb{C}G = \mathbb{C}^n$ using the basis $\{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}\}$. Then the $\mathbb{C}G$-module $\mathbb{C}G$ corresponds to a group representation of the form $\rho : G \to \text{GL}_n(\mathbb{C})$. Determine $\rho$ explicitly.

4. Let $V$ be the $\mathbb{R}$-space of $\mathbb{R}$-valued functions on $\mathbb{R}$. Consider the vector space automorphism $\sigma : V \to V : f(x) \mapsto f(-x)$.
   
   (a) Show that $\sigma$ generates a group $G$ of order 2.
   
   (b) The inclusion $G \hookrightarrow \text{Aut}_\mathbb{R} V$ is a group representation. Identify the fixed module $V^G$ with a well-known set.
   
   (c) Write out explicitly the Reynolds operator in this case.
   
   (d) Hence or otherwise, show that every function in $V$ can be written uniquely as the sum of an even and an odd function.

5. Let $R$ be a right semisimple ring. Show that the decomposition into simple modules $R_R = \bigoplus M_i$ must be finite. Hint: Show that the image of $1_R$ in each component must be non-zero.

6. Which of the following rings are semisimple? Justify your answers.
   
   i) $M_3(\mathbb{R}) \times \mathbb{C}$, ii) the ring of upper triangular matrices in $M_n(\mathbb{Q})$ in problem set 1, iii) the ring of diagonal matrices in $M_n(\mathbb{Q})$, iv) $\mathbb{F}_7G$ where $G$ is the dihedral group of order 10, v) $\mathbb{Z}G$ where $G$ is the symmetric group on 4 symbols.

7. Show for any finite abelian group $G$, that $\mathbb{C}G \cong \prod_{i=1}^{[G]} \mathbb{C}$ as $\mathbb{C}$-algebras.

8. For $G$ the cyclic group of order 3, find an explicit isomorphism $\mathbb{C}G \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}$.

9. For the cyclic group $G$ of order 4, find the Wedderburn components of $\mathbb{R}G$.

10. Let $A$ be a semisimple ring and $S$ be a simple $A$-module. Show that the isotypic component of an $A$-module $M$ corresponding to the simple $S$, is given by the sum of all submodules $N$ of $M$ which are isomorphic to $S$.

11. Let $G$ be a finite group and $A = \mathbb{C}G$. Given a left $A$-module $M$, show that $M^G$ is the isotypic component of $M$ corresponding to the trivial representation.

12. Let $G = S_4$. Find the abelianisation $G_{ab}$ of $G$. 

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13. Let $G$ be one of the two non-abelian groups of order 8 (either the dihedral group or the quaternion group). Show that $\mathbb{C}G \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

14. As mentioned in lecture 19, show that for a finite group $G$, isomorphism classes of $n$-dimensional $kG$-modules correspond to equivalence classes of representations $\rho : G \to GL_n(k)$ where $\rho \sim \rho'$ if there is some $T \in GL_n(k)$ such that $\rho'(g) = T \rho(g) T^{-1}$ for all $g \in G$.

15. Consider the dihedral group $D_n$ or order $2n$ where $n$ is even. Find the Wedderburn components of $\mathbb{C}D_n$ and the irreducible representations of $D_n$. 