**Aim lecture:** We recall the notion of a vector space which provides the context for describing linear phenomena. Throughout the rest of these lectures, $\mathbb{F}$ denotes a field.

**Defn**

A *vector space* over $\mathbb{F}$ (or $\mathbb{F}$-space) consists of a set $V$ of elements called *vectors* and two maps

1. **addition:** $V \times V \rightarrow V : (v, v') \mapsto v + v'$
2. **scalar multiplication:** $\mathbb{F} \times V \rightarrow V : (\alpha, v) \mapsto \alpha v$

such that the following axioms hold
Axioms for a vector space

### Axioms

For all \( \mathbf{v}, \mathbf{v}' \in V, \beta, \beta' \in \mathbb{F} \) we have

1. \((V, +)\) is an abelian group.
2. \((\beta \beta')\mathbf{v} = (\beta (\beta' \mathbf{v}))\)
3. \(1 \mathbf{v} = \mathbf{v}\)
4. \((\beta + \beta')\mathbf{v} = \beta \mathbf{v} + \beta' \mathbf{v}\)
5. \(\beta (\mathbf{v} + \mathbf{v'}) = \beta \mathbf{v} + \beta \mathbf{v'}\)

**Rem** It is a good example to show that (assuming 1) & 4), axioms 2) & 3) above amount to saying that the group \( \mathbb{F}^\times \) acts on \( V \).
Simple examples of vector spaces

E.g. 1 \( F^n \) is a vector space over \( F \)

E.g. 2 Geometric vectors in 3-dim space is a vector space over \( F = \mathbb{R} \).

E.g. 3 \( M_{mn}(F) \) is an \( F \)-space.
More examples

E.g. 4 $\mathbb{F}[x] = \text{the set of polynomials in the indet } x \text{ & co-eff from } \mathbb{F} \text{ is an } \mathbb{F}-\text{space.}$

E.g. 5 For a set $X$, the set of fns from $X \rightarrow \mathbb{F}$, denoted $\text{Fun}(X, \mathbb{F})$ is a vector space with pointwise addn & scalar multn.

i.e. For $f, g \in \text{Fun}(X, \mathbb{F}), \beta \in \mathbb{F}, x \in X$ we have

$$(f + g)(x) = f(x) + g(x), \quad (\beta f)(x) = \beta(f(x)).$$

E.g. 6 $\mathbb{0}$ is a vector space over any $\mathbb{F}$. 
Basic properties

You know the following (& should prove yourself if you’ve forgotten how)

### Prop

For any \( F \)-space \( V \) and \( v \in V, \beta \in F \):

1. The zero vector & negative of a vector are unique.
2. \( \beta 0 = 0 \)
3. \( 0v = 0 \)
4. \( (-1)v = -v \)
5. \( \beta v = 0 \implies \beta = 0 \) or \( v = 0 \).
One of the key points of the axioms of a vector space $V$, is that for $v_1, \ldots, v_n \in V, \beta_1, \ldots, \beta_n \in \mathbb{F}$ one can form the vector

$$\beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n.$$ 

Such a vector or expression is called a linear combination of $v_1, \ldots, v_n$.

Furthermore, axioms show such linear combinations can be manipulated arithmetically, in the way one handles $n$-tuples of reals.

E.g.
Subspaces

To get more examples of vector spaces

Prop-Defn

Let $V = \mathbb{F}$-space & $W$ be a non-empty subset. Then the following are equivalent.

1. $W$ is closed under addn & scalar multn i.e. for $w, w' \in W, \beta \in \mathbb{F}$ we have $w + w' \in W$ and $\beta w \in W$.

2. For any $w, w' \in W, \beta \in \mathbb{F}$ we have $\beta w + w' \in W$.

3. The linear combination of any set of elements of $W$ lies in $W$.

If these conditions hold, then the addn & scalar multn laws on $V$ restrict to addn & scalar multn laws on $W$ making $W$ an $\mathbb{F}$-space too. In this case we say $W$ is a subspace of $V$ & write $W \leq V$.

E.g. Any vector space $V$ has two trivial subspaces
Examples of subspaces

**E.g. 1** If one identifies $\mathbb{R}[x]$ with the set of real polynomial functions, then $\mathbb{R}[x]$ is a subspace of $\text{Fun}(\mathbb{R}, \mathbb{R})$.

**E.g. 2** Let $I \subseteq \mathbb{R}$ be an interval & $C^0(I)$ denote the set of continuous functions on $I$. Then $C^0(I)$ is a subspace of $\text{Fun}(I, \mathbb{R})$ since

**E.g. 3** Let $i$ be a positive integer or $\infty$ & $I \subseteq \mathbb{R}$ be an open interval. Let $C^i(I)$ denote the set of $i$-times continuously differentiable functions on $I$. Then $C^i(I)$ is a subspace of $C^j(I)$ for $j < i$.

**E.g. 4** For each $d \in \mathbb{N}$, the subset $\mathbb{F}[x]_{\leq d} \subseteq \mathbb{F}[x]$ of polynomials of degree $\leq d$ is a subspace.
Some basic results about subspaces

Prop

1. If $U \leq V$, $V \leq W$ then $U \leq W$. Similarly if $U \leq W$, $V \leq W$, $U \subseteq V$ then $U \leq V$.

2. If $v$ is a vector of an $F$-space then $Fv = \{\beta v | \beta \in F\}$ is a subspace.

3. Let $V$ be a vector space and $V_1, \ldots, V_r$ be subspaces. Then $\cap_{i=1}^r V_i$ is also a subspace of $V$.

4. With the above notation, $\sum_{i=1}^r V_i = V_1 + \ldots + V_r = \{\sum_i v_i | v_i \in V_i\}$ is a subspace of $V$ called the sum of the $V_i$.

Proof. All follow from checking closure axioms. We prove 4) as an example. Consider $w, w' \in \sum V_i, \beta \in F$. Then we may write $w = \sum_i v_i, w' = \sum_i v'_i$ for some $v_i, v'_i \in V_i$. Closure axioms ensure $\beta v_i + v'_i \in V_i$ so

$$\beta w + w' = \sum_i (\beta v_i + v'_i) \in \sum_i V_i.$$

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Lecture 7: Vector spaces
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Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be non-parallel vectors. Recall from MATH1141/MATH1151.

**Fact**

$\mathbb{R} \mathbf{v}$ is the line in $\mathbb{R}^3$ with parametric equation

**Fact**

$\mathbb{R} \mathbf{v} + \mathbb{R} \mathbf{w}$ is the plane in $\mathbb{R}^3$ with parametric equation
Example of sums

E.g. Let \( W = \mathbb{R}(1, 1, 0)^T + \mathbb{R}(0, 2, 1)^T, W' = \mathbb{R}(1, 1, 1)^T \). Does \( W + W' = \mathbb{R}^3 \)?
E.g. Find the intersection of the planes
\[ W = \mathbb{R}(1, 2, 1)^T + \mathbb{R}(1, 1, 1)^T, \quad W' = \mathbb{R}(0, 2, 1) + \mathbb{R}(1, 1, 0)^T. \]