This problem set covers lectures 11-14. Throughout, $k$ will denote some field and $R$ a ring.

1. Consider the sequence of $\mathbb{Z}$-modules

$$\ldots \to \mathbb{Z}/4\mathbb{Z} \overset{2}{\to} \mathbb{Z}/4\mathbb{Z} \overset{2}{\to} \mathbb{Z}/4\mathbb{Z} \overset{2}{\to} \mathbb{Z}/4\mathbb{Z} \overset{2}{\to} \ldots$$

where the maps are multiplication by 2. Show that the sequence is exact.

2. Let $R = M \oplus M'$ as $R$-modules. Hence we may write $1_R = e + e'$ where $e \in M, e' \in M'$. Show that $e$ is an idempotent of $R$.

3. A set $\{e_1, \ldots, e_n\}$ of idempotents for $R$ is said to be complete if $1 = e_1 + \ldots + e_n$ and orthogonal if $e_ie_j = 0$ for $i \neq j$. Show that if $\{e_1,\ldots,e_n\}$ is a complete, orthogonal set of idempotents and $B$ is an $(R,S)$-bimodule then we have a direct sum decomposition of right $S$-modules

$$B = e_1B \oplus \ldots \oplus e_nB.$$ 

4. Let $k$ be a field and $A$ be a finite dimensional $k$-algebra. Show that any finitely generated $A$-module $M$ is both noetherian and artinian.

5. Consider the subring $\mathbb{Z}[\frac{1}{2}] = \{\frac{n}{2^m} | n, m \in \mathbb{Z}\}$ of $\mathbb{Q}$. Note that since $\mathbb{Z}$ is a subring of $\mathbb{Z}[\frac{1}{2}]$, we may consider it as a $\mathbb{Z}$-module. Consider the quotient $\mathbb{Z}$-module $M = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$. Is $M$ noetherian? Is $M$ artinian? Justify your answer fully in either case. Hint: $\mathbb{Z}[\frac{1}{2}]$ is the union of submodules $\frac{1}{2^m}\mathbb{Z}$.

6. Which of the following rings are noetherian and/or artinian? Give full reasons for your answer. i) $\mathbb{Q}(\sqrt{7})$, ii) $M_3(\mathbb{R})$, iii) $M_3(\mathbb{Z}[i])$, iv) $k[x,y,z]/(y^2 - xyz^5)$, v) $k(x,y,z)/(y^2 - xyz^5)$, vi) $k(x,y)/(yx - qxy)$ for some $q \in k^\times$, vii) $\mathbb{Z}G$ for a finite group $G$.

7. Show that a module has finite length if and only if it is both noetherian and artinian.

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8. Consider the $k[x]$-module $M = k[x]/(x^3 - x^2)$. Show that $M$ has finite length by constructing a composition series for it. Write down the composition factors.

9. Let $R = k[x, y]$ and consider the $R$-modules $M = (R/(x, y))^3, N = R/(x^2, xy, y^2)$. Show that they have the same composition factors but are not isomorphic as $R$-modules.

10. Let $R = \mathbb{Z}[x]$ and $M = R/(6, x^2)$. Find the composition factors of $M$. Is $M$ a direct sum of simple modules?

11. This question concerns the first Weyl algebra $A_1$ which is the subalgebra of $E = \text{End}_\mathbb{C}\mathbb{C}[x]$ defined as follows. Let $\partial \in E$ be differentiation and $\lambda_x \in E$ be multiplication by $x$. Then by the universal property of free algebras, there is a unique $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}(x, y) \to E$ which sends $x$ to $\lambda_x$ and $y$ to $\partial$. Then $A_1$ is the image of $\phi$ and is often written as $\mathbb{C}(x, \partial)$. (Note that in this notation, the algebra is not the free algebra on $x, \partial$ and that $x$ really means $\lambda_x$.)

(a) Show that $\partial x = x\partial + 1$ so $A_1$ is non-commutative.

(b) Prove that $A_1 \cong \mathbb{C}(x, y)/(yx - xy - 1)$. Hint: It may be useful to show that a $\mathbb{C}$-basis for $A_1$ is $\{x^i\partial^j | i, j \in \mathbb{N}\}$.

(c) Prove that $A_1$ is noetherian.

12. Consider the matrices

$$
e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in $M_2(k)$ where $k$ is a field. They span the subalgebra $A$ of upper triangular matrices (see problem set 1, question 2).

(a) Show that $\{e_1, e_1\}$ form a complete set of orthogonal idempotents so any right $A$-module $M$ decomposes as a vector space in the form $M = Me_1 \oplus Me_2$. Show also that $(Me_1)f \subseteq Me_2$.

(b) Show that up to isomorphism, there are only two simple modules.

(c) Show that $A_A$ is the direct sum of two non-isomorphic indecomposable submodules and find their composition factors.

(d) Show that up to isomorphism, there are only 3 indecomposable $A$-modules. Hint: If $m \in M$ is non-zero, show that $km + kmf$ is a direct summand of $M$. 

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