Throughout, as in lectures, \( F \) always denotes a field.

(1) [8 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) Consider the permutation \( \sigma = [2 \ 3 \ 1] \). Then \( \sigma^2 = \text{id} \).
(ii) Let \( A \in M_{33}(\mathbb{Z}) \) be a matrix with determinant 2. Then the vector \( A^{-1}\left(\begin{array}{c} -8 \\ 4 \end{array}\right) \) has integer entries.
(iii) The row matrix \( \left( x^2 + 3x + 1 \ x^2 + 2x \ x + 1 \right) : \mathbb{R}^3 \to \mathbb{R}[x]_{\leq 2} \) defines a co-ordinate system on \( \mathbb{R}[x]_{\leq 2} \).
(iv) Let \( V^+ \leq M_{33}(\mathbb{R}) \) be the subspace of symmetric matrices and \( W \) be the subspace spanned by the matrices
\[
\begin{pmatrix}
1 & 2 & -1 \\
0 & 1 & 1 \\
0 & 0 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 0 & 0 \\
4 & 2 & 0 \\
-2 & 2 & 4
\end{pmatrix}.
\]
Then the sum \( V^+ + W \) is direct.

(2) [3 marks] Let \( P : \mathbb{R}^2 \to \mathbb{R}^2 \) be orthogonal projection onto \( \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \).

(i) Find the matrix representing \( P \).
(ii) Show that \( (\text{id} - P)^2 = \text{id} - P \).

(3) [3 marks] Consider the map \( T : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3 \) defined by
\[
Tf = \begin{pmatrix}
f(0) \\
f'(0) \\
f''(0)
\end{pmatrix}.
\]

(i) Show that \( T \) is linear by finding a representing matrix of linear maps or otherwise.
(ii) Show that \( T \) is an isomorphism.
(iii) Express the co-ordinate system \( T^{-1} : \mathbb{R}^3 \to \mathbb{R}[x]_{\leq 2} \) as a row vector with entries in \( \mathbb{R}[x]_{\leq 2} \).

PLEASE TURN OVER
(4) [3 marks]
Consider the linear map \( T : \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}^2 \) given by the matrix
\[
\begin{pmatrix}
1 & 1 & 2 \\
1 & 2 & 3
\end{pmatrix}
\]
with respect to the co-ordinate system \((1, x, x^2) : \mathbb{R}^3 \rightarrow \mathbb{R}[x]_{\leq 2}\) (and the standard co-ordinate system id : \(\mathbb{R}^2 \rightarrow \mathbb{R}^2\) on \(\mathbb{R}^2\)).

(i) Find \( T(3 - 2x^2) \).
(ii) Compute a basis for ker \( T \) and hence a co-ordinate system for ker \( T \).

(5) [3 marks] In this question, be sure to argue logically and provide complete proofs. Marks will be deducted for poorly written proofs.
Let \( T : V \rightarrow W \) be a surjective linear map and \( B \subseteq V \).

(i) Prove that if \( B \) is a spanning set for \( V \), then \( T(B) \) is a spanning set for \( W \).
(ii) If \( B \) is linearly independent, is it true that \( T(B) \) is linearly independent too?
Justify your answer fully.
Throughout, as in lectures, $\mathbb{F}$ always denotes a field. Recall also that $ev_x(f) = f(x)$.

(1) [8 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) The permutation $\sigma = [2 \ 1 \ 3]$ is even.
(ii) Consider the matrix
$$C = \frac{1}{5} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$ Then the matrix
$$C^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} C$$
represents reflection about a line in $\mathbb{R}^2$.
(iii) Let $W, W'$ be subspaces of a vector space $V$ and $T : W \oplus W' \rightarrow V$ be the natural map defined by $T(w) = w + w'$. Then $\ker T$ is isomorphic to $W \cap W'$.
(iv) The row matrix $(2x + 4 \ x - 3 \ x - 5) : \mathbb{R}^3 \rightarrow \mathbb{R}[x]_{\leq 2}$ defines a co-ordinate system on $\mathbb{R}[x]_{\leq 2}$.

(2) [3 marks] Consider the vector space $V = L(\mathbb{R}[x]_{\leq 1}, \mathbb{R})$ and note that we have the following vectors $\frac{d}{dx}, ev_1, ev_2 \in V$. Express $\frac{d}{dx}$ as a linear combination of $ev_1, ev_2$. (Recall that $ev_x(f) = f(x)$).

(3) [3 marks] Consider the subspaces
$$W = \mathbb{R}(1 + x^3) + \mathbb{R}(x + x^3), \quad W' = \mathbb{R}(1 + x + x^2 + x^3) + \mathbb{R}(3 + x^2 + 2x^3).$$
(i) Compute $W \cap W'$.
(ii) Is the sum $W + W'$ direct?

PLEASE TURN OVER
(4) [3 marks] Let $T : \mathbb{R}[x]_{\leq 1} \rightarrow \mathbb{R}[x]_{\leq 2}$ be the map defined by

$$(Tf)(x) = (x^2 - 2) \frac{df}{dx} - 3xf(x).$$

(i) Explain briefly why $T$ is linear.

(ii) Find the matrix representing $T$ with respect to the standard co-ordinate systems $(1 \ x) : \mathbb{R}^2 \rightarrow \mathbb{R}[x]_{\leq 1}$ and $(1 \ x \ x^2) : \mathbb{R}^3 \rightarrow \mathbb{R}[x]_{\leq 2}$.

(5) [3 marks] In this question, be sure to argue logically and provide complete proofs. Marks will be deducted for poorly written proofs.

Let $B = \{v_1, \ldots, v_n\}$ be a linearly independent set of vectors in the vector space $V$ and $v \in V$. Prove that $B \cup \{v\}$ is linearly independent if and only if $v \notin \text{Span}(B)$. 