**Aim lecture:** Linear maps can be combined via composition. We introduce this concept here.

**Prop**

Let \( S, T : V \rightarrow V', S', T' : V' \rightarrow V'' \) be \( \mathbb{F} \)-lin maps.

1. The composite fn \( T' \circ T : V \rightarrow V'' \) defined by \((T' \circ T)v = T'(Tv)\) is \( \mathbb{F} \)-linear.

2. \((S' + T') \circ (S + T) = S' \circ S + S' \circ T + T' \circ S + T' \circ T\) as maps from \( V \rightarrow V'' \). (Distributive law)

3. For \( \beta \in \mathbb{F} \) we have \((\beta T') \circ T = \beta(T' \circ T) = T' \circ (\beta T)\).

**Proof.** 1) Follows from checking axioms whilst 2) & 3) are computations. Note similarity with matrix arithmetic. We prove 2). For any \( v \in V \) we have

\[
[(S' + T') \circ (S + T)]v = (S' + T')(Sv + Tv) = S'(Sv + Tv) + T'(Sv + Tv) = S'(Sv) + S'(Tv) + T'(Sv) + T'(Tv) = (S' \circ S)v + (S' \circ T)v + (T' \circ S)v + (T' \circ T)v = [S' \circ S + S' \circ T + T' \circ S + T' \circ T]v
\]

so 2) follows.
Example of composite linear maps

E.g. Show that $T : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}) : f(x) \mapsto \frac{d^2 f}{dx^2} + 3f(x)$ is $\mathbb{R}$-linear.

A $\frac{d^2}{dx^2} = \frac{d}{dx} \circ \frac{d}{dx}$ is lin.$\therefore$

Hence $T$ is lin being the
Geometric example

E.g Let $P_u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be projn onto $u = \frac{1}{\sqrt{2}} (1, 1, 0)^T$. Show $P_u^2 \overset{\text{def}}{=} P_u \circ P_u = P_u$ geometrically & algebraically. Hence show algebraically the reflection $T = \text{id} - 2P_u$ satisfies $T^2 = \text{id}$.

A

Geometrically, for any $w \in \mathbb{R}u$ we have $P_\mathbb{R}u w = w$. But for any $v \in \mathbb{R}^3$ we have $P_\mathbb{R}u v \in \mathbb{R}u$ so

$P_\mathbb{R}u^2 v =$
Let \( A \in M_{lm}(F) \), \( B \in M_{mn}(F) \) so their associated linear maps are \( T_A : F^m \rightarrow F^l \) & \( T_B : F^n \rightarrow F^m \). Then

\[
T_A \circ T_B = T_{AB}.
\]

In other words, composites correspond to matrix multn.

**Proof.**

**Rem** Actually, the unusual definition of matrix multn was designed precisely to make this formula work.
Rotations

Rotation anti-clockwise through angle $\theta$ about $(0, 0) \in \mathbb{R}^2$ is given by the $2 \times 2$-matrix
Reminder on bijectivity

Prop-Defn

Recall that a function $f : X \longrightarrow Y$ is

1. **surjective or onto** if for any $y \in Y$ the eqn $f(x) = y$ always has a soln $x \in X$.
2. **injective or 1-1** if for any $y \in Y$ the eqn $f(x) = y$ has at most one soln $x \in X$.
3. **bijective** if for any $y \in Y$, there is a unique soln $x \in X$ to $f(x) = y$ i.e. $f$ is surjective & injective. This occurs iff $f$ is invertible, in which case $f^{-1}(y) = x$.

Recall also that $f \circ f^{-1} = \text{id}_Y$, $f^{-1} \circ f = \text{id}_X$ and indeed, these equations can be used to define invertibility & the inverse function.
Isomorphisms of vector spaces

Prop-Defn

An *isomorphism* of vector spaces is a bijective linear map \( T : V \rightarrow V' \). Given such an isomorphism, the inverse \( T^{-1} : V' \rightarrow V \) is also linear. We say \( V \) & \( V' \) are *isomorphic* and write \( V \cong V' \).

Proof. Good ex.

E.g. Let \( A \in M_{nn} \) be an invertible matrix. The assoc lin map \( T_A : \mathbb{F}^n \rightarrow \mathbb{F}^n \) is invertible with inverse \( T_{A^{-1}} \). Indeed for any \( w \in \mathbb{F}^n \), the unique soln to \( w = T_A v = A v \) is \( v = A^{-1} w = T_{A^{-1}} w \).

Defn

A *co-ordinate system* on an \( \mathbb{F} \)-space \( V \) is an isomorphism of the form \( C : \mathbb{F}^n \rightarrow V \).
Philosophy of isomorphisms & co-ordinate systems

In the geometric examples above, we often confused the vector space $V$ of geometric vectors in 3-dim space with $\mathbb{R}^3$ it is important to fully understand precisely what permits us to make this identification & the subtleties involved. Given a triple of non-coplanar vectors in $V$ we can put a co-ordinate system on $V$ i.e. find a co-ord system $C : \mathbb{R}^3 \rightarrow V$. Bijectivity of $C$ means that every geometric vector corresponds (via $C$) to a triple in $\mathbb{R}^3$.

To calculate with geom vectors, we can use $C^{-1}$ to pass to the corresponding co-ordinates in $\mathbb{R}^3$, then calculate using these co-ord, then use $C$ to pass back to geom vectors. If calculations only involve addn & scalar multn then linearity of $C, C^{-1}$ ensures we are fine. If the calculations involve other operations e.g. dot product, then we may need our co-ord system to be special for this to work. One subtlety is that there are usually many co-ord systems and a theme of linear algebra is they are not all the same! Some are better than others. Isomorphisms allow one more generally, to identify two vector spaces in the same way we identify $V$ with $\mathbb{R}^3$. 